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1. Introduction

By a *Q-manifold* we will mean a separable metric manifold modeled on the Hilbert cube Q. Let $f: M \to B$ be a map of a Q-manifold to an ANR. In this paper we will be concerned with the following question: *Does f fiber, i.e. is f homotopic to the projection map of a fiber bundle $M \to B$ with fiber a Q-manifold?* In general it is not true that f fibers. For example, a constant map $Q \to S^1$ does not fiber. In Theorem 1 below we treat the $[0, 1)$-stable case in which f always fibers, while Theorems 3–7 indicate some of the problems one encounters in the compact cases.

Theorem 1 is not terribly surprising. It is an extension of the well known result that Q manifolds which have the form $M \times [0, 1)$ are homeomorphic if and only if they are homotopy equivalent (see [3, Chapter V]).

THEOREM 1: *If $f: M \to B$ is a map of a Q-manifold to a locally compact* ANR, *then the composition $M \times [0, 1) \xrightarrow{proj} M \xrightarrow{f} B$ fibers.*

Of course, there is an analogue of this result for l_2-manifolds, where l_2 is separable infinite-dimensional Hilbert space.

THEOREM 2: *If $f: M \to B$ is a map of an l_2-manifold to a topologically complete separable metric* ANR, *then f fibers.*

In the compact cases below we immediately encounter obstructions to repeating the proofs of Theorems 1 and 2. By making enough connectivity assumptions so that these obstructions vanish, we obtain the following result. See §2 for a review of the undefined terms.

THEOREM 3: *Let* $f: M \to B$ be a map of a compact Q-manifold to a compact, connected ANR B *which is simple homotopy equivalent to a finite n-complex. If the homotopy fiber* $\mathcal{F}(f)$ *of* f *is homotopy equivalent to a finite n-connected complex* K, *then there is an obstruction in the Whitehead group* $\mathrm{Wh}\,\pi_1(M)$ *which vanishes iff* f *fibers. Moreover, if* $n = 1$ *we only need assume that* $\mathrm{Wh}\,\pi_1(K) = 0$, *and if* $n = 2$ *we only need assume that* K *is* 1*-connected.*

As a special case of Theorem 3 we obtain an infinite-dimensional version of Casson's fibering theorem [2].

COROLLARY: *If* $M \to S^2$ *is a map of a compact Q-manifold to* S^2 *such that* $\mathcal{F}(f)$ *is homotopy equivalent to a finite* 1*-connected complex, then* f *fibers.*

In Theorems 4–7 we specialize to the cases in which the base B is homotopy equivalent to a wedge of 1-spheres. The main tool is given in Theorem 4 and the main result is given in Theorem 5.

THEOREM 4: *Let* $(\mathscr{E}, p, B)$ *be a Hurewicz fibration such that* B *is a compact* ANR *homotopy equivalent to a wedge of* n 1*-spheres and the fiber* F *is homotopy equivalent to a finite connected complex. Then* $\mathscr{E}$ *is fiber homotopy equivalent to a compact Q-manifold fiber bundle over* B *iff an obstruction lying in a quotient of the direct sum of* n *copies of* $\mathrm{Wh}\,\pi_1(F)$ *vanishes. Given that this obstruction vanishes, there is a* 1–1 *correspondence between simple equivalence classes of such bundles and a quotient of a subgroup of* $\mathrm{Wh}\,\pi_1(F)$.

For an explanation of the last sentence in the above statement we refer the reader to §5.

THEOREM 5: *Let* $f: M \to B$ *be a map of a compact Q-manifold to a compact* ANR *which is homotopy equivalent to a wedge of* n 1*-spheres and assume that the homotopy fiber* $\mathcal{F}(f)$ *is homotopy equivalent to a finite connected complex. There are two obstructions to* f *fibering. The first one lies in a quotient of the direct sum of* n *copies of* $\mathrm{Wh}\,\pi_1\mathcal{F}(f)$. *If this obstruction vanishes, the second one is defined and lies in a quotient of* $\mathrm{Wh}\,\pi_1(M)$.

In Theorem 6 we treat the special case of Theorem 5 in which B is homotopy equivalent to S^1. Here the situation is considerably simplified and what we obtain is an infinite-dimensional version of Farrell's fibering theorem [10].

THEOREM 6: *Let $f: M \to B$ be a map of a compact Q-manifold to a compact* ANR *which is homotopy equivalent to S^1 and for which the homotopy fiber $\mathcal{F}(f)$ is homotopy equivalent to a finite connected complex. There are two obstructions to f fibering. They are independently defined and both lie in* Wh $\pi_1(M)$.

We remark that one of the obstructions obtained here is just Farrell's obstruction for the finite-dimensional case, but the infinite-dimensional nature of the problem requires another obstruction.

Finally, in Theorem 7 we classify equivalence classes of Q-manifold fiber bundle projections over nice ANRs.

THEOREM 7: *Let $f, f_1: M \to B$ be homotopic compact Q-manifold fiber bundle projections, where B is a compact* ANR *homotopy equivalent to a wedge of n 1-spheres, and let F be the connected fiber of $f: M \to B$. There are two obstructions to finding a homeomorphism $h: M \to M$ such that $fh = f_1$ and h is homotopic to the identity. The first lies in* Wh $\pi_1(F)$, *and if it vanishes the second is defined and lies in a quotient of the direct sum of n copies of $\mathcal{P}(F)$.*

Here $\mathcal{P}(F)$ is the group of all isotopy classes of homeomorphisms of F to itself which are homotopic to the identity. It is a quotient of π_0 of the concordance group of F, which has been algebraically investigated by [12]. See §2 for further details.

We now say a few words about the organization of the material in this paper. §2 contains some preliminary results and in §3 we prove Theorems 1 and 2. In §§4–8 we prove Theorems 3–7. In §9 we prove a result (Theorem 8) which calculates the kernel of a certain map of Whitehead groups. This generalizes a result of Farrell [9]. Theorem 8 may be paraphrased as follows. Let $(\mathscr{E}, p, B)$ be a Hurewicz fibration, where B is a finite wedge of 1-spheres and the fiber F has the homotopy type of a finite complex. If i is the inclusion map $i: F \hookrightarrow \mathscr{E}$, then Theorem 8 computes the kernel of

$$i_*: \text{Wh}\ \pi_1(F) \to \text{Wh}\ \pi_1(\mathscr{E}).$$

The constructions in §9 are made more geometric by replacing $\mathscr{E}$ with a finite "wedge" of mapping tori.

2. Preliminaries

If $p: E \to B$ is a map and $B_1 \subset B$, we use $E \mid B_1$ to denote $p^{-1}(B_1)$ and we let $E_b = p^{-1}(b)$, for each $b \in B$. If $p': E' \to B$ is another map, *then* $f: E \to E'$ is said to be *fiber preserving* (f.p.) provided that $f(E_b) = E'_b$, for each $b \in B$. The restriction of f to E_b is denoted by $f_b: E_b \to E'_b$. A *f.p.* map $f: E \to E'$ is said to be a *fiber homotopy equivalence* (*f.h.e.*) if there exists a *f.p.* map $g: E' \to E$ such that fg and gf are *f.p.* homotopic to their respective identities. We will abbreviate ordinary homotopy equivalence by *h.e.*

If $f: E \to B$ is any map, where B is path connected, then we define

$$\mathscr{E}(f) = \{(e, \omega) \in E \times B^I \mid f(e) = \omega(0)\}$$

(B^I is the space of paths in B.) Define $p: \mathscr{E}(f) \to B$ by $p(e, \omega) = \omega(1)$. $p: \mathscr{E}(f) \to B$ is the *mapping path fibration* of $f: E \to B$. There is a *h.e.* $g: E \to \mathscr{E}(f)$ such that $pg \simeq f$. For any $b_0 \in B$, the fiber of $\mathscr{E}(f)$ over b_0 is

$$\mathscr{F}(f) = p^{-1}(b_0) = \{(e, \omega) \mid f(e) = \omega(0), \omega(1) = b_0\}.$$

$\mathscr{F}(f)$ is called the *homotopy fiber* of $f: E \to B$.

The following result will be used several times in the sequel. For a proof see [8] for the case in which B is a countable complex and see [14] for the general case.

THEOREM 2.1: *Let* $p: E \to B$, $p': E' \to B$ *be Hurewicz fibrations, where* B *is a connected* ANR, *and let* $h: E \to E'$ *be a f.p. map such that* $h_{b_0}: E_{b_0} \to E'_{b_0}$ *is a h.e., for some* $b_0 \in B$. *Then* h *is a f.h.e.*

The above result gives us the following useful theorem.

THEOREM 2.2: *Let* $p: E \to B$, $p': E' \to B$ *be Hurewicz fibrations, where* E, B *and all the fibers have the homotopy types of countable complexes. If* $f: E \to E'$ *is a h.e. such that* $p'f \simeq p$, *then* f *is homotopic to a f.h.e.*

PROOF: Assume that B is connected and choose $b_0 \in B$, $e_0 \in E_{b_0}$. The condition $p'f \simeq p$ gives us a homotopy $H: E \times I \to B$ such that $H_0 = p$ and $H_1 = p'f$. Lifting H we get a homotopy $\tilde{H}: E \times I \to E'$ for which $\tilde{H}_1 = f$. Then $g = \tilde{H}_0: E \to E'$ is homotopic to f and g is *f.p.* The homotopy exact sequences of the two fibrations give us a commutative diagram,

$$\begin{array}{ccccccccccc}
\cdots \to & \pi_{n+1}(E, e_0) & \to & \pi_{n+1}(B, b_0) & \to & \pi_n(E_{b_0}, e_0) & \to & \pi_n(E, e_0) & \to & \pi_n(B, b_0) & \to \cdots \\
& \downarrow g_* & & \downarrow Id & & \downarrow g_{b_0} & & \downarrow g_* & & \downarrow Id & \\
\cdots \to & \pi_{n+1}(E', e'_0) & \to & \pi_{n+1}(B, b_0) & \to & \pi_n(E'_{b_0}, e'_0) & \to & \pi_n(E', e'_0) & \to & \pi_n(B, b_0) & \to \cdots
\end{array}$$

Here $e_0' = g(e_0)$ and by the five lemma $(g \mid E_{b_0})_*$ is a *h.e.* Then we apply Theorem 2.1. ■

In the sequel we will need a considerable amount of Q-manifold machinery. Our basic reference for this is [3]. It would be time consuming to give a complete description of the material from [3] which we will need, but here is a list of some of the highlights.

1. Z-sets and Z-set unknotting ([3, Theorem 19.4]).
2. The classification theorem for simple equivalences in terms of homeomorphisms on Q-manifolds ([3, Theorem 38.1]).
3. The triangulation theorem for Q-manifolds ([3, Theorem 36.2]).
4. The ANR theorem, which says that every locally compact ANR times Q is a Q-manifold ([3, Theorem 44.1]).

It will be convenient to know how to change bases in fibering problems.

THEOREM 2.3: *Consider $f: M \to B$, where M is a compact Q-manifold, and B is a compact* ANR, *and let $g: B \to B'$ be a simple equivalence of B to another compact* ANR. *Then f fibers iff gf fibers.*

PROOF: Since $g: B \to B'$ is a simple equivalence we have a homeomorphism $\beta: B \times Q \to B' \times Q$ which is homotopic to $g \times Id$. Choose a homeomorphism $\alpha: M \times Q \to M$ homotopic to the projection map. Assuming that f fibers we have a fiber bundle projection map $p: M \to B$. It is easy to check that the composition

$$M \xrightarrow{\alpha^{-1}} M \times Q \xrightarrow{p \times Id} B \times Q \xrightarrow{\beta} B' \times Q \xrightarrow{proj} B'$$

is a fiber bundle projection homotopic to gf. ■

In a similar fashion we can establish the following $[0, 1)$-stable result.

THEOREM 2.4: *Consider $f: M \to B$, where M is a Q-manifold and B is a locally compact* ANR, *and let $g: B \to B'$ be a h.e. of B to another locally compact* ANR. *Then $M \times [0, 1) \xrightarrow{proj} M \xrightarrow{f} B$ fibers iff $M \times [0, 1) \xrightarrow{proj} M \xrightarrow{f} B \xrightarrow{g} B'$ fibers.*

Here is a mild generalization of Anderson's result [1] to fiber bundles over ANRs. The result is also true for ANR Hurewicz fibrations over ANRs.

THEOREM 2.5: *Let $p_1: E_1 \to B$ and $p_2: E_2 \to B$ be compact Q-mani-*

fold fiber bundles such that B is a compact connected ANR *and let* $f: E_1 \to E_2$ *be a f.h.e. If* $b_0 \in B$, *then* $\tau(f) = i_*\chi(B)\tau(f \mid (E_1)_{b_0})$, *where* $\chi(B)$ *is the Euler characteristic of B and i is the inclusion* $(E_2)_{b_0} \hookrightarrow E_2$, *and τ denotes Whitehead torsion.*

PROOF: For the moment assume that B is a finite complex. Choose any other basepoint $b_1 \in B$. We will first prove that $j_*\tau(f \mid (E_1)_{b_1}) = i_*\tau(f \mid (E_1)_{b_0})$, where $j: (E_2)_{b_1} \hookrightarrow E_2$. Choose a path $\omega: I \to B$ from b_0 to b_1. Over $\omega(I)$ we have trivial bundles. This induces homeomorphisms $\alpha: (E_1)_{b_0} \to (E_1)_{b_1}$ and $\beta: (E_2)_{b_0} \to (E_2)_{b_1}$ so that α is homotopic to $(E_1)_{b_0} \hookrightarrow E_1$ and β is homotopic to $(E_2)_{b_0} \hookrightarrow E_2$. Thus we have a homotopy commutative diagram,

$$\begin{array}{ccc} (E_1)_{b_0} & \xrightarrow{\alpha} & (E_1)_{b_1} \\ f \mid \downarrow & & \downarrow f \mid \\ (E_2)_{b_0} & \xrightarrow{\beta} & (E_2)_{b_1} \end{array}$$

Since $\tau(\alpha) = 0$ and $\tau(\beta) = 0$ we have $\tau(f \mid (E_1)_{b_1}) = \beta_*\tau(f \mid (E_1)_{b_0})$. Since $j\beta \simeq i$ we get $j_*\tau(f \mid (E_1)_{b_1}) = i_*\tau(f \mid (E_1)_{b_0})$. Moreover, if Δ is any simplex in B we can also prove that $\tau(f \mid (E_1)_{b_0})$ and $\tau(f \mid (E_1 \mid \Delta))$ have the same image in $\mathrm{Wh}\, \pi_1(E_2)$. This follows because if $b_1 \in \Delta$, then we have a homotopy commutative diagram

$$\begin{array}{ccc} (E_1)_{b_1} & \hookrightarrow & E_1 \mid \Delta \\ f \mid \downarrow & & \downarrow f \mid \\ (E_2)_{b_1} & \hookrightarrow & E_2 \mid \Delta, \end{array}$$

where the inclusions are simple equivalences.

We now begin the proof. Let $\dim B = n$ and let B' be the $(n-1)$-skeleton of B, where $b_0 \in B'$. Then we get restricted fiber bundles

$$p_1': E_1 \mid B' \to B', \quad p_2': E_2 \mid B' \to B',$$

and a *f.h.e.* $f' = f \mid (E_1 \mid B'): E_1 \mid B' \to E_2 \mid B'$. We inductively assume that $\tau(f') = (i')_*\chi(B')\tau(f \mid (E_1)_{b_0})$, where $i': (E_2)_{b_0} \hookrightarrow E_2 \mid B'$. Let $\{\Delta_i\}_{i=1}^k$ be the n-simplexes of B. Using the Sum Theorem [6, p. 76] we have

$$\tau(f) = \chi(B')\tau(f \mid (E_1)_{b_0}) + k\tau(f \mid (E_1)_{b_0}) - \left(\sum_{i=1}^k \chi(\partial\Delta_i)\tau(f \mid (E_1)_{b_0})\right),$$

where we have omitted obvious inclusion-induced maps. Since

$$\chi(B') + k - \sum_{i=1}^k \chi(\partial\Delta_i) = \chi(B)$$

we are done for the case in which B is a finite complex. For the remainder of the proof we show how to reduce the general case to this case.

Our first observation is that if B is any compact Q-manifold, then the above proof goes through. We just replace B by $K \times Q$, for some finite complex K, and argue inductively over the skeleta of K times Q. More generally, if we multiply everything by Q we obtain Q-manifold fiber bundles $E_i \times Q \to B \times Q$, where $B \times Q$ must be a Q-manifold. We get a *f.h.e.* $f \times Id: E_1 \times Q \to E_2 \times Q$. The above special case implies that

$$\tau(f \times Id) = (i')_* \chi(B) \tau((f \times Id) \mid (E_1)_{b_0} \times Q),$$

where i' is inclusion. Projecting back to E_2 we get $\tau(f) = i_* \chi(B) \tau(f \mid (E_1)_{b_0})$ and we are done. ■

COROLLARY 2.6: *With p_i: $E_i \to B$ as above let g: $E_1 \to E_2$ be a map such that $p_2 g \simeq p_1$ and assume that* Wh $\pi_1((E_1)_{b_0}) = 0$. *If g is a h.e., then g is a simple equivalence.*

PROOF: Using Theorem 2.2 we have $g \simeq g'$, where g' is a *f.h.e.* Then

$$\tau(g) = \tau(g') = i_* \chi(B) \tau(g' \mid (E_1)_{b_0}),$$

and $\tau(g' \mid (E_1)_{b_0}) \in \mathrm{Wh}\ \pi_1(E_2)_{b_0} \cong \mathrm{Wh}\ \pi_1(E_1)_{b_0} = 0$. ■

We will also need the notion of a mapping torus. For any compactum X and map $\varphi: X \to X$, the *mapping torus* of φ is the compactum

$$T(\varphi) = X \times [0, 1]/\sim,$$

where $\sim$ is the equivalence relation generated by $(x, 0) \sim (\varphi(x), 1)$. It is clear that there is a natural map $T(\varphi) \to S^1$ so that each point-inverse is naturally identified with X.

Finally we introduce the group $\mathcal{P}(M)$ needed in Theorem 7. For any compact Q-manifold M let $\mathcal{P}(M)$ denote the group of isotopy classes of homeomorphisms of M which are homotopic to the identity. Here are some facts about $\mathcal{P}(M)$ which appear either explicitly or implicitly in [4].

1. If M is 1-connected, then $\mathcal{P}(M)$ is trivial.
2. $\mathcal{P}(S^1 \times Q) \cong Z_2 \oplus Z_2 \oplus \cdots$,
3. If M is *h.e.* to N, then $\mathcal{P}(M) \cong \mathcal{P}(N)$.
4. $\mathcal{P}(M)$ is always abelian.

If $h: M \to M$ is a homeomorphism homotopic to the identity, then h determines an isotopy class of homeomorphisms in $\mathcal{P}(M)$. To save notation we will identify h with this isotopy class in $\mathcal{P}(M)$. Thus in §8 a statement such as $f = g$ actually means that f is isotopic to g, where f and g are homeomorphisms homotopic to the identity.

3. Proofs of Theorems 1 and 2

We begin with the proof of Theorem 1. The basic step is the following result.

LEMMA 3.1: *Let N be a Q-manifold, $E \to S^n$ be a fiber bundle with fiber $N \times [0, 1)$, and let $f: S^n \times N \times [0, 1) \in E$ be a f.h.e. Then f is fiber homotopic to a homeomorphism.*

PROOF: Using Theorem 4.1 of [5] there is a *f.p.* embedding $g: S^n \times N \times [0, 1) \to E$ such that each $g_x: N \times [0, 1) \to E_x$ is a Z-embedding and such that g is fiber homotopic to f. Let $S^n \times N \times [0, 1)$ be identified with $S^n \times N \times [0, 1) \times \{0\}$ in $S^n \times N \times [0, 1) \times I$. Our strategy is to show that we have a *f.p.* homeomorphism of pairs,

$$(E, g(S^n \times N \times [0, 1))) \cong (S^n \times N \times [0, 1) \times I, S^n \times N \times [0, 1)).$$

This implies that the inclusion $g(S^n \times N \times [0, 1)) \hookrightarrow E$ is fiber homotopic to a homeomorphism, thus completing the proof of our lemma. Let $D^n \subset S^n$ be any n-cell.

ASSERTION: *There exists a f.p. homeomorphism of $D^n \times N \times [0, 1) \times I$ onto $E \mid D^n$ which agrees with g on $D^n \times N \times [0, 1)$.*

PROOF OF ASSERTION: Choose any *f.p.* homeomorphism $\alpha: D^n \times N \times [0, 1) \times I \to E \mid D^n$. We must replace α by α' so that $\alpha' \mid D^n \times N \times [0, 1) = g$. Consider the *f.p.* Z-embedding

$$g_1 = \alpha^{-1} g: D^n \times N \times [0, 1) \to D^n \times N \times [0, 1) \times I.$$

It will suffice to construct a *f.p.* homeomorphism of $D^n \times N \times [0, 1] \times I$ onto itself which extends g_1.

We now use the fact that g_1 is a *f.h.e.* Choose any $b_0 \in D^n$ and consider $(g_1)_{b_0}: N \times [0, 1) \to N \times [0, 1) \times I$, which is a *h.e.* It follows from [3, Theorem 21.2] that there exists a homeomorphism $u: N \times [0, 1) \times I \to N \times [0, 1] \times I$ extending $(g_1)_{b_0}$. Define $g_2: D^n \times N \times [0, 1) \to D^n \times N \times [0, 1) \times I$ by $(g_2)_b = (g_1)_{b_0}$, for all $b \in D^n$. Then g_2 is a "constant" *f.p.* Z-embedding. Using the homeomorphism u it is clear that g_2 extends to a *f.p.* homeomorphism of $D^n \times N \times [0, 1) \times I$ onto itself. So, to finish, all we need is a *f.p.* homeomorphism of $D^n \times N \times [0, 1) \times I$ onto itself which composes with g_1 to give g_2.

To see this, let $\theta_t\colon D^n \to D^n$ be a homotopy such that $\theta_0 = id$ and $\theta_1(D^n) = \{b_0\}$. Then define a *f.p.* homotopy

$$\beta_t\colon D^n \times N \times [0, 1) \to D^n \times N \times [0, 1) \times I$$

by $(\beta_t)_b = (g_1)_{\theta_t(b)}$. Clearly $\beta_0 = g_1$ and $\beta_1 = g_2$. Moreover, this is a *f.p.* proper homotopy. By Theorem 5.1 of [5] we conclude that there exists a *f.p.* homeomorphism r of $D^n \times N \times [0, 1) \times I$ onto itself such that $rg_1 = g_2$. This completes the proof of the assertion.

Now let G be the homeomorphism group $\mathscr{H}(N \times [0, 1) \times I, N \times [0, 1))$, the space of all homeomorphisms of $N \times [0, 1) \times I$ onto itself which are the identity on $N \times [0, 1)$. For each $b \in S^n$ let $\Phi(b)$ be the space of all homeomorphisms $\varphi\colon N \times [0, 1) \times I \to E_b$ such that $\varphi = g_b$ on $N \times [0, 1)$. This makes $E \to S^n$ into a fiber bundle with structure group G, which we call a *G-bundle* (see [16, p. 90]). We will show that E is trivial as a G-bundle. This will imply that there is a *f.p.* homeomorphism of pairs,

$$(E, g(S^n \times N \times [0, 1))) \cong (S^n \times N \times [0, 1) \times I, S^n \times N \times [0, 1)),$$

as was our strategy. To show that E is trivial for all n, all we have to do is prove that G is contractible.

Choose any $h \in G$. If $f\colon [0, 1) \times I \to [0, 1) \times I$ is any homeomorphism which is the identity on $[0, 1) \times \{0\}$, then it is easy to isotope f to a homeomorphism $f' rel [0, 1) \times \{0\}$, where f' is also the identity on $\{0\} \times I$. This same idea easily shows that h is isotopic to $h' rel\ N \times [0, 1)$, where h' is the identity on $N \times \{0\} \times I$. Using a variation of the well known Alexander trick define $h'_1 = Id$ and for $0 \leq t < 1$ define

$$h'_t = \begin{cases} Id, & \text{on } N \times [0, t] \times I \\ \varphi_t^{-1} h' \varphi_t, & \text{on } N \times [t, 1) \times I, \end{cases}$$

where $\varphi_t\colon N \times [t, 1) \times I \to N \times [0, 1) \times I$ is defined by linearly homeomorphing $[t, 1)$ to $[0, 1)$. Then h'_t defines an isotopy of h' to Id $rel(N \times \{0\} \times I) \cup (N \times [0, 1))$. All of these isotopies depend continuously on h. Thus G is contractible. ■

REMARK: The above method of proof can be used to prove that a *f.h.e.* between any two fiber bundles, with fiber $N \times [0, 1)$, is fiber homotopic to a homeomorphism.

We now use Lemma 3.1 to prove the following result.

LEMMA 3.2: *Let $\mathscr{E} \to B$ be a Hurewicz fibration over a countable complex and assume that all the fibers are h.e. to countable complexes. Then $\mathscr{E}$ is f.h.e. to a fiber bundle over B with fiber a Q-manifold.*

PROOF: Without loss of generality assume that B is connected and use [3, Theorem 28.1] to choose a Q-manifold N which is *h.e.* to the fibers of $\mathscr{E} \to B$. We will induct over the n-skeleta of B, B_n, to inductively build our fiber bundle. For $n = 0$ it is clear that $\mathscr{E} \mid B_0$ is *f.h.e.* to a fiber bundle over B_0 with fiber $N \times [0, 1)$. Passing to the inductive step assume $n \geq 0$ and let $f_1: \mathscr{E} \mid B_n \to E_1$ be a *f.h.e.*, where $E_1 \to B_n$ is a fiber bundle with fiber $N \times [0, 1)$. We will extend f_1 to a *f.h.e.* $f: \mathscr{E} \mid B_{n+1} \to E$, where $E \to B_{n+1}$ is a fiber bundle extending $E_1 \to B_n$. For simplicity of notation we assume that $B_{n+1} = B_n \cup \Delta$, where Δ is a single $(n+1)$-simplex.

By restriction we get a *f.h.e.* $f_0: \mathscr{E} \mid \partial\Delta \to E_1 \mid \partial\Delta$. By Theorem 2.1 it suffices to extend f_0 to a *f.p.* map $f_2: \mathscr{E} \mid \Delta \to E_2$, where $E_2 \to \Delta$ is a fiber bundle extending $E_1 \mid \partial\Delta \to \partial\Delta$. Since $\mathscr{E} \mid \partial\Delta$ is *f.h.e.* to $\partial\Delta \times N \times [0, 1)$, we may replace $\mathscr{E} \mid \partial\Delta$ by $\partial\Delta \times N \times [0, 1)$ and consider the following reduction of the problem: *If $f_0: \partial\Delta \times N \times [0, 1) \to E_1 \mid \partial\Delta$ is a f.h.e., then f_0 extends to a f.p. map $f_2: \Delta \times N \times [0, 1) \to E_2$.*

To see how this reduction implies the general case choose a *f.h.e.* $\alpha: \Delta \times N \times [0, 1) \to \mathscr{E} \mid \Delta$, let $\alpha_0 = \alpha \mid \partial\Delta \times N \times [0, 1)$, and let $\beta: \mathscr{E} \mid \partial\Delta \to \partial\Delta \times N \times [0, 1)$ be a fiber homotopy inverse of α_0. Given a *f.h.e.* $f_0: \mathscr{E} \mid \partial\Delta \to E_1 \mid \partial\Delta$, we get a *f.h.e.* $f_0\alpha_0\beta: \mathscr{E} \mid \partial\Delta \to E_1 \mid \partial\Delta$. The reduction implies that $f_0\alpha_0$ extends, and since β extends it follows that $f_0\alpha_0\beta$ extends. Since f_0 is fiber homotopic to $f_0\alpha_0\beta$ we conclude that f_0 extends.

To verify the reduction we first use Lemma 3.1 to see that f_0 is fiber homotopic to a homeomorphism $\alpha: \partial\Delta \times N \times [0, 1) \to E_1 \mid \partial\Delta$. Thus all we have to do is show how to extend α to a *f.p.* map $\tilde{\alpha}: \Delta \times N \times [0, 1) \to E_2$. Define

$$E_2 = (E_1 \mid \partial\Delta) \bigcup_{\alpha} (\Delta \times N \times [0, 1)),$$

where the attaching is made by α. Then α automatically extends to a *f.p.* map of $\Delta \times N \times [0, 1)$ onto E_2. ■

Finally, we will need the following result.

LEMMA 3.3: *If $f: M \to B$ is a map between locally compact* ANRs, *where B is connected, then the homotopy fiber of f has the homotopy type of a countable complex.*

PROOF: For definiteness choose a basepoint $b_0 \in B$. Let $\alpha: M \to Q \times [0, 1)$ be any closed embedding and define $f': M \to B \times Q \times [0, 1)$ by $f' = (f, \alpha)$. Choose the basepoint $b_0' = (b_0, 0, 0)$ in $B \times Q \times [0, 1)$ and consider the homotopy fiber $\mathscr{F}(f')$.

ASSERTION 1: *$\mathscr{F}(f)$ is h.e. to $\mathscr{F}(f')$.*

PROOF: Define $\varphi: \mathscr{F}(f) \to \mathscr{F}(f')$ by $\varphi(x, \omega) = (x, \omega')$, where ω' follows a straight-line path from $(f(x), \alpha(x))$ to $(f(x), 0, 0))$, for $0 \leq t \leq \frac{1}{2}$, and for $\frac{1}{2} \leq t \leq 1, \omega'$ follows the path ω in $B \times \{0\} \times \{0\} \equiv B$ from $(f(x), 0, 0)$ to b_0'. Define $\psi: \mathscr{F}(f') \to \mathscr{F}(f)$ by $\psi(x, \omega) = (x, \omega'')$, where $\omega'' = proj \circ \omega$ ($proj$: $B \times Q \times [0, 1) \to B$). We leave it as an easy exercise for the reader to prove that φ and ψ are homotopy inverses.

ASSERTION 2: *$\mathscr{F}(f')$ is an* ANR.

PROOF: Observe that f' is a closed embedding. Consider the space

$$\Omega = (B \times Q \times [0, 1), b_0')^{(I,1)} \subset (B \times Q \times [0, 1))^I,$$

the space of paths ending at b_0'. It follows from [13] that Ω is an ANR. Clearly $\mathscr{F}(f')$ is a closed subset of $M \times \Omega$. Choose $(x, \omega) \in M \times \Omega$ which is close to $\mathscr{F}(f')$. Then we must have $\omega(0)$ close to $f'(x)$. If they are sufficiently close, then there is a canonical path in $B \times Q \times [0, 1)$ from $f'(x)$ to $\omega(0)$. By composing this canonical path with ω we obtain a new path $\omega' \in \Omega$ which starts at $f'(x)$ and ends at b_0'. Thus $(x, \omega) \to (x, \omega')$ defines a retraction $r: U \to \mathscr{F}(f')$, for U some suitable neighborhood of $\mathscr{F}(f')$ in $M \times \Omega$. Therefore $\mathscr{F}(f')$ is an ANR.

Finally, it follows from [15] that the ANR $\mathscr{F}(f')$ has the homotopy type of a countable complex. ■

PROOF OF THEOREM 1: We are given a map $f: M \to B$, where B is a locally compact ANR. It follows from [15] that B is a *h.e.* to a countable complex, and therefore by Theorem 2.4 we may assume that B is a countable complex. Without loss of generality assume that B is connected. Let $p: \mathscr{E} \to B$ be the mapping path fibration with fiber $\mathscr{F}(f)$, and let $g: M \to \mathscr{E}$ be a *h.e.* such that $pg \cong f$. Using Lemma 3.2 there is a fiber bundle $q: E \to B$, with fiber a Q-manifold N, which is *f.h.e.* to $p: \mathscr{E} \to B$. We therefore obtain a *h.e.* $g': M \to E$ such that

$qg' \simeq f$. Then $g' \times id\colon M \times [0,1) \to E \times [0,1)$ is homotopic to a homeomorphism $h\colon M \times [0,1) \to E \times [0,1)$ by [3]. Clearly

$$M \times [0,1) \xrightarrow{h} E \times [0,1) \xrightarrow{proj} E \longrightarrow B$$

is a fiber bundle projection homotopic to $f \circ proj\colon M \times [0,1) \to B$. ■

PROOF OF THEOREM 2: The machinery we have used for the proof of Theorem 1 has analogues for l_2-manifolds. The knowledgeable reader can easily supply the details. ■

4. Proof of Theorem 3 and its Corollary

For the proof of Theorem 3 we will first need the following result.

LEMMA 4.1: *Let N be a compact Q-manifold, $E \to S^n$ be a fiber bundle with fiber N, and let $f; S^n \times N \to E$ be a f.h.e. If N is $(n+1)$-connected, then f is fiber homotopic to a homeomorphism. Moreover, if $n = 0$ we only need assume that* Wh $\pi_1(N) = 0$, *and if $n = 1$ we only need assume that N is 1-connected.*

PROOF: Following the proof of Lemma 3.1, *f is homotopic to a f.p. Z*-embedding $g\colon S^n \times N \to E$. It suffices to show that we have a *f.p.* homeomorphism of pairs,

$$(E, g(S^n \times N)) \cong (S^n \times N \times I, S^n \times N).$$

If $n = 0$ it follows from the assumption Wh $\pi_1(N) = 0$ that each inclusion $g_b(N) \hookrightarrow E_b$ is homotopic to a homeomorphism. Since $S^n = \{b_1, b_2\}$ this is all we need for our desired *f.p.* homeomorphism of pairs.

If $n \geq 1$ we proceed as in Lemma 3.1 and show that $E \to S^n$ may be regarded as a G-bundle, where G is the homeomorphism group $\mathcal{H}(N \times I, N)$. All we need to do is show that $E \to S^n$ is trivial as a G-bundle. For this it suffices to prove that G is $(n-1)$-connected. It follows from [4] and [11] that $\pi_0(G) = 0$ for N 1-connected, and in general $\pi_{k-1}(G) = 0$ for N $(k+1)$-connected. ■

LEMMA 4.2: *Let $\mathscr{E} \to B$ be a Hurewicz fibration over a finite n-complex and assume that all the fibers are h.e. to a compact Q-manifold N. If N is n-connected, then $\mathscr{E}$ is f.h.e. to a fiber bundle over B with fiber N. Moreover, if $n = 1$ we only need assume* Wh $\pi_1(N) = 0$, *and if $n = 2$ we only need assume N to be 1-connected.*

PROOF: Using Lemma 4.1 we can prove Lemma 4.2 just as Lemma 3.2 followed from Lemma 3.1. ■

PROOF OF THEOREM 3: We are given a map $f: M \to B$, of a compact Q-manifold to a compact, connected ANR B which is simple equivalent to a finite n-complex. By Theorem 2.3 we may assume that B is a finite n-complex. Let $\mathscr{E} \to B$ be the mapping path fibration and use Lemma 4.2 to conclude that $\mathscr{E}$ is *f.h.e.* to a fiber bundle $p: E \to B$, whose fiber is a compact Q-manifold. Thus we have a homotopy equivalence $g: E \to M$ such that $fg \simeq p$. We define our obstruction to be $\tau(g) \in \text{Wh}\, \pi_1(M)$.

To see that $\tau(g)$ is well-defined we assume that there is another such *h.e.* $g_1: E_1 \to M$, where $E_1 \to B$ is a fiber bundle whose fiber is a compact Q-manifold. It follows from Corollary 2.6 that the torsion of the composition $g^{-1}g_1: E_1 \to E$ is zero, thus $\tau(g) = \tau(g_1)$.

If $\tau(g) = 0$, then g is homotopic to a homeomorphism $h: E \to M$, and f is therefore homotopic to the bundle projection $M \xrightarrow{h^{-1}} E \to B$. On the other hand assume that f is homotopic to a bundle projection $M \to B$. The *h.e.* $g: E \to M$ must have zero torsion by Corollary 2.6. ■

PROOF OF THE COROLLARY: The homotopy sequence of $f: M \to B$ gives us an exact sequence

$$\pi_1 \mathscr{F}(f) \to \pi_1(M) \to \pi_1(S^2),$$

thus $\pi_1(M) = 0$ and $\text{Wh}\, \pi_1(M) = 0$. This implies that our obstruction to fibering is zero. ■

5. Proof of Theorem 4

We first introduce some notation which will be used throughout this section. Let $\mathscr{E} \to B$ represent a Hurewicz fibration, where B is a compact ANR *h.e.* to a wedge of n 1-spheres. Choose a basepoint $b_0 \in B$ and assume that $\mathscr{E}_{b_0}$ is *h.e.* to a finite connected complex. Let $\{\alpha_i\}_{i=1}^n$ be a collection of maps, $\alpha_i: (S^1, *) \to (B, b_0)$, such that $\{[\alpha_i]\}_{i=1}^n$ freely generates $\pi_1(B, b_0)$. Each map α_i may be regarded as a map of $(I, \partial I)$ to (B, b_0), and the homotopy lifting criterion implies that α_i can be covered by a map $\tilde{\alpha}_i: \mathscr{E}_{b_0} \times I \to \mathscr{E}$ such that $(\tilde{\alpha}_i)_0 = id$. We call

$\varphi_i = (\tilde{\alpha}_i)_1: \mathscr{E}_{b_0} \to \mathscr{E}_{b_0}$ a *characteristic map* corresponding to α_i. It is well-known that φ_i is a *h.e.* and its homotopy class is uniquely determined.

Definition of the Obstruction: Define a homomorphism

$$\theta: \text{Wh } \pi_1(\mathscr{E}_{b_0}) \to \text{Wh } \pi_1(\mathscr{E}_{b_0}) \oplus \cdots \oplus \text{Wh } \pi_1(\mathscr{E}_{b_0}) \quad (n \text{ copies})$$

by sending τ in Wh $\pi_1(\mathscr{E}_{b_0})$ to $((Id - (\varphi_1)_*)\tau, \ldots, (Id - (\varphi_n)_*)\tau)$, where $*$ as usual indicates induced homomorphisms on Whitehead groups. Choose any *h.e.* h of $\mathscr{E}_{b_0}$ to a finite complex K. We define our obstruction, $\mathcal{O}_1(\mathscr{E})$, to be the image of

$$(h_*^{-1}\tau(h\varphi_1 h^{-1}), \ldots, h_*^{-1}\tau(h\varphi_n h^{-1}))$$

in Cokernel $(\theta) = \text{Wh } \pi_1(\mathscr{E}_{b_0}) \oplus \cdots \oplus \text{Wh } \pi_1(\mathscr{E}_{b_0})/\text{Image}(\theta)$. (Here h^{-1} is a homotopy inverse of h.)

Lemma 5.1: *$\mathcal{O}_1(\mathscr{E})$ is well defined.*

Proof: Let $g: \mathscr{E}_{b_0} \to L$ be any other *h.e.* from $\mathscr{E}_{b_0}$ to a finite complex. We must prove that $(h_*^{-1}\tau(h\varphi_1 h^{-1}), \ldots, h_*^{-1}\tau(h\varphi_n h^{-1}))$ and $(g_*^{-1}\tau(g\varphi_1 g^{-1}), \ldots, g_*^{-1}\tau(g\varphi_n g^{-1}))$ have the same image in Cokernel (θ). Let $k: L \to K$ be a *h.e.* such that $kg \simeq h$. For each i we have

$$\begin{aligned} h_*^{-1}\tau(h\varphi_i h^{-1}) &= (kg)_*^{-1}\tau(kg\varphi_i g^{-1}k^{-1}) \\ &= g_*^{-1}k_*^{-1}\tau(k) + g_*^{-1}\tau(g\varphi_i g^{-1}) + (\varphi_i)_* g_*^{-1}\tau(k^{-1}), \end{aligned}$$

where the last equality follows from the formula for the torsion of a composition (see [6, p. 72]). The same formula gives us $\tau(k) + k_*\tau(k^{-1}) = 0$. Substituting this into the above equation gives us

$$h_*^{-1}\tau(h\varphi_i h^{-1}) = g_*^{-1}\tau(g\varphi_i g^{-1}) - (Id - (\varphi_i)_*)g_*^{-1}\tau(k^{-1}).$$

$$(h_*^{-1}\tau(h\varphi_1 h^{-1}), \ldots, h_*\tau(h\varphi_n h^{-1})) - (g_*^{-1}\tau(g\varphi_1 g^{-1}), \ldots, g_*^{-1}\tau(g\varphi_n g^{-1}))$$

lies in Image (θ). ■

We will need the following classification result.

Lemma 5.2: *Let $\mathscr{E} \to B$ and $\mathscr{E}' \to B$ be Hurewicz fibrations of the type described at the beginning of this section, with characteristic maps $\varphi_i: \mathscr{E}_{b_0} \to \mathscr{E}_{b_0}$ and $\varphi_i': \mathscr{E}'_{b_0} \to \mathscr{E}'_{b_0}$. Then a h.e. $h: \mathscr{E}_{b_0} \to \mathscr{E}'_{b_0}$ extends to a f.h.e. of $\mathscr{E}$ onto $\mathscr{E}'$ iff h homotopy commutes with all of the characteristic maps, i.e. $\varphi_i' h \simeq h\varphi_i$ for each i.*

PROOF: This follows immediately from Theorem C of [17]. ■

PROOF OF THEOREM 4: The proof naturally splits into two parts.

I. *Existence.* First assume that $\mathscr{E}$ is *f.h.e.* to a fiber bundle $E \to B$ with fiber a compact Q-manifold. Let $\{\psi_i\}_{i=1}^n$ be the characteristic maps of $E \to B$. If $f: \mathscr{E} \to E$ is a *f.h.e.* and $h = f \mid \mathscr{E}_{b_0}: \mathscr{E}_{b_0} \to E_{b_0}$, then by Lemma 5.2 we have $\psi_i h \simeq h\varphi_i$, for each i. Up to simple homotopy type we may regard E_{b_0} as a finite complex, so in order to prove that $\mathcal{O}_1(\mathscr{E}) = 0$ it will certainly suffice to prove that $\tau(\psi_i) = 0$. Since $E \to B$ is a fiber bundle its characteristic maps may be chosen to be homeomorphisms. But homeomorphisms of Q-manifolds are always simple equivalences.

On the other hand assume that $\mathcal{O}_1(\mathscr{E}) = 0$. Then there is a compact Q-manifold N and a *h.e.* $h: \mathscr{E}_{b_0} \to N$ such that

$$\theta(\tau) = (h_*^{-1}\tau(h\varphi_1 h^{-1}), \ldots, h_*^{-1}\tau(h\varphi_n h^{-1})),$$

for some torsion $\tau \in \mathrm{Wh}\,\pi_1(\mathscr{E}_{b_0})$. Thus $(Id - (\varphi_i)_*)\tau = h_*^{-1}\tau(h\varphi_i h^{-1})$. Choose a compact Q-manifold M and a *h.e.* $f: N \to M$ such that $\tau(f) = -f_* h_*(\tau)$. Then we calculate (again using the composition formula):

$$\begin{aligned}\tau(fh\varphi_i(fh)^{-1}) &= \tau(f) + f_*\tau(h\varphi_i h^{-1}) + f_*h_*(\varphi_i)_*h_*^{-1}\tau(f^{-1})\\ &= -f_*h_*(\tau) + f_*h_*(Id - (\varphi_i)_*)\tau + f_*h_*(\varphi_i)_*h_*^{-1}f_*^{-1}(f_*h_*(\tau))\\ &= 0.\end{aligned}$$

Let $\psi_i = fh\varphi_i(fh)^{-1}: M \to M$ and let $g_i: M \to M$ be any homeomorphism homotopic to ψ_i (which exists since ψ_i has zero torsion).

Let B' be a wedge of n 1-spheres and let b_0' be the wedge point. For each i let $T(g_i)$ be the mapping torus of g_i and let E' be the space formed by sewing the $T(g_i)$ together along their common base, M. Then we have a natural projection $p': E' \to B'$ so that

(1) $E' \to B'$ is a fiber bundle with fiber M,

(2) $E'_{b_0'}$ is the common base of the $T(g_i)$,

(3) the characteristic maps of $E' \to B'$ are $\{g_i\}_{i=1}^n$ (corresponding to loops α_i' in B').

Let $u: B \to B'$ be a *h.e.* such that $u(b_0) = b_0'$ and $u\alpha_i \simeq \alpha_i'$, for each i. Form the pull-back, $E = \{(b, e') \mid u(b) = p'(e)\}$:

$$\begin{array}{ccc} E & \longrightarrow & E' \\ p\downarrow & & \downarrow p' \\ B & \xrightarrow{u} & B' \end{array}$$

Then $p: E \to B$ is a fiber bundle with fiber M. Since $g_i \simeq fh\varphi_i(fh)^{-1}$ and since the g_i are the characteristic maps of $E \to B$ we conclude by Lemma 5.2 that $\mathscr{E}$ is *f.h.e.* to E. ■

II. *Classification.* Define G to be the subgroup of Wh $\pi_1(\mathscr{E}_{b_0})$ consisting of all elements τ such that $(n-1)\tau = (Id-(\varphi_1)_*)\tau_1 + \cdots + (Id-(\varphi_n)_*)\tau_n$, for torsions $\tau_i \in \text{Wh}\ \pi_1(\mathscr{E}_{b_0})$. We prove that the simple equivalence classes of compact Q-manifold fiber bundles over B which are *f.h.e.* to $\mathscr{E}$ are in 1–1 correspondence with the quotient group $H = \text{Kernel}\ (\theta)/(\text{Kernel}\ (\theta) \cap G)$, where two Q-manifold fiber bundles, $E_1 \to B$ and $E_2 \to B$, are in the same *simple equivalence class* if there exists a simple homotopy equivalence from E_1 to E_2 which is also a *f.h.e.* Choose a fixed compact Q-manifold fiber bundle $E \to B$ and a *f.h.e.* $f: \mathscr{E} \to E$. Choose any other compact Q-manifold fiber bundle $E_1 \to B$ and *f.h.e.* $f_1: \mathscr{E} \to E_1$. Put $h = f \mid \mathscr{E}_{b_0}$ and $h_1 = f_1 \mid \mathscr{E}_{b_0}$. Then we get a *h.e.* $hh_1^{-1}: (E_1)_{b_0} \to E_{b_0}$ and a torsion $\tau(hh_1^{-1}) \in \text{Wh}\ \pi_1(E_{b_0})$.

ASSERTION 1: $h_*^{-1}\tau(hh_1^{-1}) \in \text{Kernel}\ (\theta)$.

PROOF: It follows from Lemma 5.2 that $(hh_1^{-1})\psi_i^1 \simeq \psi_i(hh_1^{-1})$, for each i, where the ψ_i are the characteristic maps for $E \to B$ and the ψ_i^1 are the characteristic maps for $E_1 \to B$. Since $E \to B$ and $E_1 \to B$ are compact Q-manifold fiber bundles we must have $\tau(\psi_i) = \tau(\psi_i^1) = 0$. Thus

$$\tau(hh_1^{-1}) = \tau(hh_1^{-1}\psi_i^1) = \tau(\psi_i hh_1^{-1}) = (\psi_i)_*\tau(hh_1^{-1}),$$

or $(Id-(\psi_i)_*)\tau(hh_1^{-1}) = 0$. Since $h\varphi_i \simeq \psi_i h$ we can easily check that $Id-(\varphi_i)_*)h_*^{-1}\tau(hh_1^{-1}) = 0$. This proves Assertion 1.

We then define $R(h_1)$ to be the image of $h_*^{-1}\tau(hh_1^{-1})$ in H. Thus R is a function from the collection of *f.h.e.*'s $f_1: \mathscr{E} \to E_1$ to the group H. There are several properties of R which need to be established in order to finish the proof of Theorem 4.

ASSERTION 2: *R is onto.*

PROOF: Choose any $\tau \in \text{Kernel}\ (\theta)$. Thus $(Id-(\varphi_i)_*)\tau = 0$ for each i. Choose a *h.e.* g of E_{b_0} to a compact Q-manifold N such that $\tau(g) = -g_*h_*(\tau)$. (Recall that $g: E_{b_0} \to N$ can be chosen so that $\tau(g^{-1}) = -g_*^{-1}\tau(g) \in \text{Wh}\ \pi_1(E_{b_0})$ realizes any torsion in Wh $\pi_1(E_{b_0})$.) A simple torsion calculation gives us $\tau(gh\varphi_i(gh)^{-1}) = 0$. Just as in the proof of Theorem 4 (Part I) we can construct a compact Q-manifold fiber bundle $E_1 \to B$ such that $(E_1)_{b_0} = N$ and a *f.h.e.* $f_1: \mathscr{E} \to E_1$ such that $h_1 = f_1 \mid \mathscr{E}_{b_0} = gh$. Then $R(f_1)$ is the image of $h_*^{-1}\tau(hh_1^{-1})$ in H.

Computing, we have

$$h_*^{-1}\tau(hh_1^{-1}) = h_*^{-1}\tau(h(gh)^{-1}) = h_*^{-1}\tau(g^{-1}) = -h_*^{-1}g_*^{-1}\tau(g) = \tau.$$

This completes Assertion 2.

ASSERTION 3: *If* $f_1: \mathscr{E} \to E_1$ *and* $f_2: \mathscr{E} \to E_2$ *are f.h.e.'s of* $\mathscr{E}$ *to compact Q-manifold fiber bundles, then* $f_2f_1^{-1}: E_1 \to E_2$ *is a simple equivalence iff* $R(f_1) = R(f_2)$.

PROOF: Assume that $f_2f_1^{-1}$ is a simple equivalence. It follows from Theorem 2.5 that $0 = \tau(f_2f_1^{-1}) = j_*(1-n)\tau(h_2h_1^{-1})$, where j is the inclusion $(E_2)_{b_0} \hookrightarrow E_2$. Using Theorem 8 we have

$$(h_2)_*^{-1}(n-1)\tau(h_2h_1^{-1}) = (Id - (\varphi_1)_*)\tau_1 + \cdots + (Id - (\varphi_n)_*)\tau_n,$$

for torsions $\tau_i \in \mathrm{Wh}\,\pi_1(\mathscr{E}_{b_0})$. Thus $(h_2)_*^{-1}\tau(h_2h_1^{-1}) \in \mathrm{Kernel}\ (\theta) \cap G$. Computing, we have

$$\begin{aligned} h_*^{-1}\tau(hh_1^{-1}) - h_*^{-1}\tau(hh_2^{-1}) &= \tau(h_1^{-1}) - \tau(h_2^{-1}) = (h_2)_*^{-1}\tau(h_2) + \tau(h_1^{-1}) \\ &= (h_2)_*^{-1}\tau(h_2h_1^{-1}) \in \mathrm{Kernel}\ (\theta) \cap G. \end{aligned}$$

This proves that $R(h_1) = R(h_2)$.

On the other hand assume that $R(f_1) = R(f_2)$. From the above calculations we see that $(h_2)_*^{-1}\tau(h_2h_1^{-1}) \in \mathrm{Kernel}\ (\theta) \cap G$. This implies that there are torsions $\tau_1, \ldots, \tau_n \in \mathrm{Wh}\,\pi_1((E_2)_{b_0})$ such that

$$(n-1)\tau(h_2h_1^{-1}) = (Id - (\psi_1^2)_*)\tau_1 + \cdots + (Id - (\psi_n^2)_*)\tau_n,$$

where the ψ_i^2 are the characteristic maps for $E_2 \to B$. It follows from Theorem 2.5 that $\tau(f_2f_1^{-1}) = j_*(1-n)\tau(h_2h_1^{-1})$ and it follows from Theorem 9.1 that

$$j_*((Id - (\psi_1^2)_*)\tau_1 + \cdots + (Id - (\psi_n^2)_*)\tau_n) = 0. \quad \blacksquare$$

6. Proof of Theorem 5

We will need some general notation. Let $f: M \to B$ be the map given in the statement of Theorem 4. Let $p: \mathscr{E} \to B$ be the mapping path fibration of $f: M \to B$ which has fiber $\mathscr{F}(f) = \mathscr{E}_{b_0}$ and let $g: M \to \mathscr{E}$ be a *h.e.* such that $pg \simeq f$.

The First Obstruction. We define our first obstruction to be

$$\mathscr{O}_1(f) = \mathscr{O}_1(\mathscr{E}) \in \mathrm{Cokernel}\,(\theta),$$

where $\mathscr{O}_1(\mathscr{E})$ was defined in §5. Recall that $\mathscr{O}_1(f)$ vanishes iff $\mathscr{E}$ is *f.h.e.* to a compact Q-manifold fiber bundle.

PROOF OF THEOREM 5 (Part I). We show that the vanishing of $\mathcal{O}_1(f)$ is a necessary condition for f to fiber. Assume that $f \simeq f'$, where f' is the projection map of a compact Q-manifold fiber bundle. Then by Theorem 2.2 we must have g homotopic to a *f.h.e.* from the bundle f': $M \to B$ to the fibration $\mathscr{E} \to B$. Thus $\mathcal{O}_1(f) = 0$. ∎

The Second Obstruction. Assume that $\mathcal{O}_1(f) = 0$ and let h: $M \to E$ be a *h.e.* such that $qh \simeq f$, where q: $E \to B$ is a compact Q-manifold fiber bundle. Let i be the inclusion map $\mathscr{E}_{b_0} \hookrightarrow \mathscr{E}$ and define $\mathcal{O}_2(f)$ to be the image of the torsion $h_*^{-1}\tau(h)$ in Wh $\pi_1(M)/(1-n)g_*^{-1}i_*$ Kernel (θ).

LEMMA 6.1: *$\mathcal{O}_2(f)$ is well-defined.*

PROOF: Let h_1: $M \to E_1$ be an alternate choice for h. We must prove that

$$g_*h_*^{-1}\tau(h) - g_*(h_1)_*^{-1}\tau(h_1) \in (1-n)i_* \text{ Kernel } (\theta).$$

Using Theorem 2.2 we see that h_1h^{-1} is homotopic to a *f.h.e.* α: $E \to E_1$. Thus by Theorem 2.5 we calculate

$$\tau(h_1h^{-1}) = \tau(h_1) - (h_1)_*h_*^{-1}\tau(h) = (1-n)\tau,$$

where τ is the torsion of the *h.e.* $h_1h^{-1} \mid E_{b_0}$. It follows from the proof of Theorem 4 (Part II) that $(Id - (\psi_i)_*)\tau = 0$, for each i, where the ψ_i are the characteristic maps for $E_1 \to B$. So, multiplying both sides of the above equation by $g_*(h_1)_*^{-1}$ we get what we need. ∎

PROOF OF THEOREM 5 (Part II): Assume that $f \simeq f'$, where f': $M \to B$ is a compact Q-manifold fiber bundle. Since $\mathcal{O}_2(f)$ is well-defined we may choose $E = M$ and $h = id$. Clearly $\mathcal{O}_2(f) = 0$.

On the other hand assume that $\mathcal{O}_2(f) = 0$. This means that $h_*^{-1}\tau(h) = g_*^{-1}(1-n)i_*(\tau)$, for some $\tau \in$ Kernel (θ). We may write h as g_1g, where g_1: $\mathscr{E} \to E$ is a *f.h.e.* Choose a compact Q-manifold N and a *h.e.* α: $E_{b_0} \to N$ such that $\tau(\alpha) = -\alpha_*((g_1)_{b_0})_*(\tau)$. Calculating we get

$$\begin{aligned}\tau(\alpha\psi_i\alpha^{-1}) &= \tau(\alpha) + \alpha_*(\psi_i)_*\tau(\alpha^{-1}) \\ &= \tau(\alpha) - \alpha_*(\psi_i)_*\alpha_*^{-1}\tau(\alpha) \\ &= -\alpha_*((g_1)_{b_0})_*(\tau) + \alpha_*(\psi_i)_*\alpha_*^{-1}\alpha_*((g_1)_{b_0})_*(\tau) \\ &= -\alpha_*(Id - (\psi_i)_*)((g_1)_{b_0})_*(\tau),\end{aligned}$$

which is zero because $\tau \in$ Kernel (θ). (Recall that ψ_i is a characteristic map for $E \to B$, which must have 0 torsion because it can be chosen to be a homeomorphism.) Using the proof of Theorem 4 (Part

II) we can construct a compact Q-manifold fiber bundle $E_1 \to B$ such that $(E_1)_{b_0} = N$ and a *f.h.e.* $\tilde{\alpha}: E \to E_1$ extending α. Put $j: (E_1)_{b_0} \hookrightarrow E_1$ and calculate to get

$$\begin{aligned}\tau(\tilde{\alpha}g_1g) &= \tau(\tilde{\alpha}) + (\tilde{\alpha})_*\tau(g_1g)\\ &= j_*(1-n)\tau(\alpha) + (\tilde{\alpha})_*h_*g_*^{-1}(1-n)i_*(\tau)\\ &= -j_*(1-n)\alpha_*((g_1)_{b_0})_*(\tau) + (\tilde{\alpha})_*(g_1)_*(1-n)i_*(\tau),\end{aligned}$$

which is easily seen to be zero. Thus $\tilde{\alpha}g_1g: M \to E_1$ is homotopic to a homeomorphism which implies that f is homotopic to a compact Q-manifold fiber bundle projection. ■

7. Proof of Theorem 6

We first introduce some notation for this section. It follows from Theorem 2.3 that we may replace B by S^1. Let $p: \mathscr{E} \to S^1$ be the mapping path fibration of $f: M \to S^1$, where $\mathscr{F}(f) = \mathscr{E}_{b_0}$, and let $h: M \to \mathscr{E}$ be a fixed *h.e.* so that $ph \simeq f$.

We use $\varphi: \mathscr{F}(f) \to \mathscr{F}(f)$ for a characteristic map corresponding to a choice of a generator for $\pi_1(S^1)$.

The First Obstruction. The first obstruction is just the obstruction $\mathscr{O}_1(f)$ of Theorem 5. We must show that the group in which $\mathscr{O}_1(f)$ lies is isomorphic to a subgroup of Wh $\pi_1(M)$. This is the group

$$\text{Cokernel}\,(\theta) = \text{Wh}\ \pi_1\mathscr{F}(f)/(Id - \varphi_*)\ \text{Wh}\ \pi_1\mathscr{F}(f).$$

If i is the inclusion map $\mathscr{F}(f) \hookrightarrow \mathscr{E}$, then it is shown in Theorem 8 that Kernel $(i_*) = (Id - \varphi_*)\text{Wh}\ \pi_1\mathscr{F}(f)$. Thus Cokernel (θ) is isomorphic with a subgroup of Wh $\pi_1(\mathscr{E}) \cong$ Wh $\pi_1(M)$.

The Second Obstruction. We will need some more notation. Choose a finite complex K and a *h.e.* $g: \mathscr{F}(f) \to K$, and let $\psi: K \to K$ be the map $g\varphi g^{-1}$. Represent S^1 by $\{e^{2\pi it} \mid 0 \le t \le 1\}$, where $b_0 = 1$, and let $T(\psi) \to S^1$ be the natural map of the mapping torus to S^1. The fibers of $T(\psi) \to S^1$ are all naturally identified with K.

We leave it as a manageable exercise for the reader to construct a *h.e.* $\alpha: \mathscr{E} \to T(\psi)$ such that $\alpha \mid \mathscr{E}_{b_0} = g$, α takes $\mathscr{E} \mid \{e^{2\pi it} \mid \frac{1}{2} \le t \le 1\}$ to $T(\psi) \mid \{e^{2\pi it} \mid \frac{1}{2} \le t \le 1\}$, and α is *f.p.* over $\{e^{2\pi it} \mid 0 \le t \le \frac{1}{2}\}$. We then define our second obstruction to be

$$\mathscr{O}_2'(f) = h_*^{-1}\alpha_*^{-1}\tau(\alpha h) \in \text{Wh}\ \pi_1(M),$$

where $h: M \to \mathscr{E}$ is as chosen above.

LEMMA 7.1: *$\mathcal{O}_2'(f)$ is well-defined.*

PROOF: Let $g_1: \mathcal{F}(f) \to K_1$, $\psi_1 = g_1\varphi g_1^{-1}$, $\alpha_1: \mathcal{E} \to T(\psi_1)$ be alternate choices. We must prove that

$$h_*^{-1}\alpha_*^{-1}\tau(\alpha h) = h_*^{-1}(\alpha_1)_*^{-1}\tau(\alpha_1 h),$$

and for this it suffices to prove that $\tau(\alpha_1\alpha^{-1}) = 0$. (Just use the formula for the torsion of a composition.)

We may choose α^{-1} so that α^{-1} takes $T(\psi) \,|\, \{e^{2\pi i t} \,|\, \frac{1}{2} \le t \le 1\}$ to $\mathcal{E} \,|\, \{e^{2\pi i t} \,|\, \frac{1}{2} \le t \le 1\}$ and α^{-1} is *f.p.* over $(e^{2\pi i t} \,|\, 0 \le t \le \frac{1}{2}\}$. Write $T(\psi) = A \cup B$ and $T(\psi_1) = A_1 \cup B_1$, where

$$A = T(\psi) \,|\, \{e^{2\pi i t} \,|\, 0 \le t \le \tfrac{1}{2}\}, \quad A_1 = T(\psi_1) \,|\, \{e^{2\pi i t} \,|\, 0 \le t \le \tfrac{1}{2}\},$$
$$B = T(\psi) \,|\, \{e^{2\pi i t} \,|\, \tfrac{1}{2} \le t \le 1\}, \quad B_1 = T(\psi_1) \,|\, \{e^{2\pi i t} \,|\, \tfrac{1}{2} \le t \le 1\}.$$

Then $\alpha_1\alpha^{-1}$ restricts to give *h.e.*'s of A to A_1, B to B_1 and $A \cap B$ to $A_1 \cap B_1$. Using the Sum Theorem for torsion we have

$$\tau(\alpha_1\alpha^{-1}) = a\tau(\alpha_1\alpha^{-1} \,|\, A) + b\tau(\alpha_1\alpha^{-1} \,|\, B) - c\tau(\alpha_1\alpha^{-1} \,|\, A \cap B),$$

where a, b and c are inclusion-induced homomorphisms into $\mathrm{Wh}\,\pi_1 T(\psi_1)$. It is easy to see that $a\tau(\alpha_1\alpha^{-1} \,|\, A) = b\tau(\alpha_1\alpha^{-1} \,|\, B)$. Clearly $A \cap B = K' \cup K''$ (two disjoint copies of K) and $A_1 \cap B_1 = K_1' \cup K_1''$ (two disjoint copies of K_1). Computing torsions we get

$$\tau(\alpha_1\alpha^{-1} \,|\, A \cap B) = \tau(\alpha_1\alpha^{-1} \,|\, K') + \tau(\alpha_1\alpha^{-1} \,|\, K''),$$

where we have omitted the necessary inclusion-induced homomorphisms. It is easy to see that

$$c\tau(\alpha_1\alpha^{-1} \,|\, K') = c\tau(\alpha_1\alpha^{-1} \,|\, K'') = a\tau(\alpha_1\alpha^{-1} \,|\, A),$$

and therefore $\tau(\alpha_1\alpha^{-1}) = 0$ by the above formula. ■

PROOF OF THEOREM 6: We first assume that $f \simeq f'$, where $f': M \to S^1$ is the projection map of a compact Q-manifold fiber bundle. It follows from the proof of Theorem 5 (Part I) that $\mathcal{O}_1(f) = 0$. By Theorem 2.2 we have $h \simeq h': M \to \mathcal{E}$, where h' is a *f.h.e.* Since $\mathcal{O}_2'(f)$ is well-defined we may choose $\alpha: \mathcal{E} \to T(\psi) = M$ to be $(h')^{-1}: \mathcal{E} \to M$, where ψ is a characteristic homeomorphism of the bundle $f: M \to S^1$. Then $\tau(\alpha h) = 0$ and consequently $\mathcal{O}_2'(f) = 0$.

On the other hand assume that $\mathcal{O}_1(f) = 0$ and $\mathcal{O}_2'(f) = 0$. Since $\mathcal{O}_1(f) = 0$ we have a *f.h.e.* $\alpha: \mathcal{E} \to E$, where $E \to S^1$ is a compact Q-manifold fiber bundle. In the definition of $\mathcal{O}_2'(f)$ we may take $T(\psi) = E$. Then $\mathcal{O}_2'(f) = 0$ implies that we have $\tau(\alpha h) = 0$. Thus αh is homotopic to a homeomorphism. ■

8. Proof of Theorem 7

We will first need some preliminary results on homotopies. Our main result is Corollary 8.3.

LEMMA 8.1: *With M and B as in the statement of Theorem 7, let $F: M \times I \to B$ be a map such that $F_0 = F_1$. Then $F \simeq G$ rel $M \times \{0, 1\}$, where $G: M \times I \to B$ is of the form $G(m, t) = r_t F_0(m)$, for some homotopy $r: B \times I \to B$ satisfying $r_0 = r_1 = Id$.*

PROOF: Let $\Delta \subset B^I$ be the set of maps $\alpha: I \to B$ such that $\alpha(0) = \alpha(1)$. There is a natural map $p: \Delta \to B$ given by $p(\alpha) = \alpha(0)$. This map is a fibration. The fiber is a disjoint union of contractible open subsets (B is a $K(\pi, 1)$ and the fiber is ΩB.)

Let $\bar{\Delta}$ be the space obtained from Δ by identifying $\alpha \sim \alpha'$ iff α is homotopic to α' rel $\{0, 1\}$. Certainly $\bar{\Delta}$ is a covering space of B where the components of $\bar{\Delta}$ correspond to free homotopy classes of loops and the sheets in a component correspond to π_1 acting on based loops.

There is a natural map (the quotient) $q: \Delta \to \bar{\Delta}$ covering the identity on B. This map takes components in the fiber of Δ to points in the fiber of $\bar{\Delta}$ in a 1–1 fashion. By Theorem 2.1, q is a *f.h.e.* and has a fiber homotopy inverse, $q_1: \bar{\Delta} \to \Delta$. We can therefore find a *f.p.* deformation retraction $s: \Delta \times I \to \Delta$ such that $s_0 = Id$ and $s_1(\Delta) = q_1(\bar{\Delta})$.

Each $m \in M$ determines a loop in B by $m \to F_t(m)$, $0 \le t \le 1$. This defines a map $k: M \to \Delta$ such that $F_t(m) = k(m)(t)$. Define $\bar{G}: M \times I \to \Delta$ by $\bar{G}_u(m) = s_u k(m)$. Then $\bar{G}_0(m)[t] = F_t(m)$, $\bar{G}_u(m)[0] = \bar{G}_u(m)[1] = f(m)$ and $\bar{G}_1(m)$ is a path depending only on $f(m)$. Defining $G_t(m) = \bar{G}_1(m)[t]$ we have a homotopy from F_0 to F_1. Because $G_t(m)$ depends only on $f(m)$, we can write $G_t(m) = r_t F_0(m)$, for some $r: B \times I \to B$ satisfying $r_0 = r_1 = Id$. ■

REMARK: The above result is true (with the same proof) for B any $K(\pi, 1)$.

LEMMA 8.2: *Let us choose B as in Theorem 7 and let $r: B \times I \to B$ be a homotopy such that $r_0 = r_1 = Id$.*

(1) *If $n \ge 2$, then r is homotopic to the constant identity homotopy* rel $B \times \{0, 1\}$.

(2) *If $n = 1$, then r is homotopic* (rel $B \times \{0, 1\}$) *to a "standard rotation."*

PROOF: Let $\tilde{B}$ be the universal cover of B and cover r by $\tilde{r}$: $\tilde{B} \times I \to \tilde{B}$ so that $\tilde{r}_0 = Id$. $\tilde{r}_1$ is a deck transformation properly homotopic to Id. It is therefore the identity if $n \geq 2$. Thus, all loops $r_t(b)$, $0 \leq t \leq 1$, are null-homotopic for $n \geq 2$. The component of $\bar{\Delta}$ containing the null-homotopic loops covers B trivially. The cover $\bar{\Delta}$ consists of disjoint trivial sheets for $n = 1$. Thus an argument similar to Lemma 8.1 homotopes r to a constant for $n \geq 2$ and to a "standard rotation" for $n = 1$. (If $B = S^1$, a "standard rotation" is a rotation through an integral multiple of 360°. For $B \simeq S^1$, the homotopy equivalence defines a standard rotation.) ■

COROLLARY 8.3 *Let us choose M, B as in Theorem 7 and let g_1, g_2: $M \to B$ be homotopic maps. Then any two homotopies from g_1 to g_2*

(1) *are homotopic* (rel *g_1 and g_2) for $n \geq 2$, and*

(2) *differ by a "standard rotation" of B for $n = 1$.*

The First Obstruction. For convenience we will henceforth refer to the fiber bundle f_1: $M \to B$ as f_1: $M_1 \to B$. By Theorem 2.2 we see that Id: $M_1 \to M$ is homotopic to a *f.h.e.* g: $M_1 \to M$. Choose $b_0 \in B$ so that $F = M_{b_0}$. The first obstruction is $\mathcal{P}_1(f_1) = \tau(g_{b_0}) \in \mathrm{Wh}\, \pi_1(F)$, where g_{b_0}: $(M_1)_{b_0} \to F$.

LEMMA 8.4: *$\mathcal{P}_1(f_1)$ is well defined.*

PROOF: Let g': $M_1 \to M$ be another *f.h.e.* homotopic to Id. Both g and g' are obtained by lifting homotopies from f_1 to f. Thus g and g_1 depend only on the homotopy class (rel f_1 and f) of the homotopy from f_1 to f. If $n \geq 2$ we conclude by Corollary 8.3 that $g' \simeq g$ and therefore $\tau(g_{b_0}) = \tau(g'_{b_0})$. For $n = 1$ choose a characteristic map φ: $F \to F$ which is a homeomorphism. By Corollary 8.3 we have $g'_{b_0} = \varphi^k g_{b_0}$, for some $k \geq 0$. Computing we get

$$\tau(g'_{b_0}) = \tau(\varphi^k) + (\varphi^k)_* \tau(g_{b_0}) = (\varphi^k)_* \tau(g_{b_0}).$$

We showed in the proof of Theorem 4 (Part II) that $(Id - \varphi_*)\tau(g_{b_0}) = 0$. Thus $\tau(g'_{b_0}) = \tau(g_{b_0})$. ■

The Second Obstruction. Assume that $\mathcal{P}_1(f_1) = 0$. We have $\tau(g_{b_0}) = 0$ and therefore g_{b_0}: $(M_1)_{b_0} \to F$ is homotopic to a homeomorphism g_1: $(M_1)_{b_0} \to F$. Choose characteristic maps φ_i: $F \to F$, $1 \leq i \leq n$, where each φ_i is a homeomorphism. Similarly, choose characteristic maps

$\psi_i\colon F_1 \to F_1$, where $F_1 = (M_1)_{b_0}$. Define $\theta\colon \mathcal{P}(F) \to \mathcal{P}(F) \oplus \cdots \oplus \mathcal{P}(F)$ by

$$\theta(h) = (\varphi_1^{-1} h \varphi_1 h^{-1}, \ldots, \varphi_n^{-1} h \varphi_n h^{-1}).$$

It is easy to check that θ is a homomorphism since $\mathcal{P}(F)$ is abelian. We define $\mathcal{P}_2(f_1) \in \text{Cokernel}(\theta)$ to be the image of $(\varphi_1^{-1} g_1 \psi_1 g_1^{-1}, \ldots, \varphi_n^{-1} g_1 \psi_n g_1^{-1})$ In Cokernel (θ).

LEMMA 4: *$\mathcal{P}_2(f_1)$ is well-defined.*

PROOF: First assume that $n \geq 2$. Then all we have to do is show that if $g_2\colon (M_1)_{b_0} \to F$ is another homeomorphism homotopic to $g_{b_0}\colon (M_1)_{b_0} \to F$, then $\alpha = (\varphi_1^{-1} g_1 \psi_1 g_1^{-1}, \ldots, \varphi_n^{-1} g_1 \psi_n g_1^{-1})$ and $\beta = (\varphi_1^{-1} g_2 \psi_1 g_2^{-1}, \ldots, \varphi_n^{-1} g_2 \psi_n g_2^{-1})$ have the same image in Cokernel (θ). Since $\mathcal{P}(F)$ is abelian it is easy to see that

$$\varphi_i^{-1} g_2 \psi_i g_2^{-1} = (\varphi_i^{-1}(g_2 g_1^{-1}) \varphi_i (g_2 g_1^{-1})^{-1})(\varphi_i^{-1} g_1 \psi_i g_1^{-1}),$$

which implies that $\beta\alpha^{-1} = \theta(g_2 g_1^{-1})$.

For $n = 1$ let $g_2\colon (M_1)_{b_0} \to F$ be any homeomorphism homotopic to $\varphi^k g_{b_0}$. Then we must show that $\varphi^{-1} g_1 \psi g_1^{-1}$ and $\varphi^{-1} g_2 \psi g_2^{-1}$ have the same image in Cokernel (θ). We have just shown above that $\varphi^{-1} g_2 \psi g_2^{-1}$ and $\varphi^{-1}(\varphi^k g_1)\psi(\varphi^k g_1)^{-1}$ have the same image. But

$$\varphi^{-1}(\varphi^k g_1)\psi(\varphi^k g_1)^{-1} = \varphi^k(\varphi^{-1} g_1 \psi g_1^{-1})\varphi^{-k},$$

and therefore

$$(\varphi^{-1}(\varphi^k g_1)\psi(\varphi^k g_1)^{-1})(\varphi^{-1} g_1 \psi g_1^{-1})^{-1} = \varphi^k(\varphi^{-1} g_1 \psi g_1^{-1})\varphi^{-k}(\varphi^{-1} g_1 \psi g_1^{-1})^{-1}.$$

So it remains to be shown that any element of the form $\varphi^k h \varphi^{-k} h^{-1}$ lies in Image (θ), for $h \in \mathcal{P}(F)$. But this follows from interated use of the formula

$$\varphi^k h \varphi^{-k} h^{-1} = [\varphi(\varphi^{k-1} h \varphi^{-(k-1)})\varphi^{-1}(\varphi^{k-1} h \varphi^{-(k-1)})^{-1}][\varphi^{k-1} h \varphi^{-(k-1)} h^{-1}]. \quad \blacksquare$$

PROOF OF THEOREM 7: First assume that there is a *f.p.* homeomorphism $h\colon M_1 \to M$ such that $h \simeq Id$. Then $g_1 = h \mid (M_1)_{b_0}\colon (M_1)_{b_0} \to F$ is a homeomorphism and $\tau(h \mid (M_1)_{b_0}) = 0$. This proves that $\mathcal{P}_1(f_1) = 0$. For the second obstruction it can easily be argued from the existence of h that $g_1 \psi g_1^{-1}$ is isotopic to φ_i, for $1 \leq i \leq n$. (Or we can refer to [7].) Therefore $\mathcal{P}_2(f_1) = 0$.

On the other hand assume that $\mathcal{P}_1(f_1) = 0$ and $\mathcal{P}_2(f_2) = 0$. Now $\mathcal{P}_1(f_1) = 0$ implies that there is a homeomorphism $g_1\colon (M_1)_{b_0} \to F$ which is homotopic to $g \mid (M_1)_{b_0}\colon (M_1)_{b_0} \to F$, where $g\colon M_1 \to M$ is a *f.h.e.*

homotopic to Id. Now $\mathscr{P}_2(f_1) = 0$ implies that

$$(\varphi_1^{-1}g_1\psi_1g_1^{-1}, \ldots, \varphi_n^{-1}g_1\psi_ng_1^{-1}) = \theta(\alpha),$$

for some $\alpha \in \mathscr{P}(F)$. Thus $\varphi_i^{-1}g_1\psi_ig_1^{-1}$ is isotopic to $\varphi_i^{-1}\alpha\varphi_i\alpha^{-1}$, which implies that $(\alpha^{-1}g_1)\psi_i(\alpha^{-1}g_1)^{-1}$ is isotopic to φ_i, for each i. By [7] this implies that $\alpha^{-1}g_1$ extends to a *f.p.* homeomorphism of M_1 onto M. ■

9. Computation of a Kernel

Our main result is Theorem 8. We will first need the general construction of Lemma 9.1 below. For notation let $X \xrightarrow{f} B$ be a map and let $\tilde{B} \xrightarrow{p} B$ be a covering space. Form the pull-back,

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{B} \\ q\downarrow & & \downarrow p \\ X & \xrightarrow{f} & B, \end{array}$$

where $\tilde{X} = \{(x, e) \mid f(x) = p(e)\}$. Each deck transformation $\varphi: \tilde{B} \to \tilde{B}$ induces a deck transformation $\tilde{\varphi}: \tilde{X} \to \tilde{X}$ defined by $\tilde{\varphi}(x, e) = (x, \varphi(e))$.

LEMMA 9.1: *Let $X_1 \xrightarrow{f_1} B$ and $X_2 \xrightarrow{f_2} B$ be maps, $\tilde{B} \xrightarrow{p} B$ be a covering space, and let $h: X_1 \to X_2$ be a homeomorphism such that $f_2h \simeq f_1$. If the pull-back $\tilde{X}_1$ is connected, then there exists a homeomorphism $\tilde{h}: \tilde{X}_1 \to \tilde{X}_2$ such that $\tilde{h}$ covers h and $\tilde{h}$ commutes with the deck transformations of $\tilde{X}_1$ and $\tilde{X}_2$ which are induced by the deck transformations of $\tilde{B}$.*

PROOF: Since $f_2h \simeq f_1$ there is a homotopy $F: \tilde{X}_1 \times I \to B$ so that F_0 is the composition $\tilde{X}_1 \xrightarrow{q_1} X_1 \xrightarrow{f_1} B$ and F_1 is the composition $\tilde{X}_1 \xrightarrow{q_1} X_1 \xrightarrow{h} X_2 \xrightarrow{f_2} B$. Note that F_0 can be lifted to $\tilde{X}_1 \xrightarrow{\tilde{f}_1} \tilde{B}$. Therefore $F: \tilde{X}_1 \times I \to B$ can be lifted to $\tilde{F}: \tilde{X}_1 \times I \to \tilde{B}$ so that $\tilde{F}_0 = \tilde{f}_1$. This induces a map $\tilde{h}: \tilde{X}_1 \to \tilde{X}_2$ defined by $\tilde{h}(x, e) = (h(x), \tilde{F}_1(x, e))$. We leave it as an exercise for the reader to check that $\tilde{h}$ fulfills our requirements. ■

LEMMA 9.2: *Let K be a finite complex and let $\varphi: K \to K$ be a homotopy equivalence. If $T(\varphi)$ is the mapping torus of φ and i is the natural inclusion $K \hookrightarrow T(\varphi)$, then $i_*(Id - \varphi_*) = 0$, where i_* and φ_* are the induced homomorphisms on the Whitehead groups of K and $T(\varphi)$.*

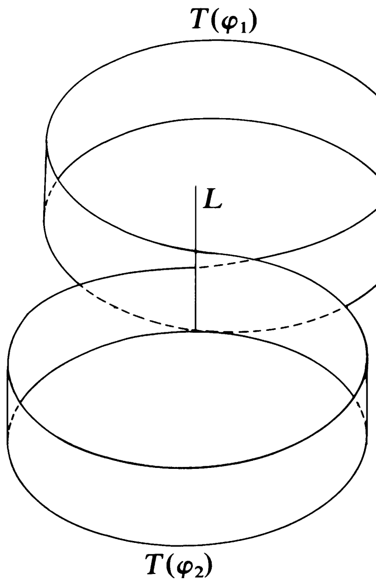
PROOF: Choose any torsion $\tau \in \mathrm{Wh}\, \pi_1(K)$. We must prove that $i_*(\tau) = i_*\varphi_*(\tau)$. By [6] we may represent τ by a pair $[L, K]$, where L is a finite complex containing K as a deformation retract. This means that $\tau = \tau(f)$, where $f: L \to K$ is any deformation retraction. It then follows that $\varphi_*\tau(f)$ may be represented by $[L \cup_\varphi K, K]$ (we assume that φ is a PL map). Applying i_* we observe that $i_*\tau(f)$ may be represented by $[L \cup T(\varphi), T(\varphi)]$ and $i_*\varphi_*\tau(f)$ may be represented by $[L \cup_\varphi T(\varphi), T(\varphi)]$. But $if \simeq i\varphi f$, and this implies that $[L \cup T(\varphi), T(\varphi)]$ and $[L \cup_\varphi T(\varphi), T(\varphi)]$ represent the same torsion in $\mathrm{Wh}\, \pi_1(T(\varphi))$. ■

LEMMA 9.3: *Let K be a finite connected complex and let $\varphi_i: K \to K$ be a homotopy equivalence, for $1 \leq i \leq n$. Define X to be the space formed by sewing the mapping tori $T(\varphi_i)$ together along $K \equiv K \times \{0\} \equiv K \times \{1\}$ in $T(\varphi_i)$. Then the kernel of the inclusion-induced map $i_*: \mathrm{Wh}\, \pi_1(K) \to \mathrm{Wh}\, \pi_1(X)$ is*

$$G = \{\tau \in \mathrm{Wh}\, \pi_1(K) \mid \tau = (Id - (\varphi_1)_*)\tau_1 + \cdots + (Id - (\varphi_n)_*\tau_n\}.$$

PROOF: It follows from Lemma 9.2 that each element of G lies in the kernel of i_*. For the other half we will assume $n = 2$. The other cases can be treated similarly.

Choose any torsion $\tau \in \mathrm{Wh}\, \pi_1(K)$ for which $i_*(\tau) = 0$. As in Lemma 9.2 we may represent τ by a pair $[L, K]$. The condition $i_*(\tau) = 0$ implies that the inclusion $X \hookrightarrow X \cup L$ is simple. Multiplying by Q and



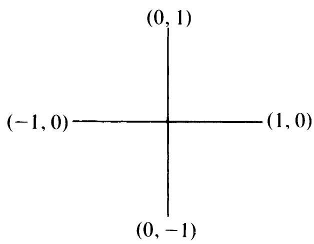
applying [3, Theorem 29.4] there is a homeomorphism $h: X \times Q \to (X \cup L) \times Q$ which is homotopic to the inclusion. Using Z-set unknotting we may assume that $h \mid X \times \{0\} = id$. There is a natural map $f: X \to B = S_1^1 \cup S_2^1$ so that K is sent to the wedge point of B and $T(\varphi_i)$ is wrapped once around S_i^1. We choose notation so that $f^{-1}(b)$ is a copy of K, for each $b \in B$, and passing down the "rays" of $T(\varphi_i)$ covers a path wrapping counterclockwise around S_i^1. That is, in the representation $T(\varphi_i) = K \times [0, 1]/\sim$, passing from 0 to 1 corresponds to going counterclockwise around S^1. Let $X_1 = X \cup L$ and define $f_1: X_1 \to B$ by the composition $X_1 \longrightarrow X \xrightarrow{f} B$, where the first map is obtained by taking a deformation retraction of L onto K. Above is a picture of X_1, where L is represented by a segment added to $K = T(\varphi_1) \cap T(\varphi_2)$.

Form the pull-backs as in Lemma 9.1,

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{B} \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{f}_1} & \tilde{B} \\ \downarrow & & \downarrow p \\ X_1 & \xrightarrow{f_1} & B, \end{array}$$

where $\tilde{B}$ is the universal covering space of B. The homeomorphism h lifts to a homeomorphism $\tilde{h}: \tilde{X} \times Q \to \tilde{X}_1 \times Q$ for which $\tilde{h} \mid \tilde{X} \times \{0\} = id$ and $\tilde{h}$ commutes with the deck transformations of $\tilde{X} \times Q$ and $\tilde{X}_1 \times Q$ which are induced by the deck transformations of $\tilde{B}$.

$\tilde{B}$ is a 1-complex such that p takes each vertex to the wedge point of B and p wraps each 1-simplex once around S_1^1 or S_2^1. Let A_1 be the following subset of the plane.



We may identify A_1 with a subcomplex of $\tilde{B}$ so that p wraps the horizontal 1-simplexes in A_1 around S_1^1 and the vertical 1-simplexes around S_2^1. Choose notation so that the positive directions on A_1 correspond to the clockwise directions on S_1^1 and S_2^1. Let T_1 be the deck transformation of $\tilde{B}$ taking $(0,0)$ to $(1,0)$ and let T_2 be the deck transformation taking $(0,0)$ to $(0,1)$.

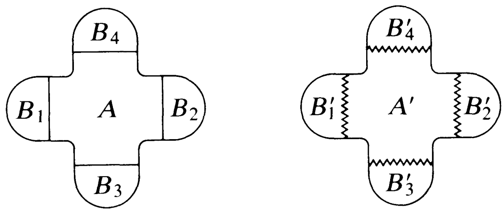
Let

$$A_{1/2} = ([-\tfrac{1}{2}, \tfrac{1}{2}] \times \{0\}) \cup (\{0\} \times [-\tfrac{1}{2}, \tfrac{1}{2}]) \subset A_1$$

and choose a finite connected subcomplex A_n of $\tilde{B}$ so large that

$$A' = \tilde{h}^{-1}(\tilde{f}_1^{-1}(A_{1/2}) \times Q) \subset \operatorname{Int} \tilde{f}^{-1}(A_n) \times Q.$$

Let $A = \tilde{f}^{-1}(A_{1/2}) \times Q$. Then A and A' divide $\tilde{f}^{-1}(A_n) \times Q$ into components as pictured.



$\tilde{f}^{-1}(A_n) \times Q$ $\tilde{f}^{-1}(A_n) \times Q$

The components are named so that $B_i \cap (\tilde{X} \times \{0\}) = B_i' \cap (\tilde{X} \times \{0\})$, $A \cap B_1 = \tilde{f}^{-1}(\{-\tfrac{1}{2}, 0)\}) \times Q$, $A \cap B_2 = \tilde{f}^{-1}(\{(\tfrac{1}{2}, 0)\}) \times Q$, $A \cap B_3 = \tilde{f}^{-1}(\{(0, -\tfrac{1}{2})\}) \times Q$, and $A \cap B_4 = \tilde{f}^{-1}(\{(0, \tfrac{1}{2})\}) \times Q$. Additionally, define $K_i = A \cap B_i \cap (\tilde{X} \times \{0\}))$ and note that each K_i has a standard identification with K. We observe that the pair $[A, K_1]$ represents the 0 torsion of $\operatorname{Wh} \pi_1(K)$ and $[A', K_1]$ represents the given torsion $\tau \in \operatorname{Wh} \pi_1(K)$.

An easy torsion calculation gives us

$$(*)[f^{-1}(A_n) \times Q, K_1] = [B_1', K] + (A', K] + (\varphi_1)_*[B_2', K] + [B_3', K] + (\varphi_2)_*[B_4', K].$$

Let $S_i\colon \tilde{X} \times Q \to \tilde{X} \times Q$ be the deck transformation induced by T_i. Since $\tilde{h}$ commutes with the induced deck transformations we observe that

$$B_1' \cup S_1^{-1}(B_2') = B_1 \cup S_1^{-1}(B_2),$$
$$B_3' \cup S_2^{-1}(B_4') = B_3 \cup S_2^{-1}(B_4).$$

Thus

$$[B_1 \cup S_1^{-1}(B_2), K] = [B_1', K] + [S_1^{-1}(B_2'), K],$$
$$[B_3 \cup S_2^{-1}(B_4), K] = [B_3', K] + [S_2^{-1}(B_4'), K].$$

It is easy to see that $[S_1^{-1}(B_2'), K] = [B_2', K]$ and $[S_2^{-1}(B_4'), K] = [B_4', K]$. Substituting all this in (*) above we get

$$(**)[\tilde{f}^{-1}(A_n) \times Q, K_1] - [B_1 \cup S_1^{-1}(B_2), K] - [B_3 \cup S_2^{-1}(B_4), K] = ((\varphi_1)_* - Id)[B_2', K] + ((\varphi_2)_* - Id)[B_4', K] + [A', K_1].$$

We now compute the left-hand side of (**). Note that

$$[B_1 \cup S_1^{-1}(B_2), K] = [B_1, K] + [B_2, K],$$
$$[B_3 \cup S_2^{-1}(B_4), K] = [B_3, K] + [B_4, K],$$
$$[\tilde{f}^{-1}(A_n) \times Q, K_1] = [B_1, K] + (\varphi_1)_*[B_2, K] + [B_3, K] + (\varphi_2)_*[B_4, K].$$

Substituting this into (**) above we get

$$((\varphi_1)_* - Id)[B_2, K] + ((\varphi_2)_* - Id)[B_4, K] = ((\varphi_1)_* - Id)[B_2', K] + ((\varphi_2)_* - Id)[B_4', K] + [A', K_1].$$

This is all we need. ■

THEOREM 8: *Let $\mathscr{E} \to B$ be a Hurewicz fibration, where B is h.e. to a wedge of n 1-spheres and the fiber $F = \mathscr{E}_{b_0}$ is h.e. to a finite connected complex. If i is the inclusion map $F \hookrightarrow \mathscr{E}$ and $\{\varphi_i\}_{i=1}^n$ is the collection of characteristic maps $\varphi_i: F \to F$, then the kernel of $i_*: \mathrm{Wh}\, \pi_1(F) \to \mathrm{Wh}\, \pi_1(\mathscr{E})$ is*

$$\{\tau \in \mathrm{Wh}\, \pi_1(F) \mid \tau = (Id - (\varphi_1)_*)\tau_1 + \cdots + (Id - (\varphi_n)_*)\tau_n\}.$$

PROOF: By taking a *h.e.* of a wedge of n 1-spheres to B and forming the pull-back, we may assume that B is a wedge of n 1-spheres, $B = S_1^1 \cup \cdots \cup S_n^1$. Choose $b_0 \in B$ to be the wedge point and let $\varphi_i: F \to F$ be the characteristic maps. Let $\alpha: \mathscr{E}_{b_0} \to K$ be a *h.e.* of $\mathscr{E}_{b_0}$ to a finite complex. Define $\psi_i = \alpha\varphi_i\alpha^{-1}: K \to K$ and form the space $X \to B$ of Lemma 9.3. We leave it as a manageable exercise for the reader to construct a *h.e.* $\beta: \mathscr{E} \to X$ such that

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\beta} & X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ \mathscr{E}_{b_0} & \longrightarrow & X_{b_0} \end{array}$$

homotopy commutes. Then Kernel (i_*) = Kernel $(j_*\alpha_*)$ and all we need is Lemma 9.3. ■

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