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SOME ZERO-DIMENSIONAL GENERIC SINGULARITIES; FINITE ALGEBRAS HAVING SMALL TANGENT SPACE

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Summary

We exhibit Artin local algebras A, quotients $A = R/I$ of the power series ring $R = k[[x_1, \dots, x_r]]$, having very small tangent space $\mathrm{Hom}(I, A)$, and hence having as flat deformations (nearby algebras) only other algebras of the same type, and same kind. Here, by the type of A, we mean the sequence of integers $T = t_0, \dots, t_j, \dots$ where $t_i = \dim_k A_i$, the size of the ith homogeneous piece of the associated graded algebra A^* of A: the type is just the Hilbert function or characteristic function of A with respect to its maximal ideal. We would like to leave the notion "same kind" vague, to stimulate the imagination of the reader, but for the second half of the paper, it means "Gorenstein algebra".

We study also the family of those zero-dimensional Gorenstein algebras, quotients of R, having a certain maximal type T, and show the family is an irreducible variety of which we parametrize an open dense subset. We show that for some types T, these Gorenstein algebras have, in general, no deformations to $k[x]/x^n$; for some T they have in general, no deformations to the trivial algebra $k \oplus \cdots \oplus k$; and we indicate why for certain T, these algebras ought to have deformations only to other Gorenstein algebras of the same type.

When $k = \mathbb{C}$ or $\mathbb{R}$, the nontrivializable algebras correspond to stable maps germs $F: (\mathbb{C}^{r}, 0) \to (\mathbb{C}^m, 0)$, (or $(\mathbf{R}^{r}, 0) \to (\mathbf{R}^m, 0)$) where the local algebra A of F is finite of length n, but where nearby map-germs F_t must have less than n points in $F_t^{-1}(0)$: there are less than n-sheets over a neighborhood of 0 in $\mathbb{C}^m$, in all map-germs F_t close to F.

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1. Deforming Artin local algebras – Introduction

We suppose $R = k[[x_1, \ldots, x_r]]$ is the ring of power series, $R' = k[x_1, \ldots, x_r]$ is the polynomial ring in r variables, and that the algebra $A = R/I = R'/I'$ where $(x_1, \ldots, x_r) \supset I' \supset (x_1, \ldots, x_r)^r$ is an Artin local algebra of length n. We can think of A as the local ring of a finite map germ, as an algebra, or as the ring of a thick point – a singular (when $n > 1$) zero-dimensional subscheme Spec A concentrated at the origin of affine r-space Spec R'.

In §1.1 we give equivalent topological, algebraic, and geometric viewpoints on the three problems, whether there exist flat deformations of A to $k \oplus \cdots \oplus k$, to $k[x]/x^n$, and to algebras of type-kind different from that of A. We also give a short history of the problem. The type of the algebra A is the sequence of integers $T = t_0, \ldots, t_j, \ldots$ where $t_i = \dim_k A_i$, the size of the ith homogeneous piece of the associated graded algebra A^* of A: the type is just the Hilbert function or characteristic function of A with respect to its maximal ideal.

In §1.2 we sketch our method for showing type $1, 4, 3, 0$ algebras have in general no deformations to any algebras of different type than A (and in particular none to $k[x]/x^n$ nor to $k \oplus \cdots \oplus k$); and we survey the results of the paper. Then in §1.3 we give an overview comparing the situation in deformation Artin algebras to that known for deforming Lie algebras, and we introduce the sections following.

1.1. Three problems and three viewpoints

We give three almost equivalent viewpoints on the 3 problems of deforming the finite local algebra A of length n to $k \oplus \cdots \oplus k$ (n copies), to $k[x]/x^n$, and to algebras of kind or type different than A. Thus, questions 2a, 3a are equivalent, and when $k = \mathbb{R}$ or $\mathbb{C}$, the answer "yes" to question 1a implies "yes" to questions 2a, 3a. For a more detailed discussion, see the Appendix.

1. *Topological*

1a. *There are nearby germs with n sheets.* Suppose $k = \mathbb{C}$ or $\mathbb{R}$ and A is the local algebra at the origin of the finite stable map germ $F: (\mathbb{C}^{r'}, 0) \to (\mathbb{C}^m, 0)$ or $(\mathbf{R}^{r'}, 0) \to (\mathbf{R}^m, 0)$. Are there small deformations of F to F_t such that for U_ϵ a small enough ball around 0, there is a ball B_δ around 0 in $\mathbb{R}^1$ or $\mathbb{C}^1$ such that for $t \in B_\delta$, the equation $F_t = 0$ has n distinct solutions in U_ϵ? Equivalently, are there map germs near F having n sheets over a well-chosen neighborhood of 0 in $\mathbb{C}^m$? (Of course, there are many stable map germs F having A as local algebra; we assume one such F has been chosen.)

1b. *Aligning.* Are there small deformations of F to F_t such that there is one solution $p_t = F_t^{-1}(0)$ to $F_t = 0$ in U_ϵ, and such that the local algebra of F_t at p is isomorphic to $k[x]/x^n$?

1c. *The type of A changes along the flat locus.* Are there small deformations of F to F_t such that there is EITHER more than one solution to $F_t = 0$ in U_ϵ, but the sum of the lengths of the local algebras of F_t at these solutions remains n, OR there is only one solution p_t to $F_t = 0$ in U, the local algebra A' of F at p_t has length n, but the type or kind of A' is different from that of A?

The statement 1a is purely topological; we don't know if the statements 1b or 1c are.

2. *Algebraic*

2a. *Trivialization.* Does the algebra A have a flat deformation to $k \oplus \cdots \oplus k$? Here we mean a deformation of the structure constants giving the multiplication in A.

2b. *Deformation to the simplest Artin algebra.* Does the algebra A have a deformation to $k[x]/x^n$?

2c. *The algebra is not almost-generic.* Does A have EITHER a deformation to an algebra A' that is not local, OR a deformation to a local algebra of type or kind different from that of A?

3. *Geometric*

3a. *Smoothing.* Is there a flat deformation of the thick point $\operatorname{Spec} A$ in affine r-space $\mathbb{A}_r$ to a smooth subscheme $\operatorname{Spec} A_t$ or $\mathbb{A}_r$? For $t \neq 0$, $\operatorname{Spec} A_t$ would consist of n distinct points, $p_1(t), \ldots, p_n(t)$ each of multiplicity one, and the algebra $A_t = (R'/\Pi m_{p_i(t)})$ where $m_{p_i(t)}$ is the maximal ideal of $p_i(t)$ in R'.

3b. *Aligning.* Is there a deformation of $\operatorname{Spec} A$ to $\operatorname{Spec} A' = \operatorname{Spec}(R'/(x_1, \ldots, x_{r-1}, x_r^n))$? (The thick point $\operatorname{Spec} A'$ consists of the origin 0 of $\mathbb{A}_r$ with an infinitesimal tangent line of order n in the x_r-direction. When a function f of R' is evaluated at $\operatorname{Spec} A'$, one obtains its value and that of its first $n-1$ partial derivatives in the x_r direction, at the origin.)

3c. *The thick point is not almost-generic.* Is there EITHER a deformation of $\operatorname{Spec} A$ to a subscheme of $\mathbb{A}_r$ not concentrated at a single point, or to a subscheme of different type at 0, OR deformations to at least two different generic subschemes of $\mathbb{A}_r$? (An almost-generic thick point $\operatorname{Spec} A$ will have deformations only to other thick points of the same type, *and* the point z parametrizing $\operatorname{Spec} A$ will lie on a single component of the Hilbert scheme $\operatorname{Hilb}^n \mathbb{A}_r$.)

Notice that the algebra $k[x]/x^n$ has the deformation $k[x]/(x^n - t)$ which is isomorphic to $k \oplus \cdots \oplus k$ when $t \neq 0$, since then $x^n - t$ has n distinct roots. We do not know whether conversely the local algebra A being trivializable implies it has a deformation to $k[x]/x^n$.[1] Thus an answer "yes" to questions 1b, 2b, 3b implies "yes" to questions 1a, 2a, 3a, respectively.

Our viewpoint is geometric. The problem of deforming zero dimensional singular subschemes of $\mathbb{A}_r$ is interesting for two reasons:

i. There is a rumor that the problem of deforming s-dimensional schemes Y^S in X^r is related to that of deforming zero-dimensional schemes in an $(r-s)$ dimensional variety: it is embedding codimension that indicates the difficulty. Work of Schaps concerning the Cohen–Macaulay subschemes of codimension 2, the example of Mumford of an irreducible reduced curve in $\mathbb{P}^4$ having no nonsingular deformation; and when X^r is regular local, the result of Buchsbaum–

[1] (Added in proof) Kleppe [27] shows the converse is false: he shows every 3-generator Gorenstein algebra is trivializable; but Theorem 3.35 shows not all these have a deformation to $k[x]/x^n$.

Eisenbud concerning the Pfaffian structure of height 3 Gorenstein ideals, bear out the rumor.

ii. There is a scheme Hilb^n $\mathbb{A}_r$ parametrizing the length n zero-dimensional subschemes of $\mathbb{A}_r$. We use the tangent space Hom(I, A) to the point z in Hilb^n $\mathbb{A}_r$ parametrizing the subscheme Spec $A =$ Spec R'/I', to study the deformations of A.

We now give a short account of previous work. For 2-generator algebras, Hartshorne (unpublished), then Fogarty (1968) [8], Schaps (1970) [21], Briançon–Galligo (1972) [2], and Laksov (1975) [17] all gave various explicit deformations of A to $k \oplus \cdots \oplus k$. Briançon then showed in 1972 (see thèse, 1976) [1], that 2-generator local algebras A have deformations to $k[x]/x^n$, when k is algebraically closed. J. Briançon and J. Damon remark that work of Levine–Eisenbud shows the topological degree of the mapping, $\mathbb{R}^2 \to \mathbb{R}^2$ given by $(xy, x^{2a} - y^{2b})$ has absolute value 2; and the degree of the mapping (y, x^{2a+2b}) is 0; hence $\mathbb{R}[x, y]/(xy, x^{2a} - y^{2b})$ has no deformation to $\mathbb{R}[x]/(x^{2a+2b})$. Damon–Galligo used the trivialization result in 1975 to show that the type of a 2-generator algebra A is a C^0 invariant for a C^∞ stable map germ F having discrete algebra type of which A is the local algebra. (See also (Damon [2] and [4]) where these results are extended.)

For r-generator algebras, Damon–Galligo have exhibited certain algebras $A = R/I$ where I is a tower-like ideal, having trivializations (1975). On the other hand, in 1972 Iarrobino exhibited a nontrivializable local algebra: the argument was that the dimension of a certain family of algebras (computed via Hilb^n $\mathbb{A}_r$) is larger than the dimension rn of the open set U in Hilb^n $\mathbb{A}_r$ parametrizing trivial algebras. For comparison here, we summarize that example: suppose $r \geq 3$ and consider the family of algebras $\{A_V = R/(V, m^{j+1})\}$ where V is a vector space of degree j forms having size dim $V = \#R_j/2$, half the size of the space of all degree j forms in R. (We will use throughout $\#S$ to denote the vector space dimension of S over k.) The dimension of the family $\{A_V\}$ (the number of parameters) is dim Grass$(\#R_j/2, \#R_j) = (\#R_j/2)^2 \cong c(j^{r-1})^2$. The length n of each algebra A_V is $n = (\#R_0 + \cdots + \#R_{j-1} + \#R_j/2) \simeq c'j^r$. Thus the dimension of the family is $(c''n^{2-2/r}) > (rn)$ when $n \geq 3$ and n is large; hence the general algebra of the family cannot be trivializable since the dimension of U parametrizing sets of n distinct points in $\mathbb{A}_r$ is only rn. The existence of such nontrivializable algebras with 3 generators also implies the existence of "generic" local algebras with 3-generators, for which the answer to questions 1c, 2c, 3c is "no": the subscheme parametrized by a generic point z of a component of Hilb^n $\mathbb{A}_r$ other than $\bar{U}$ is a family of sets of thick points and points; the algebra A corresponding to one of the thick points will be "generic" and local (an "elementary component" of Hilb^n $\mathbb{A}_r$ in the language of [14]). However, till now, there were no specific examples known of such "almost generic" or generic algebras.

1.2. Examples of "generic" Artin algebras

We here give two specific examples of such "generic" algebras – in lengths 8 and 10, and for $r = 4$; and we indicate, identify, many more such components, up to a verification of the independence of certain linear conditions that we describe. The only method we've so far found to verify the independence of the linear conditions is to calculate them, and we verified the two simplest examples (where one might expect the conditions to be most degenerate). Our method is to bound the size of the tangent space Hom(I, A) to Hilb^n $\mathbb{A}_r$ at a point z of Hilb^n $\mathbb{A}_r$ parametrizing a local graded Argin algebra $A = R/I$; we show that for certain types of algebras A the tangent space is so small that the only deformations of A are other algebras of the same type. The two explicit examples are

1. Algebras of type $1, 4, 3, 0$ generated by 7 general enough quadratic forms in 4 variables.
2. Gorenstein algebras of type $1, 4, 4, 1$ generated by 6 polynomials of order 2 in 4 variables.

Precisely speaking, a generic algebra B of type 1, 4, 3, 0 is $B = k[x, y, z, w](\{a_{ij}\})/(g_1, \ldots, g_7)$, where $\{a_{ij} | 1 \leq i \leq 7,\ 1 \leq j \leq 3)$ are 21 variables and $g_i = u_i + a_{i1}z^2 + a_{i2}zw + a_{i3}w^2$, with $u_1, \ldots, u_7 = x^2$, xy, xz, xw, y^2, yz, yw respectively. We prove any deformation B' of B has in turn B as a deformation; and B is parametrized by a generic point of a component Z of the Hilbert scheme Hilb^n $\mathbb{A}_4$ parametrizing length 8 zero-dimensional subschemes, of affine 4-space; the component Z parametrizes only "thick points" – or in other words, singular length 8 subschemes concentrated at some point of $\mathbb{A}_4$. Thus Spec B is a "generic" 0-dimensional singular subscheme of $\mathbb{A}_4$. An almost generic algebra A of type $(1, 4, 3, 0)$ is for us a quotient $A = k(x, y, z, w)/I$ "close to" a generic algebra, in the sense A has deformations only to other algebras of the same type and kind. The point parametrizing A lies on a single component Z of Hilb^n $\mathbb{A}_r$, and an open subset of Z parametrizes only "thick points" of the same type as A; the coefficients of $(f_1, \ldots, f_7)$ defining I are in k. A generic point of the component Z parametrizes a generic algebra B of type $(1, 4, 3, 0)$, which is a deformation of A. We will use these words henceforth somewhat loosely; in particular we'll employ "generic" with quotes to mean almost generic.

These examples are not rigid, since there are moduli for the isomorphism classes of these algebras; we do not know if there exist rigid finite Artin algebras. Our method depends heavily on a natural gradation of the tangent space $\mathrm{Hom}(I, A)$ when I is graded, described and used by M. Schlessinger in (Schlessinger, 1973 [23]). We believe that this dependence of our simple method on this graded structure is what prevents our detection by it of "generic" algebras in 3 variables, (which are certainly not homogeneous, and which will require a somewhat finer method). However, there is evidence that over a closed field, 8 is very likely the smallest colength in which such "generic" local algebras exist: algebras of length no more than 5 have trivializations; algebras of length 6 and most types do (it remains to check the type $1, 3, 1, 1$); algebras of length 7 and types $1, 4, 2$ or $1, 3, 3$ (the types where there are moduli of isomorphism classes) also have trivializations (see [7]).

Our argument proving genericity for the $1, 4, 3, 0$ algebras is simple: suppose I is an ideal generated by 7 quadratic forms in R, and that $E \to F \to I \to 0$ is the beginning of a free resolution for I over R; then there are graded maps $\mathrm{Hom}(I, A) \to \mathrm{Hom}\, F, A \xrightarrow{\theta} \mathrm{Hom}\, E, A$ such that $\mathrm{Hom}(I, A)$ is the kernel of θ. The zero part $\mathrm{Hom}_0(I, A)$ of the tangents correspond to choosing a slightly different set of 7 quadratic forms; the positive part is 0; we show using the sequence that if I is sufficiently general, $\mathrm{Hom}_{-1}(I, A)$ is 4-dimensional, just large enough for the trivial tangents corresponding to the 4 independent partial derivatives; we prove a general lemma showing that then also $\mathrm{Hom}_{-2}(I, A) = 0$. It follows that for linear (tangent) reasons only, the ring A has deformations only to other rings of the same type. The ring $A = R/I$, with $I = (f_1, \ldots, f_7) = (x^2 + z^2,\ xy + w^2,\ xw,\ xz + wz,\ y^2 + z^2,\ yw,\ yz + w^2)$ is general enough to "work" here. We then generalize, but show something weaker; if I is the ideal generated by d forms of degree j in r variables, where d, j and r are chosen properly, then $(\dim_k(\mathrm{Hom}_{-1}(E, A)) + r) \geq \dim(\mathrm{Hom}_{-1}(F, A))$: there are enough conditions, *if independent*, to ensure the ring A has deformations only to other rings of the same type.

Our argument for the Gorenstein $1, 4, 4, 1$ algebras is similar. We begin with a graded $1, 4, 4, 1$ Gorenstein algebra $A = R/I$; we show that the zero part $\mathrm{Hom}_0(I, A)$ corresponds to choosing a slightly different graded Gorenstein algebra; the positive part $\mathrm{Hom}_1(I, A)$

corresponds to choosing a non-graded Gorenstein algebra A' having A as associated graded algebra, and we show that if I is general enough, the negative part $\mathrm{Hom}_{-1}(I, A)$ consists only of the trivial tangents. The graded Gorenstein algebra $A = R/I$, with $I = (f_1, \ldots, f_6) = (yz - x^2 - w^2,\ xz - y^2 - w^2,\ wz - x^2 - y^2,\ yw - 2x^2 - z^2,\ xw - 2y^2 - z^2,\ xy - 2w^2 - z^2)$ is general enough to work here: thus, it has as deformations only other Gorenstein algebras (not necessarily graded) of the same type 1, 4, 4, 1.

We then generalize, as before, to the largest Gorenstein algebras having symmetric type T, socle of degree j and r generators. First, we show the family $\mathrm{Gor}\, T \subset \mathrm{Hilb}^n R$ of such algebras is irreducible, and generically rational, of a dimension we calculate. (Theorem 3.34.) To prove this, we show the open subset $\pi^{-1}(G\,\mathrm{Gor}\, T)$ parametrizing those algebras quotients of R whose associated graded algebra is Gorenstein, is in fact a locally trivial fibration $\pi: \pi^{-1}G\,\mathrm{Gor}\, T \to G\,\mathrm{Gor}\, T$ with fibre an affine space, over $G\,\mathrm{Gor}\, T$, parametrizing the graded Gorenstein algebras of type T (Lemmas 3.3A and 3.3B). The variety $G\,\mathrm{Gor}\, T$ is itself an open set in the projective space $\mathbb{P}(R_j)$ (Theorem 3.31). The dimension calculation shows that for $r \geq 8$ and all j, or for $r \geq 4$ and $j \geq 9$, the general Gorenstein algebra of type T is not smoothable. (Theorem 3.35.) We then in section 3.4 extend the argument of section 2, and indicate for which types one might expect to prove genericity using the method of small tangent spaces.

1.3. Overview

Our work shows that there is a similarity in the structure of $\mathrm{SLalg}(k^n)$ parametrizing semilocal commutative algebras, and the structure of $\mathrm{Lie}(k^n)$ parametrizing Lie algebras of length n, as summarized for example in the thèse of Monique Levy-Nahas: different "kinds" of algebras determine the generic points of components of the parameter variety. (See Monique Levy-Nahas [18], M. Vergne [25].) It's just a matter of finding the right "kinds" to accurately describe the components. Perhaps this search will suggest new ways of divying up the finite commutative algebras, into kinds other than "type T", or "Gorenstein of type T". Our work also indicates that there are two elementary components of $\mathrm{Hilb}^{10} \mathbb{A}_5$, hence that the number of components of $\mathrm{Hilb}^n \mathbb{A}_5$ grows exponentially with n. (See [14].)

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We now outline the sections. In section 2.1 we give our main argument on smallness of the tangent space for the case $A = R/I$, where I is generated by the elements of a d-dimensional vector space of quadratic forms. We then calculate the pairs (d, r) where the argument should work. We give the example in section 2.2 – a calculation of the independence of the 24 conditions in 28 unknowns arising in the simplest case of algebras of type 1, 4, 3, 0. In section 2.3 we generalize the discussion of §2.1 to vector spaces of degree j forms, when r is large compared to j. We also show that if $\mathrm{Hom}_{-1}(I, A)$ has dimension r, and I is general enough, then all the negative part $\mathrm{Hom}_{-}(I, A)$ vanishes (Lemma 2.31).

In section 3.1 we give our argument for the Gorenstein algebras of type $1, r, r, 1$ when $r \geq 4$. In section 3.2 we give the calculation of independence in the simplest algebra we found that works for $r = 4$. Verifying the example involves finding the somewhat complicated relations between the 6 generators of a graded type 1, 4, 4, 1 Gorenstein ideal I, then showing the nonsingularity of the appropriate 20×20 matrix: the key is to choose an ideal I symmetric enough so that the relations can be hand-calculated, but general enough so that the conditions we use are independent. Then in section 3.3 we discuss the Gorenstein ideals of thin symmetric type T, showing their existence (Theorem 3.31), parametrizing them (Lemmas 3.33A and 3.33B, Theorem 3.34), and showing their nonsmoothability for large r (Theorem 3.35). We then in section 3.4 study their tangent space, comparing the sizes of $\mathrm{Hom}(F, A)$ and $\mathrm{Hom}(E, A)$ in Theorem 3.36. In these more complicated algebras, we need to use more of the presentation, to estimate the tangent space. Section 3.4 is conjectural, depending on an assumption concerning the presentation of the general Gorenstein algebra of type T.

Finally, in an appendix, we give the relation between dim Alg T and dim Hilb T, the dimensions of the parametrizations by structure constants and by the Grassmanian, of Artin local algebras of type T. We do this mainly to clarify the relation between these two parametrizations. One application of the appendix is that the known bounds on $\dim \mathrm{Hilb}^n \mathbb{A}_r$ (See [14]), [12]) determine bounds on $\dim \mathrm{SLalg}_r(k^n)$.

2. Graded algebras with small tangent space

We consider rings $A = R/I$ where I is the ideal generated by a sufficiently general d-dimensional vector space V of quadratic forms in the power series ring $R = k[[x_1, \ldots, x_r]]$ of dimension $r > 2$. We give a plan of argument showing that if d is well chosen for r, these rings A have no rigid deformation (and in particular they have no smooth deformation). The only deformation of such rings A, in cases where the argument can be carried out, will be algebras $A' \approx R/I'$ where I' is generated by another d-dimensional vector space of quadratic forms in R. If $E \to F \to I \to 0$ is the beginning of a minimal free resolution of I over R, we may assign degrees to the basis elements of F and E such that the maps have degree 0. We also can grade $\mathrm{Hom}(F, A)$, $\mathrm{Hom}(E, A)$, and $\mathrm{Hom}(I, A)$: we say $t \in \mathrm{Hom}_s(F, A)$ if degree $t(f) =$ degree $f + s$ when f is "homogeneous". Then $\theta: \mathrm{Hom}(F, A) \to \mathrm{Hom}(E, A)$ is a graded map with kernel $\mathrm{Hom}(I, A)$, and the first order deformations $T^1(A)$ are just $\mathrm{Hom}(I, A)/A(\partial/\partial x_1, \ldots, \partial/\partial x_r)$. The argument is simply that if V is not too special the linear conditions $\theta(t) = 0$ determining the degree -1 part $T^1_{-1}(A)$ of the first order deformations, ought to be irredundant. If so, and if the dimension d of V is well chosen, the -1 part of $\mathrm{Hom}(I, A)$ contains only the trivial tangents, the -2 and positive parts vanish, and the degree 0 part comes from deformations of the generators of I to other generators of the same degree. $\mathrm{Hom}(I, A)$ is the tangent space to the point z parametrizing I on the Hilbert scheme parametrizing ideals in $k[x_1, \ldots, x_r]$; we can explicitly describe there all the deformations producing degree 0 elements of $\mathrm{Hom}(I, A)$ or producing the trivial degree -1 elements; and there is no room in the tangent space for extra deformations. Thus the algebra A will be "generic". The quotients of R that are deformations of A, will be described by an open U in the Grassmannian $\mathrm{Grass}(d, R_2)$ parametrizing all d-dimensional V in R_2. The component of $\mathrm{Hilb}^n \mathbb{A}_r$ including a point z parametrizing A, will contain the open set $U \times \mathbb{A}_r$, and will be smooth and reduced at its generic point. Of the coordinates in U, there are $(r^2 - 1)$ parameters of $\mathrm{Pgl}(r-1)$ acting on $\mathrm{Grass}(d, R_2)$, and the rest $(d(\mathrm{cod}\, V) - (r^2 - 1))$ are the moduli of isomorphism classes of algebras A' of the same type near A. The closure $\overline{U \times \mathbb{A}_r}$ in $\mathrm{Hilb}^n \mathbb{P}_r$ will be an example of an elementary component of $\mathrm{Hilb}^n \mathbb{P}_r$ – one parametrizing only irreducible 0-dimensional subschemes of $\mathbb{P}_r$ (see [13]). (Here the colength n is $1 + r + \mathrm{cod}\, V$.) The dimensions d that work ought to be all those between about $\frac{1}{3}$ the size of R_2 ($\frac{1}{3}$ the dimension of the space of quadratic forms), and $(\#R_2 - 3)$. However, we have not found the key, the proof that the linear conditions $\theta(t) = 0$ that we find are usually irredundant, save in the simplest case $T = 1, 4, 3, 0$. So at the moment the method is mainly a scouting tool for finding elementary components of $\mathrm{Hilb}^n\, \mathbb{P}_r$. Any particular values of interest can be put into a computer and checked. The number of conditions is about rd, which ranges from about $r^3/6$ to $(r^3/2 + r^2/2)$.

In section 2.1 we give in more detail the general argument and describe which d should work; in section 2.2 we verify that it works in the simplest case – giving a colength 8 elementary "generic singularity", the quotient of $k[[x, y, z, w]]$ by the ideal generated by 7 general quadratic forms. To prove the example, we simply solve the 24×24 system of linear conditions defining $T^1_{-1}(A)$ when $I = (f_1, \ldots, f_7) = (x^2 + z^2,\ xy + w^2,\ xw,\ xz + wz,\ y^2 + z^2,\ yw,\ yz + w^2)$. In section 2.3 we comment on higher degree extensions of the same intuition.

2.1. Conditions satisfied by the degree −1 tangents

We view $A = R/I$ as the ring of the subscheme $X = \mathrm{Spec}\, A$ concentrated at the origin 0 of $\mathbb{A}_r = \mathrm{Spec}\, k[x_1, \ldots, x_r]$; the affine space $\mathbb{A}_r$ is embedded in $\mathbb{P}_r$; $\mathrm{Hilb}^n\, \mathbb{P}_r = H$ parametrizes the length n dimension 0 subschemes of $\mathbb{P}_r$. If $z \in \mathrm{Hilb}^n\, \mathbb{P}_r$ parametrizes X, then $N_X = \mathrm{Hom}(I, A)$ is the tangent space to H at z; the isomorphism classes of the first order deformations $T^1_x = T^1(A)$ is defined by the exact sequence

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{A}_r} \xrightarrow{\ \phi\ } N_X \longrightarrow T^1_x \longrightarrow 0.$$

The map ϕ takes the vector field $\chi = \Sigma\, f_i\, \partial/\partial x_i$ to $\phi\chi$ which maps $f \in I$ to the class of $\Sigma\, f_i\, \partial/\partial x_i \bmod I$; the image is the trivial first order deformations, arising from the Hilbert scheme H by merely changing the point in $\mathbb{P}_r$ where the singularity is defined from the origin to a nearby point in $\mathbb{P}_r$, and acting by an element of $\mathrm{Aut}\, R$. All the degree 0 tangents – mapping the generators of I to $A_2 = R_2/V$ – occur by changing $V = \langle f_1, \ldots, f_d\rangle$ to a nearby vector space $V' = \langle f_1', \ldots, f_d'\rangle$ in R_2; there are no nonzero positive degree tangents, provided $I \supset m^3$ (which will hold in our examples); if we can show $\mathrm{Hom}_{-1}(I, A)$ is just the trivial tangents and $\mathrm{Hom}_{-2}(I, A) = 0$, then the Hilbert scheme H near z is just $U \times \mathbb{A}_r$, with U an open in the Grassmannian G.

Suppose I is graded and the sequence $E \to F \to I \to 0$ begins a minimal graded free resolution of I, and that E_0 denotes the submodule of trivial relations; then there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(I, A) \longrightarrow \mathrm{Hom}_R(F, A)$$
$$\xrightarrow{\theta} \mathrm{Hom}_R(E/E_0, A) = \mathrm{Hom}(E', A).$$

The degree -1 tangents, those in $\mathrm{Hom}_{-1}(I, A)$ arise from $\mathrm{Hom}(F, A_1)$ where $A_1 = R_1$ is the linear part of A. Our goal is to show these lie in the image of ϕ, for V well-chosen. We count: there are rd elements of $\mathrm{Hom}(F, A_1)$. The relations in E/E_0 are the degree 3 relations ($\Sigma B_iF_i = 0$ with $B_i \in R_1$), and are the kernel of multiplication map: $(E/E_0)_3 = \mathrm{Ker}(R_1 \otimes V) \xrightarrow{M} R_3$. Thus if M is surjective

$$\dim(E_3') = \dim(R_1 \otimes V) - \dim R_3 = (rd) - \binom{r+2}{3}.$$

Now θ maps $(\mathrm{Hom}\, F, A_1)$ to $\mathrm{Hom}(E/E_0, A_2)$: If $t \in \mathrm{Hom}(F, A_1)$ takes f_i to $\ell_i \in A_1$ and $e \in E'$ is the degree 3 relation $\Sigma B_if_i = 0$, then $\theta(t)(e) = \Sigma B_i\ell_i \in A_2$. That $\theta(t) = 0$ is equivalent to the vanishing of s linear conditions whose variables are the rd coefficients of t. Here $s = \#(\mathrm{Hom}(E_3', A_2) = \dim E_3' \cdot \dim A_1 = (rd - \binom{r+2}{3})(\mathrm{cod}\ V)$. We know there is at least the solution space $\phi\langle \partial/\partial x_1, \ldots, \partial/\partial x_r\rangle$ which is r-dimensional if V is sufficiently general. If V is chosen general enough, the s linear conditions will, we expect, be as independent as possible, and we expect $\dim T^1_{-1}(A) = \dim \mathrm{Hom}_{-1}(F, A) - (\dim R_1 \otimes V - \dim R_3)(\mathrm{cod}\ V) - r$ (or zero when the quantity is negative). In other words we expect

$$\text{(2.1.1)} \qquad \dim T^1_{-1}(A) = \max\left(rd - \left(rd - \binom{r+2}{3}\right)\left(\binom{r+1}{2} - d\right) - r, 0\right)$$
$$= \max(h(r, d), 0).$$

We now show, assuming char $k \neq 2$

LEMMA 2.1: *If (with the notation above)* $\dim V \geq 2$, *and* $T^1_{-1}(A) = 0$, *then also* $T^1_{-2}(A) = 0$.

PROOF: Suppose $t \in \ker \theta: \mathrm{Hom}(F, A_0) \to \mathrm{Hom}(E', A_1)$. We choose the basis $\{f_i\}$ of V such that $t(f_1) = 1$, $t(f_2) = 0$. Then

$\langle x_1t, \ldots, x_rt\rangle \subset \mathrm{Hom}(F, A_1)$ is an r-dimensional subspace (the images of f_1 being independent) coming from $\mathrm{Hom}(I, A)$, so must be the same subspace as $\phi\langle \partial/\partial x_1, \ldots, \partial/\partial x_r\rangle$ since $T^1_{-1}(A) = 0$. This would imply all the partials are 0 on f_2. $\Rightarrow\Leftarrow$ ■ (See also Lemma 2.31)

We give here a table of solutions of

$$(2.1.2) \quad h(r, d) \leq 0, \text{ with } h(r, d) = rd - \left(rd - \binom{r+2}{3}\right)\left(\binom{r+1}{2} - d\right) - r.$$

We have shown that when $d = (\#R_2 - 2)$, the algebra $A = R/(V)$ has smooth deformations, so that value of d is omitted from the table.

Table of Solutions to Key Inequality: $h(r, d) \leq 0$, and $d \neq (\#R_2 - 2)$

$r = \#$ variables	$d =$ dimension	$n =$ colength	
4	7	8	
5	$8, \ldots, 12$	$13, \ldots, 9$	respectively
6	$11, \ldots, 18$	$17, \ldots, 10$	respectively
7	$14, \ldots, 25$	$22, \ldots, 11$	respectively.

When $r \geq 5$, and d satisfies

$$(2.1.3) \qquad (r+2)(r+1)/6 + 2 \leq d \leq \#R_2 - 3 = r(r+1)/2 - 3$$

then $h(r, d) \leq 0$. When r is very large, the constant 2 on the left of (2.1.3) can be reduced to 1, then to slightly over $\frac{1}{2}$.

In other words, we expect that if one chooses any number between about $\#R_2/3$ (see (3) for precise limits) and $(\#R_2 - 3)$ of general enough quadratic forms in R, the ideal I in R they generate has no deformations other than ideals I' similarly formed.

Of course, when completed, this is a first order argument: there could be more values of d yielding "generic singularities". (The skeptic would say there could be less!) Incidentally, this style argument gives no examples in 3 variables: at the time of writing, although we know there are generic singularities of colength no more than 102 in 3 variables (probably in colengths much less), we have no *explicit* examples of embedding codimension 3.

The line of argument suggested can be used to limit the size of $\mathrm{Hom}(I, A)$, even when it doesn't show $T^1(A)$ is 0. Emsalem remarks that in certain cases this gives further examples of non-smoothable ideals, or algebras having no deformation to $k[x]/x^n$; we showed in §1.2 that to have a smooth deformation, $\#(\mathrm{Hom}(I, A))$ must be at least

rn; and to have a deformation to an s-dimensional component of $\mathrm{Hilb}^n\, \mathbb{P}_r$ or of $\mathrm{Hilb}^n\, R$, the dimension $\#(\mathrm{Hom}(I, A))$ must be at least s. When $r = 4$, and $d = 6$, $n = 9$, the argument above suggests $\#(\mathrm{Hom}_{-1}(I, A)) = 4.6 - (24 - 20)(4) = 8$ and similarly (a slight extension) that $\#(\mathrm{Hom}_{-2}(I, A)) = 0$; we know $\#(\mathrm{Hom}_0(I, A) = \dim \mathrm{Grass}(6, R_2) = 24$, hence we expect $\#\mathrm{Hom}(I, A) = 32$ instead of the needed $rn = 36$ to have smooth deformations. That this argument does not work in the very special case cod $V = 2$ can be seen when $r = 3$ and $d = 4$; here the formula (1) would predict 16 for $\#\mathrm{Hom}(I, A)$. But it turns out there is one extra independent degree -1 first order deformation in $T^1_{-1}(A)$, and there is even a non-zero element of $T^1_{-2}(A)$. Thus $\dim \mathrm{Hom}(I, A) = 18$ as needed, and in fact the "generic" ideal $I = (x^2, y^2, xz - z^2, yz - z^2)$ of type $(1, 3, 2)$ has smooth deformations.[1]

We propose the verification of this argument above for the d and r of the table (or (2.1.3)) as a nontrivial problem. A natural approach is to try to construct inductively a sequence of examples, one ideal (as simple as possible) for each pair d, r in the table, and to show inductively that they work. (The example we give in the section 2.2 following is the first step in the induction!) We mention in passing that the status of this general sort of problem in 3 or more variables is rather poor. A similar problem, also in general unsolved, is "given d, r, j, i what is the expected dimension $\#R_iV$, for V a "generic" d-dimensional vector space of degree j forms?

2.2. Example of a length 8 algebra having as deformations only other algebras of the same type

We now give the example of an "almost generic" algebra of type, $1, 4, 3, 0$, and we show it is not rigid by describing the 6 moduli of isomorphism classes of $1, 4, 3, 0$ algebras. The irredundancy of the conditions described in section 2.1 amount to the nonvanishing of a determinant whose entries are polynomials in the coefficients of the

[1] Algebras of type $1, r, 2, 0$ are smoothable. For the general case see (Ems–Iar). When $r = 3$ we give the proof here. An open dense set of these algebras are $A = R/I$ with $I \cong (x^2, y^2, xz - z^2, yz - z^2)$. The two degree 3 relations are $z(\underline{x}^2 - \underline{y}^2) - (x + z)(\underline{xz - z})^2 + (y + z)(\underline{yz - z^2}) = e_1$ and $(x - z)(\underline{yz - z^2}) - (y - z)(\underline{xz - z^2}) = e_2$; these and the trivial relations generate all the relations. The deformation of I in the polynomial ring $k[x, y, z]$ to $I(t)$ replacing y^2 by $y^2 + t(x - y)$ and replacing the first relation e_1 by $(e_1 - t(\underline{xz - z^2}) + t(\underline{yz - z^2}))$ is certainly flat, since all relations in I extend to $I(t)$. When $t \neq 0$, the part $I(t)_0$ of $I(t)$ resting at the origin has colength 4 and Hilbert function $1, 2, 1, 0$. Being an ideal in essentially 2 variables it is smoothable; the rest of $I(t)$ has colength $6 - 4 = 2$, so is smoothable. Thus $I(t)$ for $t \neq 0$ is smoothable, and so is I.

f_i's generating I. Thus it suffices to produce a single example to validate the discussion for a given pair d, r. The guideline is, the example has to be simple enough to calculate easily, but complex enough to work. Our example has $r = 4$, $d = 7$, $n = 8$, and is the ideal $I = f_1, \ldots, f_7 = (x^2 + z^2, xy + w^2, xw, xz + wz, y^2 + z^2, yw, yz + w^2)$. It is easy to check $R_1 \otimes V \to R_3$ is surjective. In order to be able to write down the $(7 \cdot 4 - 20) = 8$ relations easily, we chose which combinations of generators to use in getting w^3, w^2z, wz^2, z^3: respectively $wf_7 - zf_6$, $wf_4 - zf_3$, $wf_5 - yf_6$, and $zf_5 - yf_7 + wf_6$. We assumed f_i is deformed to $t(f_i) = f_i + L_i$, with $L_i = L_{i1}x + L_{i2}y + L_{i3}w + L_{i4}z$, and L_{ij} constants in k. By subtracting off multiples of $\phi(\partial/\partial x_i)$ (thus working in $T^1_{-1}(A)$) we may assume $L_{11} = L_{21} = L_{31} = L_{41} = 0$. We then write the inner product of each linear relation $e = \Sigma\, B_i f_i$ with the deformation, reducing immediately the result $\Sigma\, B_i L_i$ to the complementary basis w^2, wz, z^2 to V in R_2, thus calculating $\theta(t)(e)$ in A_2. Since $h(4,7) = 0$ each condition – resulting coefficient on w^2 or wz, or z^2, must count, and must not be in the span of previous conditions for the example to work (if $h(r,d) = -s$, with $s = 0$, we'd expect s redundant conditions in an example that works). Naturally, we begin with the simplest relations. We include the rest of the calculation for completeness.

CALCULATION: To start $L_{11} = L_{21} = L_{31} = L_{41} = 0$

Relation e_1: Involving 3, 6, $yf_3 - xf_6$

$$\theta(t)(e_1) = L_{32}y^2 + L_{34}yz - x^2L_{61} - xyL_{62} - xzL_{64}$$

(terms xw, yw are in I so are 0 in A_2).

$\theta(t)(e_1)$ written in a basis of w^2, wz, z^2

$$A_2 = (-L_{32} + L_{62})w^2 + (L_{64} - L_{34})wz + L_{61}z^2.$$

CONDITIONS: $L_{32} = L_{62}$, $L_{34} = L_{64}$, $L_{61} = 0$.

Relation e_2, 23(67), $wf_2 - yf_3 - wf_7 + zf_6 = w(f_2 - f_7) - yf_3 - zf_6$

$$\theta(t)(e_2) = (L_{23} - L_{73})w^2 + (L_{24} - L_{74} + L_{63})wz - L_{32}y^2 + (L_{62} - L_{34})yz + L_{64}z^2$$

$L_{23} - L_{73} - (L_{62} - L_{64}) = 0$, $L_{24} - L_{74} + L_{63} = 0$, $L_{64} + L_{62} = 0$

on $-w^2$, wz, z^2

Relation e_3, 24(36), $zf_2 - yf_4 + zf_6 - wf_4 + zf_3$

$$L_{43} + L_{22} - L_{44} + 2L_{62} = 0,\ L_{23} + L_{63} - L_{44} + L_{33} = 0,\ L_{24} + 2L_{64} + L_{42} = 0,$$
on $-w^2$, zw, and z^2.

Relation e_4, 13(56), $wf_1 - xf_3 - wf_5 + zf_6$

$$\theta(t)(e_4) = w^2(L_{13} - L_{53}) + wz(L_{14} - L_{54} + L_{63}) - xyL_{32}$$
$$- xzL_{34} + yzL_{62} + z^2L_{64}$$
$$L_{64} = 0,\ L_{13} - L_{53} = 0,\ L_{14} - L_{54} + L_{63} = 0 \quad \text{on } z^2,\ w^2,\ zw.$$

Status after 4 relations: $L_{11} = L_{21} = L_{31} = L_{41} = L_{32} = L_{34} = L_{62} = L_{64} = L_{61} = 0$, $L_{23} = L_{73}$, $L_{13} = L_{53}$, $L_{24} = -L_{42}$, $-L_{42} - L_{74} + L_{63} = 0$, $L_{22} + L_{43} - L_{44} = 0$, $L_{14} - L_{54} + L_{63} = 0$, $L_{33} + L_{73} - L_{44} + L_{63} = 0$.

Relation e_5, 27(34), $zf_2 - xf_7 - wf_4 + zf_3 + wf_3$

$$\theta(t)(e_5) = yzL_{22} + wz(L_{23} + L_{33} - L_{44}) + w^2(L_{33} - L_{43}) + z^2L_{24} - x^2L_{71} - xyL_{72}$$
$$- xzL_{74} \mod V$$
$$= wz(L_{33} - L_{44} + L_{73} + L_{74}) \quad + z^2(L_{71} - L_{42}) \ + w^2(-L_{22} + L_{33} - L_{43} + L_{72})$$

substitute for L_{33} substitute for L_{22}, L_{33}

$$wz(L_{74} - L_{63}) + \ z^2(L_{71} - L_{42}) + w^2(L_{43} - L_{44} + L_{44} - L_{63} - L_{73} - L_{43} + L_{72})$$

Conclude $L_{63} = L_{74} \Rightarrow L_{42} = 0$, $L_{24} = 0$ $\quad L_{71} = 0 \quad$ $L_{72} - L_{73} - L_{74} = 0$

Relation e_6, 25(46) $yf_2 - xf_5 - wf_6 + zf_4 - wf_5 + yf_6$

$$Y^2L_{22} - w^2(L_{53} + L_{63}) + wz(L_{43} - L_{54}) - x^2(L_{51}) - xy(L_{52}) - xzL_{54} + z^2L_{44}$$
$$= w^2(L_{52} - L_{53} - L_{63}) + wz(L_{43} - L_{54} + L_{54}) + z^2(L_{44} - L_{22} + L_{51})$$
$$L_{52} - L_{53} - L_{74} = 0, \qquad L_{43} = 0 \Rightarrow L_{22} = L_{44}, \qquad L_{51} = 0$$

Relation e_7, 14(3567), $zf_1 - xf_4 + zf_3 - zf_5 + yf_7 - wf_6$

$$zy(L_{12} - L_{52} + L_{74}) + wz(L_{13} + L_{33} - L_{53}) + z^2(L_{14} - L_{54})$$
$$- xzL_{44} + y^2L_{72} - w^2L_{63}$$
$$= z^2(L_{14} - L_{54} - L_{72}) + wz(L_{13} + L_{33} - L_{53} + L_{44})$$
$$+ w^2(L_{52} - L_{12} - L_{74} - L_{63})$$

Substitute for L_{14}. Substitute for L_{13} $\quad -(L_{12} - L_{52} - 2L_{74})$

$$L_{72} = -L_{74} \Rightarrow L_{73} = -2L_{74},\ L_{33} = -L_{44} \Rightarrow 2L_{44} = -L_{74},$$
$$L_{12} - L_{53} - 3L_{74} = 0$$

Relation e_8, 12(347), $yf_1 - xf_2 + wf_3 - zf_7 + wf_4 - zf_3$

$$\begin{aligned}\theta(t)(e_8) &= y^2L_{12} + yz(L_{14} - L_{72}) - xyL_{22} + w^2L_{33} + wz(L_{44} - L_{33} - L_{73}) \\ &\quad - z^2L_{74} \\ &= wz(L_{44} - L_{33} - L_{73}) + z^2(-L_{74} - L_{12}) \\ &\quad + w^2(L_{33} + L_{22} + L_{72} - L_{14}) \\ &\quad 2L_{44} - L_{73} = 0 \\ &\quad L_{74} = 0 \qquad \text{by } e_6\end{aligned}$$

$$L_{63} = L_{72} = L_{73} = L_{44} = L_{33} = 0,\ L_{22} = 0.\ L_{12} = 0,\ L_{53} = 0 \Rightarrow L_{52} = 0,$$
$$L_{14} = 0,\ L_{54} = 0.$$

A check of the L_i shows that t is 0, hence $T^1_{-1}(A) = 0$, as claimed. Thus the algebra A has deformations only to other algebras A' isomorphic to R/I' where I' is the ideal generated by 7 linearly independent quadratic forms in R, and thus A is an example for the discussion in section 2.1.

We now describe the 6 moduli for isomorphism classes of 1, 4, 3, 0 algebras, or in other words, the 6 moduli for orbits of 7-dimensional vector spaces of quadratic forms in $k[x, y, z, w]$ under the action of Pgl(3). We can instead classify the dual spaces V of 3-dimensional forms (see [15]) or [7] for a description of the duality). Three quadrics in $\mathbb{P}_3$ determine 8 points of intersection but 7 of the points in general enough position suffice to determine both a vector space V and the 8th point common to the quadrics of V. This is true since passing through a point is a linear condition on the 10 coefficients of the quadratic form in 4 variables, so the condition of passing through 7 of the points determines a 3-dimensional vector space V of forms. The set of 7 points can be chosen almost arbitrarily (parametrized by an open in $\mathrm{Symm}^7(\mathbb{P}_3)$). Under the action of Pgl(3) five points of $\mathbb{P}_3$ in general enough position can be moved to $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(1,1,1,1)$. The 6 coordinates of the two remaining points of the seven are the 6 parameters for moduli of isomorphism classes of the vector spaces V, and also of the general type 1, 4, 3, 0 algebras.

2.3. Ideals generated by vector spaces of degree j forms

We consider algebras of the form $A = R/I$, where I is the ideal generated by a d-dimensional vector space V of degree j forms in

$R = k[[x_1, \ldots, x_r]]$. First we give a lemma reducing the work of finding "generic" algebras of this kind, to showing the size $\dim_k \mathrm{Hom}_{-1}(I, A)$ is r coming from the trivial tangents; we thus reduce the work to a check on the degree -1 part of the tangent space (Lemma 2.31). We then fix j, and describe for large r, the range of values d where the function $h(d, r, j) = (\dim_k(\mathrm{Hom}_{-1}(F, A)) - \dim_k(\mathrm{Hom}_{-1}(E', A)) - r)$ is zero or negative. (Lemma 2.32.) These are values of d, r, j with r large, for which we expect our argument to work: if V is general enough, the corresponding algebra A should have deformations only to other algebras of the same type. Finally we show that $h(d, r, j)$ is non-positive when $r = 5$, $j = 2$, and d is the integer $\#R_j/2$ or $(\#R_j/2+1)$, in Lemma 2.33.

We now generalize Lemma 2.1.

LEMMA 2.31: *Suppose $r, j \geq 2$, and the ideal $I = (V)$ is generated by a degree j vector space of forms V. If* char $k \neq 0$ *and divides $(j-2)$, assume further that V contains a form f that cannot be written as a product $(c)(x_1^2 + \cdots + x_r^2)g(x_1^p, \ldots, x_r^p)$ with c a constant, and g a polynomial. Then $T^1_{-1}(A) = 0$ implies* $\mathrm{Hom}_{-s}(I, A) = 0$ *for $s > 1$.*

PROOF: It suffices to show $\mathrm{Hom}_{-2}(I, A) = 0$. Suppose, by way of contradiction that $\mathrm{Hom}_{-2}(I, A)$ contains the non-zero homomorphism h. Then the multiples $x_1h, \ldots, x_rh$ form an r-dimensional subspace of the trivial tangents T in $\mathrm{Hom}_{-1}(I, A)$; since $\#T \leq r$, the two spaces must be equal. Thus, there is a nonsingular $r \times r$ matrix of constants M such that the vector $(x_1h, \ldots, x_rh) = (\partial/\partial x_1, \ldots, \partial/\partial x_r)M$. It is easy to see that if $(x_1', \ldots, x_r') = (x_1, \ldots, x_r)A$ is a change of coordinates, then in new coordinates the vector $(x_1'h, \ldots, x_r'h) = (\partial/\partial x_1', \ldots, \partial/\partial x_r') \cdot A^TMA$. After extending k to its algebraic closure, we may choose A such that $A^TMA = $ identity. Writing $x_1, \ldots, x_r$ for these new coordinates, we conclude that for each f in V, and for each i, the form $x_ih(f) = \partial f/\partial x_i$. By Euler's formula in R, the sum $\Sigma_i x_i^2 h(f) = j \cdot f$.

Now if char $k = p$ and divides j, we conclude $h(f) = 0$ for all the generators of I, hence $h = 0$. Otherwise, we take the partial derivatives of the Euler identity, and use $x_ih(f) = \partial f/\partial x_i$ to conclude

$$\begin{aligned} 2x_ih(f) + x_i^2 \cdot \partial h(f)/\partial x_i &= j \cdot (\partial f/\partial x_i) = j \cdot x_ih(f) \\ (j-2)x_ih(f) &= x_i^2 \cdot (\partial h(f)/\partial x_i) \\ (j-2)h(f) &= x_i(\partial h(f)/\partial x_i). \end{aligned} \tag{2.3}$$

If $\operatorname{char} k = p$ and divides $(j-2)$, all partials of $h(f)$ are 0, and $h(f) = g(x_1^p, \ldots, x_r^p)$ which with the Euler formula above contradicts our assumption there is an f in V that cannot be written $c(\Sigma x_i^2)g(x_1^p, \ldots, x_r^p)$. Otherwise, after writing $h(f)$ as a polynomial in x_i with coefficients polynomials in the remaining variables, it is easy to see that (2.3) implies $h(f) = cx_i^{(j-2)}$ for each i, hence $h(f) = 0$ for each f in V, and the homomorphism h is 0. The contradiction with our assumption $h \neq 0$ completes the proof. ■

The condition on V when $(\operatorname{char} k)|(j-2)$ is not a restraint for us, since we are concerned with general vector spaces V of dimension d in R_j and most spaces V satisfy the condition.

Notice that $h(d, r, j) = (\#R_{j-1})d - (rd - \#R_{j+1})(\operatorname{cod} V) - r$.

LEMMA 2.32: (*Range of dimension d which should work, when r is large compared to j.*) *Suppose j is fixed, and the constants* $a > 1/(j+1)$ *and* $b > (j+1)/j!$ are chosen. There is an $r(a, b)$ *such that when* $r > r(a, b)$ *and* $a(\#R_j) < d < (\#R_j - br^{(j-2)})$ *imply* $h(d, r, j) \leq 0$.

PROOF: When d is large, almost $\#R_j$, then the difference $h(d, r, j) \approx (\#R_{j-1})(\#R_j) - (r(\#R_j) - (\#R_{j+1}))(\operatorname{cod} V)$ which is $h(d, r, j) \approx (r^{j-1}/(j-1)!)(r^j/j!) - (jr^{j+1}/(j+1)!))(\operatorname{cod} V)$. It suffices to choose $\operatorname{cod} V > (r^{j-2}/(j-1)!)((j+1)/j)$ to make $h(d, r, j)$ nonpositive. When d is somewhat smaller, equal to the fraction $a(\#R_j)$, then $h(d, r, j) \approx (\#R_{j-1})(a(\#R_j)) - (ra(\#R_j) - (\#R_{j+1}))((1-a)(\#R_j))$; then for large r, $h(d, r, j)/(\#R_j) \approx ar^{j-1}/(j-1)! - (\mathrm{rar}^j/j! - r^{j+1}/(j+1)!)(1-a)$. It suffices to choose $a > 1/(j+1)$ to ensure that the second, negative term dominates. ■

When r and j are fixed, $h(d, r, j)$ is a parabola whose minimum occurs when the first derivative $(\partial/\partial d)h(d, r, j)$ is 0, or when d is $d_{\min} = (\#R_{j+1} - (\#R_{j-1}) + r(\#R_j))/2r$. A check when $r = 4$ shows that $j = 2$ is the only value where $h(d, r, j)$ can be negative: $h(d_{\min}, 4, j)$ has leading term $j^4/48$ and for small j greater than 2, $h_{\min}$ is positive. However, in 5 variables, for each j there are values d where $h(d, r, j)$ is negative.

LEMMA 2.33: *When* $r = 5$, *and* $j \geq 2$, *the difference* $h(d, r, j)$ *is negative for d equal to the integer* $\#R_j/2$ *or* $(\#R_j + 1)/2$.

PROOF: When $j = 2$, $d = 8 = (15+1)/2$ works. Since the actual minimum $d_{\min}$ is larger than $\#R_j/2$, it suffices to show $h(\#R_j/2, 5, j)$ is nonpositive. But $h(\#R_j/2, r, j) = (\#R_{j-1} + \#R_{j+1} - r(\#R_j)/2)(\#R_j)/2$, and for $r = 5$ the first factor is $(\binom{j+3}{4} + \binom{j+5}{4} - \binom{5}{2}\binom{j+4}{4})$, which is negative when $j \geq 3$.

Finally, we note that when $j = 3$, the difference $h(d, r, j)$ is negative for $r = 5$ and $17 \leq d \leq 29$; and also for $r = 6$ and $24 \leq d \leq 50$. This suggests (just to be specific) that general enough algebras of types $1, 5, 15, 6, 0$ to $1, 5, 15, 18, 0$ (this last is $d = 17$) and of types $1, 6, 21, 6, 0$ to $1, 6, 21, 32, 0$ will be generic.

3. Gorenstein algebras with small tangent space

We now consider zero-dimensional Gorenstein rings $A = R/I$, quotients of $R = k[[x_1, \ldots, x_r]]$; and we suppose at first A has type $T = 1, r, r, 1$. When $r = 3$, the ideal I defining A is a complete intersection, hence is smoothable: in that case, the algebra A has a deformation to a direct sum of fields. We give an argument indicating that when $r \geq 4$, A has in general deformations only to other Gorenstein ideals of the same type. In particular, we show that when $r = 4$, and $I = (yz - x^2 - w^2,\ xz - y^2 - w^2,\ wz - x^2 - y^2,\ yw - 2x^2 - z^2,\ xw - 2y^2 - z^2,\ xy - 2w^2 - z^2)$, then $A = R/I$ has deformations only to other Gorenstein ideals of type $T = 1, 4, 4, 1$. Thus $\bar{U}$, the Zariski-closure of the variety U parametrizing these Gorenstein ideals, is a component of $\mathrm{Hilb}^n R$; likewise, the globalization $\bar{\mathcal{U}}$ (closure of $\mathcal{U}$, which is a locally trivial bundle over projective space $\mathbb{P}_r$, fibred by U) is a component of $\mathrm{Hilb}^n \mathbb{P}_r$.

The main idea of the proof is to show that when $r \geq 4$ and I is a general-enough graded Gorenstein ideal of type T, then A ought to satisfy first $T^1_{-1}(A) = 0$; secondly $\#T^1_0(A) = \dim G\,\mathrm{Gor}\,T$ (parametrizing graded Gorenstein ideals of type T); and lastly $\#T^1(A) = r$, which is the dimension of the fiber of $\mathrm{Gor}\,T$ over $G\,\mathrm{Gor}\,T$ (under the natural map taking an ideal to its associated graded ideal). Thus, there is room in the tangent space $\mathrm{Hom}(I, A)$ to $\mathrm{Hilb}^n \mathbb{P}_r$ at the point z parametrizing at I, only for deformations of I to other Gorenstein ideals of the same type, (concentrated at a nearby point of $\mathbb{P}_r$). As in section 2, to show this, we count the conditions on the tangents, coming from the degree-3 relations among the generators of I. We show that there are exactly the right number of conditions so that $T^1_i(I)$ ought to have the sizes specified, when I is general enough. In section 3.1 we outline this argument. In section 3.2 we verify the independence of the linear conditions for the ideal I specified above, hence in general for Gorenstein ideals of type $1, 4, 4, 1$. There are 4 moduli for isomorphism classes of graded Gorenstein ideals of type $1, 4, 4, 1$, so again we have a "generic", algebra which is not rigid, and has no deformations to a rigid algebra.

We then consider Gorenstein rings of the particular type $T = 1, r, \binom{r+1}{2}, \ldots, \binom{r+s-1}{s}, \ldots, \binom{r+1}{r}, r, 1$ with degree of socle $j = 2s$, and those of type $T = 1, r, \ldots, \binom{r+s-1}{s}, \binom{r+s-1}{s}, \ldots, r, 1$ with degree of socle $j = 2s + 1$. We begin in section 3.3 by showing Gor T and G Gor T are nonempty, irreducible, of a dimension we calculate. It was known that once a system of parameters for R is chosen, a graded height-r Gorenstein ideal in R containing m^{j+1} but not m^j (in other words, such that A has degree j socle), corresponds uniquely to a form f of degree j in R – the annihilating form of I (See [15]). We show that conversely, if f belongs to a certain open set U of the projective space $P(R_j)$ parametrizing degree j forms, then the graded Gorenstein ideal annihilated by f has type T above. In other words, most graded zero-dimensional Gorenstein rings with r generators and degree j socle have type T.

We show an ideal I in the fibre of Gor T over I^* in G Gor T is determined uniquely by the choice of an annihilator function $h = f_{s+1} + \cdots + f_j$ with $f_j = f$, the function annihilating I^*. Here f_{j-i} is well-defined $\bmod(J^i f)$. We conclude that G Gor T, the fibre of Gor T over G Gor T, and hence also (Gor $T) \cap \pi^{-1}(G$ Gor $T)$ are irreducible as varieties; and that the dimension of Gor T is

$$\dim \operatorname{Gor} T = \sum_{i=s+1}^{j} \# I_i = \sum_{i=s+1}^{j} \left(\binom{r+i-1}{i} - \binom{r+j-i-1}{j-i} \right).$$

We conclude in Theorem 3.34 that for most such types T when $r \geq 4$, the general Gorenstein algebra of type T has no trivialization.

Then in the conjectural section 3.4, we assume I is graded and that the minimal resolution of I has the degrees it ought to have, and outline a calculation of $T^1(A)$. We show modulus our assumptions that the number of conditions imposed by the relations is exactly the number needed, so that, if independent, they force the non-negative part $T^1_*(A) = \dim \operatorname{Gor} T$. For odd $j \geq 3$ and $r = 4, 5$ the conditions, if independent, force the negative part $T^1_{-1}(A)$ to be zero. However, we have not confirmed the independence of the conditions, or the assumptions of section 3.4 in any case beyond $j = 3$, $r = 4$, that we complete in section 3.2.

3.1. Gorenstein algebras of type $T = 1, r, r, 1$

We first describe the family Gor T parametrizing Gorenstein ideals of type T in R; then we plan the calculation showing $T^1(I)$ is small. We assume here char $k = 0$ or char $k > 3$. This restriction can be avoided by using divided powers.

Once a system of generators $x_1, \ldots, x_r$ for R is chosen, we define a pairing ψ: $R_i \times R_j \to R_{j-i}$ from the forms of degrees i and j to those of degree $j - i$. On monomials, if v is a vector of r nonnegative integers of length j, and u is a vector of length i, then

$$\psi(x^u, x^v) = \partial x^v / \partial x^u = x^{(v-u)}(v_1)_{u_1}, \ldots, (v_r)_{u_r}.$$

Notice that if $u_i > v_i$, $\psi(x^u, x^v) = 0$. We extend the definition bilinearly to $R_i \times R_j$.

The graded Gorenstein ideals I of type T are the annihilators $I(f)$ under the pairing of certain cubic forms f – namely those cubic forms involving essentially all the r variables [See [15]. These cubic forms are parametrized by an open set of the projective space $\mathbb{P}(R_3)$, thus

$$G \text{ Gor } T \hookrightarrow (\mathbb{P}(R_3))$$

is an open immersion, and $\dim(G \text{ Gor } T) = (\binom{r+2}{3} - 1)$.

It is easy to see that the *general* Gorenstein ideals of type T having Gorenstein associated graded ideal, are the annihilators $I(h)$ of polynomials $h = h_2 + f$, where $h_2 \in R_2$, $f \in R_3$, and $I(f)$ also of type T, is the associated graded ideal of $I(h)$. (See section 3.3 for details.)

We now wish to parametrize the ideals $I(h)$, so we determine to what extent h is unique. Recall from [15] that $I_2(f)$, the graded degree 2 part of $I(f)$, is the vector space of degree 2 forms annihilating $J^1 f = \partial f / \partial x_1, \ldots, \partial f / \partial x_r$ under the pairing ψ. The following lemma is a special case of Lemma 3.33A.

Lemma 3.1A. *If $I(h)$ and $I(h')$ are two ideals of type T, as above, such that $h = h_2 + f$ and $h' = h'_2 + f'$ are the decompositions to forms, and both $I(f)$ and $I(f')$ have type T, then $I(h) = I(h') \Leftrightarrow \exists c \in k^*$* such that $f = cf'$, and $(h_2 - ch'_2) \in J^1 f$.

Proof of $\Leftarrow$. By linearity, we may assume $f = f'$, and $(h_2 - h'_2) \in J^1 f$; by symmetry it suffices to show $I(h') \subset I(h)$. Suppose $g = g_2 + g_3 \in I(h')$. If $g_2 = 0$, then $g \in I_3(f) = I_3(h) \subset I(h)$. Otherwise, $g_2 \in$

$I_2(h') = I_2(f)$, and this implies $\psi(g_2, f) = 0$ and that g_2 annihilates $J^1 f$. Thus, $\psi(g, h) = \psi(g_2, h_2) + \psi(g_3, f) = \psi(g_2, h_2') + \psi(g_3, f) = \psi(g, h') = 0$ and $g \in I(h)$.

PROOF OF $\Rightarrow$. Clearly, $I(h) = I(h') \Rightarrow$ the associated graded ideals $I(h)^* = I(h')^* \Rightarrow \exists c \in k^*$ such that $f = cf'$. Suppose $g_2 \in I_2(f)$; then $\exists g_3 \in R_3$ with $g_2 + g_3 \in I(h) = I(h')$, and hence $0 = (g_2 + g_3)(h - ch') = g_2(h_2 - ch_2')$. Thus $(h_2 - ch_2') \in \mathrm{Ann}(I_2(f)) = \mathrm{Ann}(R_2 \cap \mathrm{Ann}(J^1 f))$. Since $\psi | R_2 \times R_2 \to k$ is an exact pairing, we conclude that $(h_2 - ch_2') \in J^1 f$. This completes the proof. ■

An immediate corollary is

LEMMA 3.1B. *The natural map* π: (Gor $T \cap \pi^{-1} G$ Gor $T) \to$ Gor T, *coming from the map taking an ideal to its associated graded ideal, makes* (Gor $T \cap \pi^{-1} G$ Gor T) *into a locally trivial bundle over* G Gor T, *with fibre the affine space of dimension* $\binom{r}{2}$. *The map has a natural section coming from the inclusion of graded ideals in all ideals.* ■

We now outline the calculation of $T^1(A)$, for a graded Gorenstein ideal I of type T. The ideal I has $\binom{r}{2} = \binom{r+1}{2} - r$ generators of degree 2 in any minimal generating set. In general, these will generate I and if so, there will be $e = (r\binom{r}{2} + 1 - \binom{r+2}{3})$ linear relations among them. The 1 extra linear relation compared to an ideal of type $1, r, r, 0$, plays an important role here. We first estimate $\#(T^1_{-1}(A))$ by considering as before in section 2 the identity

$$\mathrm{Hom}_{-1}(I, A) = \mathrm{Ker}\, \theta_{-1}\colon \mathrm{Hom}_{-1}(F, A) \to \mathrm{Hom}_{-1}(E', A).$$

The dimension $\#\mathrm{Hom}_{-1}(F, A)) = \binom{r}{2} r$ and the dimension $\#(\mathrm{Hom}_{-1}(E', A)) = er$, thus the expected dimension of $T^1_{-1}(I)$ is $(\#\mathrm{Hom}_{-1}(I, A) - r)$, or

$$\text{expected } \#T^1_{-1}(A) = \max\left(0, \left(r\left(\binom{r}{2} - e\right) - r\right)\right) = 0 \quad \text{if } r \geq 4.$$

We now likewise bound the dimension of $\mathrm{Hom}_0(I, A)$. The dimension $\#(\mathrm{Hom}_0(F, A)) = \binom{r}{2} r$ and the dimension $\#(\mathrm{Hom}_0(E', A)) = e$, thus the expected dimension of $\mathrm{Hom}_0(I, A)$ is

$$\text{expected } \#\mathrm{Hom}_0(I, A) = \binom{r}{2} r - e = \binom{r+2}{3} - 1 = \text{dimension } G \text{ Gor } T.$$

Evidently, $\#\mathrm{Hom}_1(I, A) = \#\mathrm{Hom}_1(F, A) = \binom{r}{2}$, the dimension of the fibre of $(\mathrm{Gor}\ T \cap \pi^{-1} G\ \mathrm{Gor}\ T)$ over $G\ \mathrm{Gor}\ T$.

Clearly, if I is generated in degree 2, and the various linear conditions above imposed by the linear relations are independent, then I has deformations only to other Gorenstein ideals of the same type. We let $U \subset (\mathrm{Gor}\ T \cap \pi^{-1} G\ \mathrm{Gor}\ T)$ be the open set in $\mathrm{Gor}\ T$ parametrizing I such that I has $\#T^1_{-1}(A) = 0$ and $\#(\mathrm{Hom}(I, A))$ is that expected above. When U is nonempty, $\bar{U}$ is a component.

3.2. A Gorenstein algebra of type $T = 1, 4, 4, 1$ having deformations only to other Gorenstein algebras of the same type

We verify the argument of section 3.1 for the graded Gorenstein ideal $I = I(f)$, where $f = 6(xyz + ywz + xwz + 2xyw) + (x^3 + y^3 + z^3 + w^3)$. First, a word about choosing an example to calculate: on the one hand, the argument depends on f being chosen general enough; on the other hand, the calculation needed becomes rapidly more difficult as f becomes less symmetric. Thus, we try to find an f as symmetric and as simple as possible, that works. Simpler choices of f than that above do not work!

The vector space $J^1 f$, annihilated by the generators of I, is

$$J^1 f = (yz + wz + 2yw + x^2/2,\ xz + wz + 2xw + y^2/2,$$
$$yz + xz + 2xy + w^2/2,\ xy + yw + xw + z^2/2).$$

The generators of I are $I = (h_1, \ldots, h_6)$ with

$$h_1 = xz - y^2 - w^2,\ h_2 = yz - x^2 - w^2,\ h_3 = wz - x^2 - y^2,$$
$$h_4 = yw - 2x^2 - z^2,\ h_5 = xw - 2y^2 - z^2,\ h_6 = xy - 2w^2 - z^2.$$

We let σ denote the cyclic permutation (xyw) taking h_1 to h_2, h_2 to h_3, h_3 to h_1, and permuting cyclically h_4, h_5, h_6 also. We will abuse notation and let h_i also denote a basis element of the free R-module F on 6 generators, according to context.

It is not hard to verify, once found, that a basis of the linear relations among the generators, is e_1, $e_2 = \sigma e_1$, $e_3 = \sigma^2 e_1$, e_4, and $e_5 = \sigma e_4$, where

$$e_1 = (x + 2y + 2w)(h_2 - h_3) - (z + y + w)(h_6 - h_5),$$

and

$$\begin{aligned} e_4 = & (-154xh_1 + 11yh_2 + 143wh_3) + (-56xh_2 - 52yh_3) \\ & + (56yh_1 + 52wh_2) + (28xh_4 - 2yh_5 - 26wh_6) \\ & + (41xh_5 - 72yh_6 - 77wh_4) + (85xh_6 + 63yh_4 - 40wh_5) \\ & + (154zh_1 - 11zh_2 - 143zh_3). \end{aligned}$$

The bracketed coefficients of e_4 permute under σ, and it serves as a check on our calculation of the linear relations, that evidently,

$$\begin{aligned} (1+\sigma+\sigma^2)e_4 &= 108(1+\sigma+\sigma^2)e_1 \\ &= 108(1+\sigma+\sigma^2)(xh_6 - xh_5 + xh_3 - xh_2). \end{aligned}$$

We now show the linear independence of the 20 linear conditions imposed on $t \in \mathrm{Hom}_{-1}(F, A)/\langle \partial/\partial x, \partial/\partial y, \partial/\partial w, \partial/\partial z\rangle$ by the requirement t be in the kernel of θ. This task is facilitated since the 20×20 matrix involved breaks into 4×4 blocks, because of the symmetry in the first 3 relations. To begin, we suppose as in section 2, that $t(h_i) = L_i = L_{i1}x + L_{i2}y + L_{i3}w + L_{i4}z$. Since $J^1h_1 = R_1$, in considering t mod the trivial tangents (the partials), we may assume $L_1 = 0$. We proceed by substituting L_i for h_i in each relation, and calculating the image in $A_2 = R_2/I_2$, modding out by the generators. We now give the matrix M resulting. On the left we mention the relation in question (each giving 4 linear conditions), and the basis used for $A_2 = R_2/I_2$. Blanks are zeroes. (See table on next page)

We calculated the determinant of the aforegoing matrix M in two different ways on the University of Texas Computation Center CDC 64-6600 system computer which works with 48 bits (14 decimal places) and gives usually over 10 digit accuracy. The modified "LU (lower, upper triangulator) decomposition" method working with real numbers gives

$$\text{determinant } M = (-5.0158315356) \times 10^{24}$$

and the product of the complex eigenvalues (none of which was smaller than 0.2 in modulus) was computed as

$$\prod(\text{eigenvalues}) = (-5.0158315355) \times 10^{24} + i(1.1873\ldots) \times 10^{12}.$$

The performance indices of both programs was good and they give the same result to 10 digits; we conclude that the determinant of M is nonzero, and that the 20 linear conditions are independent. We thank

		L_{21}	L_{22}	L_{23}	L_{24}
e_1	y^2				
	x^2				
	x^2-w^2		0		
	z^2				
e_2	w^2	−4	−4	−1	−4
	y^2	−2	−2	−4	−3
	y^2-x^2	0	0	0	0
	z^2	−3	−3	−4	0
e_3	x^2	1	4	4	4
	w^2	4	2	2	3
	w^2-y^2	0	0	0	0
	z^2	4	3	3	0
e_4	x^2	−56	93	11	63
	y^2	93	11	−123	−4
	w^2	11	−123	52	−45
	z^2	63	−4	−45	−11
e_5	y^2	154	56	−154	−154
	w^2	266	−154	0	−98
	x^2	−154	154	266	56
	z^2	56	−154	−98	154

		L_{31}	L_{32}	L_{33}	L_{34}
e_1	y^2	4	1	4	4
	x^2	2	4	2	3
	x^2-w^2	0	0	0	0
	z^2	3	4	3	0
e_2	w^2				
	y^2		0		
	y^2-x^2				
	z^2				
e_3	x^2	−1	−4	−4	−4
	w^2	−4	−2	−2	−3
	w^2-y^2	0	0	0	0
	z^2	−4	−3	−3	0
e_4	x^2	0	143	−247	91
	y^2	143	−52	−143	143
	w^2	−247	−143	143	−52
	z^2	91	143	−52	−143
e_5	y^2	11	−56	93	63
	w^2	−123	93	11	−4
	x^2	52	11	−123	−45
	z^2	−45	63	−4	−11

		L_{41}	L_{42}	L_{43}	L_{44}
e_1	y^2	1		−3	
	x^2	−1	1	−1	1
	x^2-w^2	0	4	−2	2
	z^2	1	0	−1	−1
e_2	w^2	3	−1		
	y^2	1	1	−1	−1
	y^2-x^2	2	0	−4	−2
	z^2	1	−1	0	1
e_3	x^2				
	w^2				
	w^2-y^2		0		
	z^2				
e_4	x^2	28	−154	126	−14
	y^2	−154	63	56	−49
	w^2	126	56	−77	91
	z^2	−14	−49	91	0
e_5	y^2	−144	85	−52	−98
	w^2	170	−52	−72	59
	x^2	−26	−144	170	13
	z^2	13	−98	59	0

		L_{51}	L_{52}	L_{53}	L_{54}
e_1	y^2				
	x^2		0		
	x^2-w^2				
	z^2				
e_2	w^2	−3	1		
	y^2	−1	−1	1	1
	y^2-x^2	−2	0	4	0
	z^2	−1	1	0	−1
e_3	x^2	0	3	−1	0
	w^2	−1	1	1	−1
	w^2-y^2	−4	2	0	−2
	z^2	0	1	−1	1
e_4	x^2	41	−80	−4	−42
	y^2	−80	−2	82	1
	w^2	−4	82	−40	39
	z^2	−42	1	39	0
e_5	y^2	126	28	−154	−14
	w^2	56	−154	63	−49
	x^2	−77	126	56	91
	z^2	91	−14	−49	0

		L_{61}	L_{62}	L_{63}	L_{64}
e_1	y^2	−1		3	
	x^2	1	−1	1	−1
	x^2-w^2	0	−4	2	−2
	z^2	−1	0	1	1
e_2	w^2				
	y^2		0		
	y^2-x^2				
	z^2				
e_3	x^2	0	−3	1	0
	w^2	1	−1	−1	1
	w^2-y^2	0	−1	1	−1
	z^2	0	−1	1	−1
e_4	x^2	85	−52	−144	−98
	y^2	−52	−72	170	59
	w^2	−144	170	−26	13
	z^2	−98	59	13	0
e_5	y^2	−4	41	−80	−42
	w^2	82	−80	−2	1
	x^2	−40	−4	82	39
	z^2	39	−42	1	0

D. Kincaid, Hunter Ellinger, and Andy Martin for help with the programming and debugging.

We now show $\#\mathrm{Hom}_0(I, A) = 19$. The size $\#\mathrm{Hom}_0(E, A) = \#\mathrm{Hom}(E, A_3) = 5$; the size $\#\mathrm{Hom}_0(F, A) = \#\mathrm{Hom}(F, A_2) = 24$; and we must show there are 5 independent linear conditions imposed on $t \in \mathrm{Hom}_0(F, A)$ by the requirement $t \in \ker \theta$. We show for $t(h_1) = B_1x^2$, $t(h_2) = B_2y^2$, $t(h_3) = B_3z^2$, $t(h_4) = B_4(y^2 + w^2)$, $t(h_5) = B_5x^2$, and $t(h_6) = B_4w^2$ with $B_i \in k$, that $t \in \ker \theta \Rightarrow t = 0$, which will complete our proof. Since I contains all the monomials x^2y, xz^2, zx^2, etc. (all $x_i^2x_5$ with $i \neq s$), as well as $x^3 - y^3$, $x^3 - z^3$, etc., in order to calculate the image of t in $\mathrm{Hom}(e_i, A_3)$, it suffices to substitute $t(h_1), \ldots, t(h_6)$ in the relation e_i and to sum the resulting cubic coefficients – the coefficients on $x^3, \ldots, z^3$. In this way we obtain for $\theta(t)$

$$\theta(t)(e_1) = 2B_2,\ \theta(t)(e_2) = 2B_1,\ \theta(t)(e_3) = 2B_1 - 2B_2 - (B_5 - B_4),$$
$$\theta(t)(e_4) = -154B_1 + 11B_2 - 143B_3 + (63 - 72 - 26)B_4 + 41B_5,$$
$$\theta(t)(e_5) = 143B_1 - 154B_2 - 11B_3 + (85 - 72 - 2)B_4 - 77B_5.$$

Thus $\theta(t) = 0$ on e_1, e_2, and e_3 implies $B_1 = B_2 = 0$, and $B_4 = B_5$. Then $\theta(t) = 0$ also on e_4 and e_5 implies $143B_3 = B_5$ and $11B_3 = -66B_5$, respectively. Thus $t \in \ker \theta \Rightarrow t = 0$. This shows $\#\mathrm{Hom}_0(I, A) = 19 = \dim G \text{ Gor } T$, and completes the proof that A has deformations only to other Gorenstein rings of the same type T.

3.3. Parametrizing Gorenstein algebras of symmetric, maximal types

By a Gorenstein ideal of maximal type in R, we mean an ideal I for which there is an integer s satisfying $m^{s+1} \supset I \supset m^{2s+2}$. By symmetric type we mean that if j is the highest power of the maximal ideal m, not contained in I (in other words, j is the degree of the socle of R/I^*) then $T(I) = t_0, \ldots, t_i, \ldots$ with $t_i = t_{j-i}$. A Gorenstein ideal may have a maximal, nonsymmetric type: for example, if $f = x^2 + y^3$ in $k[[x, y]]$, the ideal $I(f) = (xy, 3x^2 - y^3)$ has type $1, 2, 1, 1, 0$. But a symmetric, maximal type is uniquely determined by s, and the number j defined above, is $2s$ or $2s + 1$. The types we study are thus

If $j = 2s$,

(3.3a)
$$T = 1, r, \binom{r+1}{2}, \ldots, \binom{r+s-2}{s-1}, \binom{r+s-1}{s},$$
$$\binom{r+s-2}{s-1}, \ldots, \binom{r+1}{2}, r, 1$$

If $j = 2s + 1$,

(3.3a bis)

$$T = 1, r, \binom{r+1}{2}, \ldots, \binom{r+s-1}{s}, \binom{r+s-2}{s-1}, \ldots, \binom{r+1}{2}, r, 1.$$

The lengths $n(T) = \Sigma\, t_i$ of these types are respectively $n(T) = 2\binom{r+s}{s} - \binom{r+s-1}{s}$ when $j = 2s$, and $n(T) = 2\binom{r+s}{s}$ when $j = 2s + 1$. The types can also be characterized by the condition

$$t_i = \min(\# R_i, \# R_{j-i}). \tag{3.3b}$$

They are also the symmetric types of maximal length $n(T) = \Sigma\, t_i$, for which $t_j = I$, and $t_{j+1} = t_{j+2} = \cdots = 0$.

We first study graded Gorenstein ideals of these types in R, and later we'll study more general Gorenstein ideals of these types.

The graded Gorenstein ideals of type T in R are parametrized by a subscheme $G\,\mathrm{Gor}\, T$ of $\mathrm{Hilb}^{n(T)} R$. There is a 1-1 correspondence between graded Gorenstein ideals I in R (not necessarily of type T) such that $A = R/I$ has degree j socle, and forms of degree j in R (up to non-zero constant multiple). Using the ψ of section 3.1, this correspondence is (see [15]).

$$\begin{aligned} f &\leftrightarrow I(f) = \text{all forms } g \text{ such that } \psi(g, f) = 0 \\ I &\leftrightarrow \langle f \rangle = (\mathrm{Ann}\, I_j) \cap R_j = \{f \text{ such that } \psi(I_j, f) = 0\}. \end{aligned} \tag{3.3c}$$

For such ideals,

$$I_i = \mathrm{Ann}(J^{g-i} f), \text{ the annihilator of } J^{j-i} f \text{ in } \psi | R_i \times J^{j-1} f \to k.$$

Here, in turn, $J^{j-i} f = \psi(R^i, f) =$ vector space spanned by all ith partials of f. The type of such graded Gorenstein ideals is always symmetric, and consequently $\# I_i = t_i = t_{j-i} = \# J^{j-i} f \le \min(\# R_i, \# R_{j-i})$. Thus the graded Gorenstein ideals having the type T of (3.3a, b), are those annihilating a degree j form and having the *maximum* type consistent with the symmetry $t_i = t_{j-i}$. Thus clearly "$\# J^{j-i} f = \# R_i$" is an open condition "$\# J^{j-i} f \ge (\# R_i - 1)$" on the projective space $\mathbb{P}(R_j)$ parametrizing degree j forms in R, so for the T of (3.3a) there is an open immersion $G\,\mathrm{Gor}\, T \hookrightarrow \mathbb{P}(R_j)$. We now show $G\,\mathrm{Gor}\, T$ is nonempty; this together with the 1-1 correspondence (3.3c) shows G Gor T is dense in $\mathbb{P}(R_j)$.

THEOREM 3.31: *If T satisfies* (3.3a), *there is an open dense immersion $G\,\mathrm{Gor}\,T \hookrightarrow \mathbb{P}(R_j)$, from the scheme parametrizing graded Gorenstein ideals of type T, to the projective space on R_j. The variety $G\,\mathrm{Gor}\,T$ is irreducible, rational, and has dimension $(\#R_j - 1) = \binom{r+j-1}{j} - 1$.*

PROOF: We need to show $G\,\mathrm{Gor}\,T$ is nonempty[1].

CASE $j = 2s$: It suffices to construct an f such that $\#J^s f = \#R_s$; for then $J^s f = R_s$, and for $u > 0$, $J^{s+u} f = R_{s-u}$: the symmetry of the type T of $I(f)$ then shows H must be the T of (3.3a). Order the monomials $\mu_1, \ldots$ having degree s alphabetically: $x_1^s < x_1^{s-1}x_2 < \cdots < x_1^{s-1}x_r < x_1^{s-2}x_2^2 < \cdots < x_{r-1}^2 x_r^{s-2} < x_{r-1}x_r^{s-1} < x_r^s$; and we let J be the matrix whose i, j entry is the coefficient of $\psi(\mu_i, f)$ on the monomial μ_j. We will show, by induction on r, a stronger result,

CLAIM: *Given $j = 2s$, an even integer, and given an $\#R_s \times \#R_s$ square matrix C of constants, there is a form f of degree j such that* $\det(J - C) \neq 0$.

PROOF OF CLAIM: If $r = 1$, given a constant c, we choose $f = c'x^{2s}$ such that $c'\binom{2k}{k}(k!) \neq c$. Suppose the lemma is true for all even j in $(r-1)$ variables $x_2, \ldots, x_r$ and that $z, x_2, \ldots, x_r$ are the new r variables (thus $z = x_1$), that $R = k[[z, x_2, \ldots, x_r]]$, that $R' = k[[x_2, \ldots, x_r]]$, and suppose that C is a square $\#R_s \times \#R_s$ matrix of constants. We will choose in order $f_0, f_2, \ldots, f_{2s}$ with $f_{2i} \in R'_{2i}$, such that $f = f_0 z^{2s} + f_2 z^{2s-2} + \cdots + f_{2s}$ works: $\det(J - C) \neq 0$. We decompose the matrices J, C, $M = J - C$ and all others used, into rectangular blocks J_{uv}, C_{uv}. The rows and columns of $J, C, M, \ldots$ are labeled by the monomials of R_s. The block J_{uv} contains all the entries of J for which the z-power of the row-label is $s - u$, and the z-power of the column label is $s - v$.

The idea of the proof is that the part of $(J - C)$ in the upper left corner: $(J - C)_{uv}$ with $u, v \leq n$ depends only on the $f_0, \ldots, f_{2n}$ terms of f, and we can choose in order $f_0, \ldots, f_{2n}$ so the upper left corner is diagonalizable. Assume, to start, that $f = \Sigma\, a_I z^{i_1} x_2^{i_2} \ldots x_r^{i_r}$ with coefficients a_I independent variables if i_1 is even, 0 if i_1 is odd.

[1] (Added in proof) E.L. Green has independently shown that $G\,\mathrm{Gor}\,T$ is nonempty, in "Complete intersections and Gorenstein ideals", to appear, Journal of Algebra.

	z^3	$z^2x, z^2y,$	$zx^2, zxy, zy^2,$	x^3, x^2y, xy^2, y^3
z^3	$(6_3)f_0$	$J_{01} = 0$	$J_{02} = (4_3)f_2$	$J_{03} = 0$
z^2x z^2y	0	$J_{11} = (4_2)J^1f_2$	0	$J_{13} = 2J^1f_4$
zx^2 zxy zy^2	$4J^2f_2$	0	$J_{22} = 2J^2f_4$	$J_{23} = 0$
x^3 x^2y xy^2 y^3	0	$J_{31} = J^3f_4$	0	$J_{33} = J^3f_6$

FIG. 1. Matrix $J = J^sf$ when $j = 2s = 6$, and $r = 3$.

We choose in sequence $f_0, f_2, \ldots, f_{2s}$ in R', calling the resulting matrices $J^{(0)}, J^{(1)}, \ldots, J^{(s)}$ with $J^{(j)} = J^sf$ after substituting the coefficients of $f_0, \ldots, f_{2i}$ for the pertinent a_I, namely those a_I in which $i_1 \geq j - 2i$. We let $M^i = J^{(i)} - C$. The purpose of this notation is to allow us to perform matrix operations to almost-diagonalize the upper left corner of M^i *after* choosing $f_0, \ldots, f_{2i}$, while also indicating the effects of these operations on the rest of the matrix M^i *before* actually choosing $f_{2i+2}, \ldots, f_{2s}$. First, we choose $f_0 \in R'_0 = k$ such that $M^0_{00} \neq 0$. Since $M^0_{00} \neq 0$, we may reduce M^0_{01} and M^0_{10} to 0 by matrix operations involving the first row and column, obtaining thereby a new matrix $C^1 = S_1M^0S_1^{-1}$ similar to M^0 by the matrix $S_1 \in G1(k)$. Choose now f_1 and thus M^1, so that the determinant $|C^1_{11}| \neq 0$, when evaluated at f_1, and let $B^1 = S_1M^1S_1^{-1}$.

At stage n, we have just chosen f_n and M^n, and $B^n = S_nM^nS_n^{-1}$ such that $S_n \in G1(k)$, and B^n is almost diagonal in its upper left corner: this means the early diagonal blocks B^n_{uu} with $u \leq n$ are each in $G1(k)$, and the early off-diagonal blocks B_{uv} with $u \neq v$, and $u, v \leq n$ are all zero. Then we use matrix operations involving the early rows and columns to reduce the blocks $B^n_{n+1,u}$ and $B^n_{u,n+1}$ with $u \leq n$, to zero, obtaining thereby a new matrix $C^{n+1} = (S_{n+1}S_n^{-1})B^n(S_{n+1}S_n^{-1})^{-1} = S_{n+1}M^nS_{n+1}^{-1}$ for some $S_{n+1} \in G1(k)$. We choose f_{n+1} (and hence also M^{n+1}) so that the determinant $|C^{n+1}_{n+1,n+1}| \neq 0$. This key step involves solving $|J^{n+1}f_{2n+2} - C'| \neq 0$ over R', for C a matrix of constants, and we use the induction hypothesis. We then let $B^{n+1} = S_{n+1}M^{n+1}S_{n+1}^{-1}$, and notice that B^{n+1} is almost diagonal in its upper left corner, which is the set of blocks B_{uv} with $u, v \leq n + 1$. Finally, after choosing $f_{s'}$ the matrix M^s is similar to B^s which has off-diagonal blocks zero and diagonal blocks invertible, so $M^s = (J^{(s)} - C)$ is invertible. This proves the claim, and the theorem when $j = 2s$.

CASE $j = 2s + 1$: It suffices to show there is an f with $\#J^s f = \#R_s$, for then by symmetry $\#J^{s+1}f = \#R_s$, implying $J^{s+1} = R_s$; the discussion under the case $j = 2s$ applies to show $T(I(f))$ must be the T of (3.3a). For this we need a slight generalization of the claim proven above. There, the general J_{uv} block of J was $J_{uv} = ((2s - u - v)!/(s - u)!)J^u f_{2v}$. We need a similar claim for the matrix J' where $J'_{uv} = ((2s + 1 - u - v)!/(s - u)!)J^u f_{2v}$. Clearly the proof is the same. We apply the result with $C = 0$ to show there is a form $f = f_0 z^{2s+1} + f_2 z^{2s-1} + \cdots + f_{2i} z^{2s+1-2i} + \cdots + f_{2s} z$ with $|J^s f| \neq 0$. This completes the proof of the theorem. We remark that the proof works when characteristic $k = p > j$; or in lower characteristics, if we replace the derivatives and duality used by one without coefficients – if we use divided powers. Thus the theorem as stated is true in all characteristics. ■

Note. The matrix $J = J^s f$, when degree $f = 2s$, was studied of old under the name catalecticant of f (Grace–Young p. 66 [10]), but we do not know whether it was known to be in general non-zero. Much of the duality used here was known to Macaulay, a modern reference being [16]. Our variation is to give the Gorenstein ring A more structure by considering A as a quotient $A = R/I$ of R. If $I \supset m^{j+1}$ we then choose a dualizing module $\mathrm{Hom}(A, k)$ in R depending only on a choice of system of parameters for R (equivalent to a choice of the pairing ψ). Since A is Gorenstein, the dualizing module is simple and $\mathrm{Hom}(A, k) = Jf$ for some polynomial f. When A is graded there is a unique form $\langle f \rangle$ up to scalar multiple generating $\mathrm{Hom}(A, k)$. When A is not graded, but its associated graded algebra A^* is also Gorenstein of maximal symmetric type T, we will choose a unique polynomial f generating $\mathrm{Hom}(A, k) = Jf$ (but depending on A^*), in Lemma 3.3A. This added structure allows us to parametrize G Gor T and also Gor $T \cap \pi^{-1} G$ Gor T, the family of Gorenstein ideals of type T in R having a Gorenstein associated graded ideal.

DEFINITION: Dualizing module $\mathrm{Ann}\, I$ of $A = R/I$. We suppose a system of parameters $x_1, \ldots, x_r$ for R is chosen, and that ψ is the pairing of section 3.1. If I has finite colength in R we let $\mathrm{Ann}\, I = \{g$ in R such that $\psi(I, g)(0) = 0\}$. The action of A on $\mathrm{Ann}\, I$ is $a \cdot g = \psi(a, g)$, which is also in $\mathrm{Ann}\, I$ since $\psi(I, (\psi(a, g)))(0) = \psi(Ia, g)(0) = \psi(I, g)(0) = 0$. The A-module isomorphism, $\mathrm{Ann}\, I \to \mathrm{Hom}(A, k)$ is $g \to \psi(', g)(0)$.

PROOF OF ISOMORPHISM: Suppose $I \supset m^{j+1}$. The bihomomorphism $\psi(\cdot, \cdot)(0)$ on $(R_0 \oplus \cdots \oplus R_j) \times (R_0 \oplus \cdots \oplus R_j)$ to k, is clearly an exact pairing, and $g \in \operatorname{Ann} I \Rightarrow$ degree $g \leq j$. Thus $\#\operatorname{Ann} I = \#(R/I) = \#A = \#\operatorname{Hom}(A, k)$. The homomorphism $\operatorname{Ann} I \to \operatorname{Hom}(A, k)$ is injective, since if g annihilates both I and R/I it annihilates R and must be 0; thus the homomorphism is an isomorphism.

When I is Gorenstein, $\operatorname{Hom}(A, k) = \operatorname{Ann} I$ is a simple extension of k: thus, $\operatorname{Ann} I$ is a simple A-module generated by a function f, and $\operatorname{Ann} I = A \cdot f = Jf$, the vector space of all partials of f. The polynomial f has degree no more than j, if I contains m^{j+1}. Also $f' = uf = \psi(u, f)$ is a generator, if u is a unit in A. If now $I \not\supset m^j$, the highest degree form f_j of f is uniquely defined up to multiplication by a non-zero constant. It is non-zero since otherwise $I_j = R_j \cap \{h \mid \psi(h, \operatorname{Ann} I)(0) = \}$ would include R_j. We now explain the relation between $I(f_j)$ and I.

Lemma 3.32: *If I is a Gorenstein ideal containing m^{j+1} but not m^j, and f with top degree form f_j generates* $\operatorname{Ann} I$, *then* $I(f_j) = \{g \mid \psi(g, f_j) = 0\}$ *is the unique graded Gorenstein ideal containing I^* but not containing m^j. Also, $I(f_j) = (I_j : R_j) + \cdots + (I_j : R_0) + m^{j+1}$. If I^* is also Gorenstein, then $I^* = I(f_j)$.*

Proof: We show first that $I(f_j) \supset I^*$, the associated graded ideal of I; then we show there are no epimorphisms among graded Gorenstein rings having degree j socle. If $h_i \in I_i$, then $h_i R_{j-1} \subset I_j \subset I$, since $m^{j+1} \subset I$. Thus $0 = \psi(h_i R_{j-i}, f)(0) = \psi(h_i, J^{j-1} f)(0) = \psi(h_i, J^{j-i} f_j)$, since only the degree i terms of g contribute to $\psi(h_i, g)(0)$. Thus $h_i \in I(f_j)$, by a remark above. These graded Gorenstein rings with degree j socle, quotients of R, correspond 1-1 with codimension 1 vector spaces of forms in R_j (that annihilate a form f_j), and thus there are no nontrivial inclusions. The rest of the lemma, in fact all, is a trivial consequence of the definitions in [15]: for if I^* is a graded ideal, the ancestor ideal $\overline{I_j} = (I_j : R_j) \oplus \cdots \oplus (I_j : R_0)$ of I_j contains $I_0 \oplus \cdots \oplus I_j$.

We now suppose $I^* = I(f_j)$ is a fixed graded Gorenstein ideal of maximal symmetric type T. We now choose a unique f such that $I = I(f)$.

Lemma 3.33A: *Suppose $I(f_j)$ is a graded Gorenstein ideal, having type T of* (3.3a), *and suppose that $V_s, \ldots, V_{j-1}$ is an arbitrary sequence of complementary vector spaces to $J^{j-i} f_j$ in R_i: thus $V_i \oplus J^{j-i} f_j = R_i$ for $i = s, \ldots, j-1$. Then there is a* 1-1 *correspondence*

{Gorenstein ideals I having $I^* = I(f_j)$}

$\Leftrightarrow$ *{polynomials $f = f_s + \cdots + f_{j-1} + f_j$ having $f_s \in V_s, \ldots, f_{j-1} \in V_{j-1}$}.*

The correspondence is

$$I \Rightarrow \text{the generator } f \text{ of } \operatorname{Ann} I \text{ with } f_s \in V_s, \ldots, f_{j-1} \in V_{j-1}$$
$$\text{and } f \Rightarrow \text{the ideal } I(f) = \{\text{all } g \in R \mid \psi(g, f) = 0\}$$

PROOF OF $\Rightarrow$: Given I, we show there is a unique f generating $\operatorname{Ann} I$ and satisfying the condition. Begin with an f generating $\operatorname{Ann} I$ and having top degree term f_j. Suppose further that $u < j$ is chosen such that $f_{j-1}, \ldots, f_{u+1}$ are in $V_{j-1}, \ldots, V_{u+1}$ respectively, but $f_u = f'_u + f''_u$ with $f'_u \in V_u$ and $f''_u = h \circ f_j \in J^{j-u} f_j$. Then $(1-h) \circ f$ will have top terms $(1-h) \circ f \equiv f_j + f_{j-1} + \cdots + f_{u+1} + f'_u \bmod (R_{u-1} + \cdots + R_0)$, and $(1-h)f$ also generates $\operatorname{Ann} I$ since $h \in m$. Continuing in this way down to $u = 0$ (we take $V_s = \cdots = V_0 = 0$), we find a generator of $\operatorname{Ann} I$ satisfying the condition.

If f and f' satisfy the conditions and both generate $\operatorname{Ann} I$, suppose $(f - f')_i$ is the top non-zero term in $f - f'$. If $h \in I$ has initial degree i, then $0 = \psi((f - f'), h)(0) = \psi((f - f')_i, h_i)$, thus $(f - f') \in \operatorname{Ann} I_i \cap R_i = J^{j-i} f_j$ by an earlier remark; this contradicts the choice of $(f - f')_i$ in V_i, the complementary space to $J^{j-i} f_j$.

PROOF OF $\Leftarrow$: Suppose f satisfies the condition, then the ideal $I = I(f) = \{g \mid \psi(g, f) = 0 = \{g \mid \psi(g, Jf)(0) = 0\}\}$ and f generated $\operatorname{Ann}(I(f))$. By Lemma 3.32, $I(f_i) \supset I^*$, and the type $T = T(I(f_j)) \leq T(I)$ in the sense $t_i \leq t_i(I)$ for all i. It suffices to show $T \geq T(I)$ for then $T = T(I)$ and I^*, being included in the ideal $I(f_j)$ of same colength, is equal to $I(f_j)$. The inequality $T \geq T(I)$ follows from

CLAIM: *If I is a Gorenstein ideal in R containing m^{j+1} but not m^j, and T is the type of* (3.3a), *then for each $u \leq j$, $\Sigma_u^j t_i \geq \Sigma_u^j t_i(I)$.*

PROOF OF CLAIM: The ideal $I = \operatorname{Ann} Jf$ for some degree j polynomial. We associate to Jf a graded J-ideal $(Jf)^* = \bigoplus_0^j (Jf)_i$ where $(Jf)_i = ((Jf \cap (R_i + \cdots + R_0)) + (R_{i-1} + \cdots + R_0))/(R_{i-1} + \cdots + R_0)$. It is easy to show $(Jf)_i$ is the annihilator of I_i in the pairing $(\cdot, \cdot)$ from $R_i \times R_i$ to k, using the fact $\psi(g_i + g_{i+1} + \cdots, \ h_i + h_{i-1} + \cdots)(0) = \psi(g_i, h_i)$. It follows that $\#((Jf)_i) = t_i(I)$. The only partials that can contribute to $(Jf)_u + \cdots + (Jf)_j$ are those of order no more than $j - u$, thus $\Sigma_u^j t_i(I) \leq \Sigma_0^{j-u} \# R_i = \Sigma_u^j t_i$ if $u > s$. If $i \leq s$, $\#((Jf)_i) \leq \# R_i = t_i$; this completes the proof of the claim, for all u, and of the Lemma 3.33A. ∎

An immediate corollary of the above Lemma is

LEMMA 3.33B: *The natural map* $\pi: (\text{Gor } T \cap (\pi^{-1} G \text{ Gor } T)) \to G \text{ Gor } T$ *makes the former into a locally trivial bundle over* G Gor T, *having as fibre an affine space of dimension* $(\Sigma_s^{j-1} (\# I_i))$ *and having a natural global section* i: G Gor $T \to$ Gor T.

PROOF: It suffices to notice that the same choices of V_i will work – be complementary to $J^{j-i} f_j$ – for an open neighborhood of degree j forms f_j: over that neighborhood, the map π is trivial with fibre the product of affine spaces each having dimension equal to dim V_j. ■

The subscheme $\pi^{-1}(G \text{ Gor } T)$ is open in Gor T, since the Gorenstein ideal I is in $\pi^{-1} G$ Gor T iff I_j belongs to an open set of the codimension 1 vector spaces in R_j – namely the open set U in $(P(R_j)^*$ corresponding to the open set U parametrizing f_j where $I(f_j)$ has type T (see Theorem 3.31, and Lemma 3.32). We now show that $\pi^{-1}(G \text{ Gor } T)$ is dense in Gor T. We suppose char $k = 0$, for the proof, but we expect the proof extends to characteristic p.

THEOREM 3.34: *If T is a type of* (3.3a), *then* Gor T *parametrizing Gorenstein ideals of type T, contains* $\pi^{-1} G$ Gor T *as an open dense subscheme*; Gor T *is irreducible, and has dimension*

$$\dim \text{Gor } T = \begin{cases} \binom{r+j}{j} - \binom{r+s}{s} & \text{if } j = 2s+1 \\ \binom{r+j}{j} - 2\binom{r+s}{s} + \binom{r+s-1}{s} & \text{if } j = 2s. \end{cases}$$

PROOF: It suffices to show $\pi^{-1} G$ Gor T is dense: the dimension calculation and irreducibility are then immediate consequences of Theorem 3.31 and Lemma 3.33B. Suppose that I is a Gorenstein ideal of type T. Then $I = I(f)$ for some polynomial f of degree j, with $f_j \neq 0$. Let $f(t) = f_j(t) + \cdots + f_0(t)$ be a 1-parameter family of polynomials such that $\text{limit}_{t \to 0} f(t) = f$, and such that for $t \neq 0$, the top term $f_j(t)$ is in the set U of Theorem 3.31: then $I(f_j(t))$ has type T for $t \neq 0$. Then by Lemma 3.32 and Lemma 3.33, $I(f(t))^* = I(f_j(t))$, for $t \neq 0$. Thus I is in the closure of $\pi^{-1}(G \text{ Gor } T)$.

The dimension of the component $\bar{U}$ of $\text{Hilb}^n \mathbb{A}_r$ containing points parametrizing smooth subschemes is rn; the dimension of the component of $\mathrm{Hilb}^n R$ containing points parametrizing quotients of R isomorphic to $k[x]/x^n$ is $(r-1)(n-1)$ (for this, see [13]); we conclude from the dimension formula of Theorem 3.34, and an easy calculation:

THEOREM 3.35: *The general Gorenstein algebra of type T in r-variables has no trivial deformations when $j \geq 3$ in 8 or more variables, when $j \geq 5$ in 7 or 6 or 5 variables, and when $j \geq 9$ in 4 variables. It has no deformations to $k[x]/x^n$ when $j \geq 3$ in 5 or more variables, when $j \geq 5$ in 4 variables, and when $j \geq 7$ in 3 variables.*

Here j is the integer satisfying $A_j \neq 0$ but $A_{j+1} = 0$.

3.4. Some Gorenstein algebras that should be generic

In order to prevent the miasma of conjectures from spreading, we isolate them in this section. From assumptions on the resolution of the general graded Gorenstein algebra $A = R/I$ of maximal symmetric type T, we calculate an expected dimension $\#\mathrm{Hom}_s(I, A) = \#I_{j-s}$ for $s \geq 0$ and all r, j; and we calculate the expected dimension $\#\mathrm{Hom}_{-1}(I, A) = r$ for $r = 4$ and j odd. This leads us to predict that the general Gorenstein algebra of maximal symmetric type T, can be shown "generic" by our methods when $r \geq 4$, and j odd. The example of section 3.2 is the simplest case, but we add no further examples and prove nothing but implications among conjectures here. We include the section for the interest of the conjectures, and for the curious calculation in the proof of Claim 1.

We assume $A = R/I$ is a general graded Gorenstein algebra of maximal type T, and that $0 \to F_r \to \cdots \to F_1 \to F_0 = R \to A \to 0$, is a minimal graded free resolution of A over R, where the homomorphisms are graded of degree 0. The difference in the calculation of this section and previous, is that we must take into account more terms of the resolution to calculate the expected sizes of $\mathrm{Hom}(I, A)$. The conjectures that follow are not about particular algebras, but about those parametrized by an open (we hope nonempty) set in G Gor T. We assume first

CONJECTURE 1: If $i < i'$, the degrees of generators of F_i are less than the degrees of the generators of $F_{i'}$.[1]

[1] (Added in proof) R. Stanley notes that here $\deg F_r = j + r$, and that Conjecture 1 is true when j is even.

We then consider the sequence

$$\operatorname{Hom}(I, A) \longrightarrow \operatorname{Hom}(F_1, A) \xrightarrow{\theta^1} \operatorname{Hom}(F_2, A)$$

$$\xrightarrow{\theta^2} \cdots \xrightarrow{\theta^{r-1}} \operatorname{Hom}(F_r, A) \longrightarrow \operatorname{Ext}^1(I, A) \longrightarrow \cdots$$

where the maps θ^i are graded, $\theta^i = \bigoplus_s \theta^i_s: \bigoplus_s \operatorname{Hom}_s(F_i, A) \to \bigoplus_s \operatorname{Hom}_s(F_{i+1}, A)$, the composition $\theta^i\theta^{i-1} = 0$, and on the left $\operatorname{Hom}(I, A)$ is the kernel of θ^1. We then assume

CONJECTURE 2: For $s \geq 0$, θ^i_s is surjective to the kernel of θ^{i+1}_s in $\operatorname{Hom}_s(F_{i+1}, A)$. From these we will show

CLAIM 1. *Conjectures* 1 *and* 2 *imply* $\#\operatorname{Hom}_s(I, A) = \#I_{j-s}$ *for* $s \geq 0$.

This would imply that the non-negative-graded part $\operatorname{Hom}_{0,+}(I, A)$ of the tangent space to A is just large enough to account for deformations of the algebra A to (not necessarily graded) Gorenstein algebras of type T.

We then suppose $r \geq 4$, and j is odd ≥ 3, and assume

CONJECTURE 3. The homomorphism θ^i_{-1} is surjective for $i > 1$; and the image of θ^1_{-1} has maximal size consistent with $\#\operatorname{Hom}_{-1}(I, A) \geq r$.

We then show when $r = 4$ only

CLAIM 2. *Conjectures* 1 *and* 3 *imply that the* -1 *graded piece of the first deformation space,* $T^1_{-1}(A)$ *is zero for general graded Gorenstein algebras* A *of maximal types* T *with* $r \geq 4$ *and with* j *odd* ≥ 3.

Then Lemma 2.31 implies $T^1_{-s}(A) = 0$ for $s > 1$, and we conclude from Claims 1 and 2 that A has deformations only to Gorenstein algebras of type T. Thus the general Gorenstein algebra A' of type T would be generic. The calculations we've made (not included) indicates that maximal types T with $r \geq 4$, j odd ≥ 3 are the only for which our "small tangent space" method could work to produce "generic" Gorenstein algebras. But Theorem 3.5 might be considered as evidence that most symmetric maximal types T produce "generic" Gorenstein algebras.

It remains to discuss Claims 1 and 2.

There is a standard result (see [26] Vol. II) that the term t_i^r in the rth difference sequence of the type T of the graded algebra A is $t_i^r = \Sigma(-1)^s$ (#generators of F_s having degree i). Thus, assuming Conjecture 1 which implies the sum has a single nonzero term, we may read off the number and degrees of the generators of the F_s. For example, the 4th difference of $T = 1, 4, 4, 1$ is $(t_7^4, \ldots, t_0^4) = (1, 0, -6, 5, 5, -6, 0, 1)$. Thus there are 6 generators to I having degree 2, 5 relations among the generators in degree 3, and 5 more in degree 4; there are 6 relations among the relations happening in degree 5, and 1 basis element of F_4 with degree 7. The qualitative behavior of the $\{t_i^r\}$ depends on the parity of r and of j. When r is even and j odd, the non-zero terms consist of the extreme terms $t_0^r = t_{j+r}^r = 1$ and a middle cluster of r symmetric nonzero values occurring in adjacent degrees, whose signs alternate expect for the middle two. Thus F_1 has basis elements of degree $(j+1)/2, \ldots, F_{r/2-1}$ has degree $(r+j-3)/2$; $F_{r/2}$ has basis elements in the two middle degrees, and so on to F_{r-1} of degree $((j-1)/2 + r)$. When also $r = 4$, the sequence

$$(3.4)\quad (t_{j+4}^4, \ldots, t_0^4) = \left(1, 0, \ldots, 0, -\binom{j+1}{2}, \left(\binom{j+1}{2} - 1\right), \left(\binom{j+1}{2} - 1\right), -\binom{j+1}{2}, 0, \ldots, 0, 1\right).$$

Notice finally that $\mathrm{Hom}_s(F_r, A) = 0$ for $s-1$ and $r-2$, since F_r has degree $j+r$, and $A_{j+1} = 0$.

Proof of Claim 1: We let $F_{i,u}$ denote the part of the module F_i having basis elements of degree u. Then when $s \geq 0$

$$\begin{aligned}
\#\mathrm{Hom}_s(I, A) &= \sum_{i=1}^{i=r} (-1)^{i+1} \#\mathrm{Hom}_s(F_i, A) \\
&= \sum_1^r \sum_{u=j/2}^{u=j} (-1)^{i+1} \#\mathrm{Hom}(F_{i,u}, A_{s+u}) && \text{as terms of degree} > j \text{ are } 0 \\
&= \sum \sum (-1)^{i+1} (\#F_{i,u} \cdot \#A_{s+u}) && \text{by Conjecture 2} \\
&= \sum \sum (-1)^{i+1} (\#F_{i,u} \cdot \#A_{j-s-u}) && \text{since } T \text{ is symmetric} \\
&= \sum \sum (-1)^{i+1} (\#F_{i,u} \cdot \#R_{j-s-u}) && \text{since } s+u \geq j/2 \text{ and } T \text{ is maximal} \\
&= \#I_{j-s} && \text{by a standard identity.}
\end{aligned}$$

We now turn to Claim 2. Numerical evidence strongly supports it for $r \geq 4$, but we have explicitly verified it only for $r = 4$ and $r = 5$, of which we include here the case $r = 4$.

PROOF OF CLAIM 2 WHEN $r = 4$: We must show $S = \Sigma_1^4 (-1)^{i+1} \#\mathrm{Hom}_{-1}(F_i, A) \leq 4$. Letting $k = (j/2) + \frac{1}{2}$ the size $\#A_{k-1} = \#A_k = \#R_{k-1} = \binom{k+2}{3}$, and likewise the size $\#A_{k+1} = \binom{k+1}{3}$ and $\#A_{k+2} = \binom{k}{3}$. Thus by (3.4) the sum S is

$$\begin{aligned} S &= \binom{j+1}{2}\binom{k+2}{3} - \left(\binom{j+1}{2} - 1\right)\binom{k+2}{3} - \left(\binom{j+1}{2} - 1\right)\binom{k+1}{3} \\ &\quad + \binom{j+1}{2}\binom{k}{3} \\ &= \binom{k+2}{3} + \binom{k+1}{3} - \binom{j+1}{2}\binom{k}{2}, \\ &\leq 0 \text{ for } j \geq 3. \end{aligned}$$

It may be useful to reconsider the example where $T = 1, 4, 4, 1$, and A is proven generic in section 3.2. The sizes $\#\mathrm{Hom}(F_1, A) = 24$, $\#\mathrm{Hom}_{-1}(F_{2,3}, A) = 5 \cdot 4 = 20$, $\#\mathrm{Hom}_{-1}(F_{2,4}, A) = 5 \cdot 1 = 5$, and $\mathrm{Hom}_{-1}(F_3, A) = 0$. We showed before that $\theta = pr_3\theta'_{-1}$: $\mathrm{Hom}_{-1}(F_1, A) \to \mathrm{Hom}_{-1}(F_{2,3}, A)$ is surjective; we also expect that $\theta' = pr_4\theta'_{-1}$ $\mathrm{Hom}_{-1}(F_1, A) \to \mathrm{Hom}_{-1}(F_{2,4}, A)$ is surjective; but the conditions imposed on an element t by $\theta(t) = \theta'(t) = 0$ are of course dependent: in other words $\mathrm{Hom}_{-1}(F_1, A) \to \mathrm{Hom}_{-1}(F_2, A)$ is not surjective. More generally, when r is even, the two pieces $\ker\theta$ and $\ker\theta'$ coming from $F_{2,k}$ and $F_{2,k-1}$ are just independent enough, we believe, to force $T^1_{-1}(A) = 0$.

Appendix: Comparison of two parametrizations of finite-length semi-local algebras

We explain the equivalences among questions 1–3 of 1.1. We begin with some general comments and a comparison of the topological and geometric viewpoints. We then compare the algebraic and geometric viewpoints. The last discussion yields a dimension result: Proposition A5. Dimension((Hilb^n $k[[x_1, \ldots, x_r]]$) $\cap$ desingularizable algebras) $<$ $(rn - r)$.

In the algebraic and geometric viewpoints, by deformation we mean "flat deformations"; topologists don't usually restrict themselves to flat deformations, but those involved in question 1 are flat since the length of the algebra is constant. Questions 2 and 3 use the Zariski topology on the scheme Slalg(k^n, 1) or Hilb^n $\mathbb{A}_r$ respectively (see later). When $k = C$ or R, we let 2′a, 3′a, etc. denote the same questions as 2a, 3a, etc. but with the complex or real topology on the parameter schemes. We'll say B is a 1-parameter deformation of A in the complex topology if there is a 1-parameter flat family of algebras $A(t)$ with $A(0) = A$, and with $A(t) = B$ for t in a punctured neighborhood of 0. We'll say B is a deformation of A (in the complex topology; or "deformation of A in the extended sense" in the Zariski topology) if every neighborhood of the point parametrizing A contains a point parametrizing an algebra isomorphic to B. We claim $1i \Leftrightarrow 2'i \Leftrightarrow 3'i$, for $i = a, b, c$. Since the Zariski topology is weaker than the complex or real topology, "yes" to 2a (using the extended sense of deformation) implies "yes" to 2′a or 3′a. To be precise, geometrically the algebra $A(T)$ over $K(T)$ is a deformation of $A(0)$ if $A[T]$ is flat over $k[T]$, the polynomial ring in one variable. This is the sense of Note 2, and of the deformation $k(T)[X]/(X^n - T)$ of $k[X]/X^n$ found in §1.1. What we prove about the algebras A of sections 2.2 and 3.2 is that every irreducible Zariski open W in the parameter variety Slalg(k^n, 1) or in Hilb^n $\mathbb{A}_r$ containing a point z parametrizing the algebra A is such that the geometric points of an open in W parametrize only local algebras, of the same type and kind as the algebra A. Thus A has no deformations (not even infinitesimally) to algebras of type or kind different than A, in any of the senses above. Thus, for these algebras the answers to questions 2c and 3c are "no"; likewise they are counterexamples to questions 2′c, 3′c, and 1c.

We now discuss the relation of questions 1 and 3′. A finite mapping germ $F: C^{r'}$ to C^m is given by its local algebra, a quotient $A = k[x_1, \ldots, x_{r'}]/I$ *plus* generators $f_1, \ldots, f_m$ of the ideal I defining A. There is always a stable map germ having its local algebra isomorphic to a given algebra; deformations of stable germs are themselves stable, and right-left equivalence of stable germs is the same as isomorphism of their local algebras. A small 1-parameter deformation of the stable germ F to $F(t) = f_1(t), \ldots, f_m(t)$ (where $t \in$ an open U containing $\mathbb{C}$ or R), satisfying the condition that for all $t \in U - 0$, the algebra $A(T) = k[x_1, \ldots, x_{r'}]/(f_1(t), \ldots, f_m(t))$ has length n, in fact induces a flat deformation of the quotient A to $A(t)$, since constant length is a criterion of flatness. The deformation of A to $A(t)$ is parametrized by an arc in Hilb^n $k[x_1, \ldots, x_{r'}]$, because of the universal property of Hilb^n. Conversely, a flat 1-parameter deformation of I to $I(t)$ will satisfy $I(t) = (f_1(t), \ldots, f_m(t))$ for t in some neighborhood of 0 (see [Tjurina]). We have merely noted that 1-parameter deformations of A can be recognized by what happens to a fixed set of generators of the ideal I defining A as a quotient of $k[x_1, \ldots, x_{r'}]$. This suffices to show question 1 $\Leftrightarrow$ question 3′.

We now compare questions 2 and 3. We first define the first of two schemes, Slalg$(k^n, 1)$ and Hilb^n $\mathbb{A}_r$ which we use, and show they are equivalent from the standpoint of our deformation questions. Choose an element "1" $= v_1$ of the vector space k^n, and chose a complementary basis $v_2, \ldots, v_n$ to v_1. An associative commutative (hence semilocal) algebra with underlying vector space k^n and identity "1" is defined by a multiplication law $v_i v_j = \Sigma_{s=1}^{s=n} c_{ijs} v_s$ with $c_{ijs} \in k$. The c_{ijs} must satisfy certain polynomial conditions, such as $c_{ijs} = c_{jis}$, which define the subscheme Slalg$(k^n, 1)$ of affine n^3 space in the variables X_{ijs}. We let Lalg$_T(k^n, 1)$ be the subscheme parametrizing Artin local algebras of type $T = t_0, t_1, \ldots,$ with $t_0 = 1$ and $n = \Sigma\, t_i$. We now give a well-known lemma, showing that the minimal number of generators of A is semicontinuous on Slalg$(k^n, 1)$.

LEMMA A1: *If $A(z_0)$ is the algebra parametrized by $z_0 \in$ Slalg$(k^n, 1)$, and has generators $x_1, \ldots, x_r \in k^n$, then $x_1, \ldots, x_r$ generate all k-algebras $A(z)$ parametrized by points of an open set U containing z_0.*

PROOF: Suppose $1, u_1, \ldots, u_{n-1}$ are a set of monomials in $x_1, \ldots, x_r$ which span $A(z_0)$. Then they are linearly independent and if 1, $u_1(z), \ldots, u_{n-1}(z)$ denote the same monomial functions of $x_1, \ldots, x_r$ in $A(z)$, these monomials are linearly independent in an open neighborhood U of z_0 in Slalg$(k^n, 1)$. For points $z \in U$, the monomials are independent, hence they span k^n and $x_1, \ldots, x_r$ generate $A(z)$. ■

With the notation above, suppose $A(z_0) = k[X_1, \ldots, X_r]/I$ corresponds to the point w_0 of Hilb^n $\mathbb{A}_r$ = Hilb^n(Symm E) with $E = x_1, \ldots, x_r$. Then if U is the neighborhood of lemma A.1, by the universal property of Hilb^n $\mathbb{A}_r$ there is a morphism $\pi\colon U \to$ Hilb^n $\mathbb{A}_r$ given by $A(z) \to k[X_1, \ldots, X_r]/I(z)$ where $I(z)$ is the ideal of polynomials that are 0 when evaluated at $x_1, \ldots, x_r$ in $A(z)$.

COROLLARY A2: *The morphism π is surjective onto a neighborhood of w_0 in* Hilb^n $\mathbb{A}_r$, *and is an open morphism on U.*

PROOF: Clearly π is surjective into the locus of prime ideals I where the intersection $I \cap \langle 1, u_1, \ldots, u_{n-1}\rangle = 0$, or the locus of prime ideals where n sections $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, u_{n-1})$ of the rank n locally free sheaf $(\mathcal{O} \oplus \mathcal{O} \cdots \oplus \mathcal{O})$ remain independent; this locus is an open neighborhood of w_0. It follows also that the image of an open subset U' of U is open. ■

The equivalence of questions 2 and 3 follows from the Lemma A1 and its Corollary.

We now consider local algebras of type T and compare the dimensions of $\mathrm{Lalg}_T(k^n, 1)$ and $Z_T = \mathrm{Hilb}_T^n R$ parametrizing type T quotient algebras of the power series ring $R = k[[X_1, \ldots, X_r]]$. By dimension of the scheme X at the point x, noted $\dim(X, x)$ we mean $\inf(\dim U | U$ a neighborhood of $x)$, where $\dim U$ is the dimension of the largest component of U. We will use the notation (X, x) to denote a small neighborhood of x in X. We let $\mathcal{F}$ denote a filtration $k^n = F_0 \supset F_1 \supset \cdots \supset F_n = 0$ compatible with the type T: thus $t_i = \#F_i - \#F_{i+1}$. We let $\mathrm{Lalg}(k^n, 1, \mathcal{F})$ parametrize local algebras on k^n with maximal ideal $m = F_1$ such that $m^i = F_i$. The following lemma is self-evident.

LEMMA A3: *Suppose $z_0 \in \mathrm{Lalg}(k^n, 1, \mathcal{F})$ parametrizes the algebra $A(z_0)$,e generated by $x_1, \ldots, x_r$; and that w_0 parametrizes the corresponding point of Z_T,e and that $U' = U \cap \mathrm{Lalg}(k^n, 1, \mathcal{F})$ where Ue is the neighborhood of Lemma A1. Then the morphism $\pi: U' \to \pi(U')$ is a fibration. If $w \in \pi(U')$ parametrizes the quotient $B = R/I$ then the fibre $\pi^{-1}(w)$ parametrizes vector space isomorphisms of B to k^n, taking 1 to 1, X_i to x_i, and preserving the filtration by taking m^iB to F_i.* ■

We can now compare the dimension of the family of local algebras of type T, in the two parametrizations.

PROPOSITION A4: *Suppose z_0 is a point of $\mathrm{Lalg}_T(k^n, 1)$ parametrizing the algebra $A(z_0)$ of type $T = (1, r, t_2, \ldots, 0, 0, 0, \ldots)$ and having generators $x_1, \ldots, x_r$; and suppose w_0 is the corresponding point of $Z_T = \mathrm{Hilb}_T^n R$. Then $\dim(\mathrm{Lalg}_T(k^n, 1), z_0) = \dim(Z_T, w_0) + (n - r)(n - 1)$.*

PROOF: The difference $(\dim(\mathrm{Lalg}_T(k^n, 1), z_0) - \dim(Z_T, w_0))$ is the sum of the dimension of the fibre of U' over (U'), and the dimension of the family of filtrations on k^n near the filtration induced by $A(z_0)$. An isomorphism as in Lemma A3 from B to k^n preserving the filtration is an isomorphism mapping a complementary space E_i to F_{i+1} in F_i, to F_i, hence

$$\text{dim of fibre } \pi^{-1}(w) = \sum_2^n (\#E_i)(\#F_i) = \sum_2^n t_i(n - t_0 - \cdots - t_{i-1}).$$

To choose a filtration, we may begin by choosing the smallest piece F_s, then choosing successively the image of F_{s-1} in $k^n/F_s, \ldots$, the image of F_1 in k^n/F_2. The d-dimensional subspaces of a b-dimensional vector space are parametrized by the Grassman variety Grass(d, b), of dimension $d(b-d)$ as a variety. Thus the dimension of the variety giving filtrations on k^n, is

$$\begin{aligned}\dim(\text{Filtrations}) &= \sum_s^1 (\#(F_i/F_{i+1})(\#(k^n/F_{i+1}) - \#(F_i/F_{i+1})) \\ &= \sum_s^1 t_i(n - t_i - t_{i+1} - \cdots).\end{aligned}$$

The sum of the two contributions to the difference in dimensions is

$$\begin{aligned}\Delta \dim &= t_1(n - t_1 - t_2 - \cdots) + \sum_2^s t_i(2n - t_0 - \cdots - t_s) \\ &= t_1 + \sum_2^s t_i(n) = r + (n-1-r)n \\ &= (n-1)(n-r) \quad \text{as claimed.}\end{aligned}$$ ■

We let Triv $\subset$ Slalg$(k^n, 1)$ parametrize trivial algebras on k^n with fixed identity, in other words algebras isomorphic to $k \oplus \cdots \oplus k$. We note that the dimension dim(Triv)$=n(n-1)$. For, such a trivial algebra is uniquely determined by the choice of n different 1-dimensional subspaces $V_1, \ldots, V_n$ of k^n (such that no proper sum of them contains 1); given such a decomposition of k^n, we may write $1 = \Sigma\, \lambda_i$, $\lambda_i \in V_i$, and the multiplication law of the resulting algebra is determined by $\lambda_i\lambda_j = \delta_{ij}\lambda_i$. On the other hand, the dimension of the open W in Hilb^n $\mathbb{A}_r$ parametrizing nonsingular subschemes of $\mathbb{A}_r$, is rn, the number of ways of choosing n distinct points in affine r-space. We regard R as the completed local ring at the origin of $\mathbb{A}_r$. We conjecture that $\dim((\text{Hilb}^n R) \cap \bar{W}) \overset{?}{=} (n-1)(r-1)$, but can conclude from our discussion the weaker result

Proposition A5: *The dimension* $\dim((\text{Hilb}^n R) \cap \bar{W}) < (rn - r)$.

Proof: The closure $\overline{\text{Triv}}$ is a component of Slalg$(k^n, 1)$, having dimension $n^2 - n$. Trivial algebras of length more than 1 have no deformations to local algebras, so the intersection ($\overline{\text{Triv}} \cap$ Lalg$(k^n, 1)$) is a proper closed subscheme of $\overline{\text{Triv}}$ and has dimension smaller than $(n^2 - n)$. We conclude from Lemmas A1, A4, and Corollary A2 that if

T is a type with $t_1 = r$, then

$$\begin{aligned}\dim(Z_T \cap \bar{W}) &= (\dim(\mathrm{Lalg}_T(k^n, 1) \cap \mathrm{Triv}) - (n-1)(n-r)) \\ &< ((n^2 - n) - (n-1)(n-r)),\end{aligned}$$

or that $\dim(Z_T \cap \bar{W}) < r(n-1)$.

We now suppose $r > r'$ and $T = (1, r', t_2, \ldots)$; we consider Z_T parametrizing ideals of type T in R, and Z'_T parametrizing ideals of type T in $R' = k[[X_1, \ldots, X_{r'}]]$. If the ideals I and I' of type T in R and R' respectively, satisfy $I \cap R' = I'$, and correspond to the points z, z' of Z_T and Z'_T, then we claim

$$\dim(Z_T, z) = \dim(Z'_T, z') + (r - r')(n-1).$$

From the claim, and from the first part of the proof, we would conclude $\dim(Z_T, z) < (r'(n-1) + (r-r')(n-1)) = r(n-1)$, which is the assertion of the Proposition.

Proof of claim: The variables $X_1, \ldots, X_{r'}$ generate the algebra $A = R/I$, and will continue to generate algebras near A. Suppose $1, u_1, \ldots, u_{n-1}$ are monomials in $X_1, \ldots, X_{r'}$ spanning algebras near A. An ideal J in R near I, such that J has type T and $J \cap R' = J'$, is such that J' also has type T, and there are constants $\alpha_{is} \in k$, with $r' < i \leq r$ and $1 \leq s \leq n-1$, such that

$$J = \left(J', \left\{\left(X_i + \sum_{s=1}^{n-1} \alpha_{is} u_s\right) \middle| r' < i \leq r\right\}\right).$$

Conversely, an ideal J' of type T in R', and an arbitrary set of constants α_{is} determine as above an ideal J of type T in R. It is now easy to verify that the mapping $J \to J'$ induces a morphism (Z_T, z) to (Z'_T, z') with fibre an affine space of dimension $(r - r')(n-1)$. This proves the claim and completes the proof of the Proposition. ■

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