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THE DECOMPOSITION AND SPECIALIZATION OF ALGEBRAIC FAMILIES OF VECTOR BUNDLES

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Introduction

We consider vector bundles, by which we mean torsion free, coherent sheaves on a nonsingular projective variety X. These vector bundles subsume the "classical" vector bundles (locally free sheaves), but they are themselves locally free over large open sets. We will show that many of the pleasant properties of bundles over curves are in fact true for these sheaves over X of any dimension. In particular, we will obtain the canonical decomposition of bundles by flags whose factors are stable in Mumford's and Takemoto's sense [14, 16]. This was first done for curves by Harder and Narasimhan [10], so we call these flags Harder-Narasimhan Flags (HNF's).

The HNF's lead to certain convex polygons, and we investigate the behavior of these flags and polygons when a bundle moves in an algebraic family over X. It turns out that there is a basic semi-continuity theorem which states that the polygons rise under specialization. This is applied to construct a map from the set of algebraic families of bundles on X, parametrized by a Noetherian scheme S, to the semi-group of non-negative algebraic cycles on S. The fibres of this map turn out to be the equivalence classes of families of bundles under the identification: two families of bundles are equivalent when their associated polygons on each fibre of Xx_kS over S agree.

I have been informed that Tjurin [17] constructed HNF's for the case of curves, and that he also mentioned the elementary properties (A), (B), (C) of §2. However, in his work, the emphasis is not on the

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flags constructed, nor did he emphasize the controlling effect of the notion of slope.

§1. Generalities on vector bundles

By a vector bundle on a locally Noetherian scheme X, we mean a torsion-free coherent sheaf on X. We shall always assume that X has property R_1 of Serre [4], that is, for every point $x \in X$ of codim $\leqq 1$, the local ring $\mathcal{O}_{X,x}$ is regular. It follows from this assumption that there exists a non-empty open set, U, containing all points of codim $\leqq 1$ such that our vector bundle is actually a locally free sheaf on U (i.e., a "classical" vector bundle on U). The reasons for using this more general notion of vector bundle are amply explained in Langton's article [12]; not the least of these reasons being that every vector bundle in the above sense has a complete flag. (Of course, a sub-bundle is a subsheaf such that the associated quotient sheaf is again a bundle.) Moreover, Gieseker has recently settled some moduli problems using this notion of bundle, [2].

Actually, some of our results are valid in a more general situation; and, in fact, are best stated in this generality. We consider the full subcategory of the coherent sheaves on X consisting of those coherent sheaves for which there is a non-empty open set $U \subseteq X$ such that

(1) U contains all points of codim $\leqq 1$, and

(2) The sheaf restricted to U is locally free.

We then localize this category by considering as isomorphisms those maps which are isomorphisms in codim $\leqq 1$. When considering isomorphisms in this sense, we shall write L-isomorphism (L standing for "local"), and all propositions involving this set-up will have the prefix "L" attached. Similar remarks apply to all constructions with L-vector bundles. As an example, observe that if E is a vector bundle on X, then $\Lambda^r E$ need not be a bundle on X (except if $r = \operatorname{rk} E$), whereas if E is an L-vector bundle on X, then certainly $\Lambda^r E$ is again an L-vector bundle on X for every r.

Assume now that X is irreducible, and keep this assumption throughout the rest of the paper. The rank of a bundle E, denoted $\operatorname{rk}(E)$, is the rank of its fibre at the generic point of X; the same definition works for L-bundles. If $f: E \to E'$ is a homomorphism of vector bundles (or L-vector bundles), then the image of f, in the sheaf theoretic sense, is a vector bundle (resp. an L-bundle) although *not* in general a subbundle of E'. The rank of f, $\operatorname{rk}(f)$, is defined to be the rank of $\operatorname{Im} f$. Notice that the rank of a bundle (or a map) is equal to the rank of its associated L-bundle (resp. L-map).

PROPOSITION L1: *If V, W are bundles (resp. L-bundles) over X, and if f is a homomorphism (resp. L-homomorphism), then the following are equivalent:*

(1) $\operatorname{rk}(f) = t \neq 0$,

(2) $\Lambda^r f = 0$ *in the L-category for* $r > t$, *and* $\Lambda^t f \neq 0$.

The proof of this, being elementary and standard, will be left to the reader.

REMARK: If f is a homomorphism of bundles V, W, then the following are equivalent:

(1) $f = 0$

(2) $f = 0$ in the L-category

(3) $\operatorname{rk}(f) = 0$.

Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If (3) holds, then $(\operatorname{Im} f)_x = (0)$, where x is the generic point of X. Hence, $\operatorname{Im} f$ is a torsion subsheaf of W; but W is torsion-free; thus (1) holds.

Now let $f: V \to W$ be a homomorphism of vector bundles over X. The kernel of f is a vector bundle and the image of f being a subsheaf of W is also a vector bundle. We have the exact sequence of vector bundles over X:

$$0 \to V_1 \to V \to V_2 \to 0$$

in which $V_1 = \ker f$ and $V_2 = \operatorname{Im} f$. However, $W/\operatorname{Im} f$ need not be a bundle. If $t(W/\operatorname{Im} f)$ is the torsion subsheaf of $W/\operatorname{Im} f$, let

$$W_2 = (W/\operatorname{Im} f)/t(W/\operatorname{Im} f).$$

We obtain a bundle, W_2, over X and a surjection $W \to W_2$. Let $W_1 = \ker(W \to W_2)$, then W_1 is a subbundle of W and we have a diagram of bundles over X with exact rows:

$$(*) \qquad \begin{tikzcd} 0 \arrow[r] & V_1 \arrow[r] & V \arrow[r] \arrow[d, "f"'] & V_2 \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 & W_2 \arrow[l] & W \arrow[l] & W_1 \arrow[l] & 0. \arrow[l] \end{tikzcd}$$

We will refer to (*) as the *canonical factorization* of the map $f: V \to W$. Let us call a map of bundles $f: V \to W$ of *maximal rank* if $\operatorname{rk} f = \operatorname{rk} V$. Then the map φ of diagram (*) has maximal rank as is obvious. This proves

PROPOSITION 2: *If $f: V \to W$ is a homomorphism of vector bundles* *over X, then f admits a canonical factorization* (*) (*above*) *in which the map φ has maximal rank.*

Note that Langton [12] calls W_1 the subbundle generated by $\operatorname{Im} f$.

§2. Slopes and stability

We assume that X is a non-singular projective variety over an algebraically closed field, k. Let H denote a very ample divisor class on X and let E be a vector bundle on X. The Chern classes of E, $c_1(E)$, $c_2(E), \ldots, c_r(E)$, are then defined and $c_1(E)$ is the unique extension to X of the divisor class (= line bundle) $\Lambda^r(E \restriction U)$ where r is the rank of E and U is an open set containing all points of codim $\leqq 1$ on which E is locally free. The intersection number $(c_1(E) \cdot H^{n-1})$ makes sense and is the *H-degree of E*. (Here, $n = \dim X$.) We shall usually delete the H and merely write $\deg E$ for the degree of E. (For a full discussion of these matters see [11] and [12].) The rational number

$$\mu(E) = \frac{\deg E}{\operatorname{rk} E} \tag{1}$$

will be called the *slope of E* (or *H-slope of E* if necessary). Takemoto (inspired by Mumford) [16] calls a bundle, E, *H-stable* (resp. *H-semi-stable*) iff for every non-trivial subbundle F of E, we have

$$\mu(F) < \mu(E) \text{ (resp. } \mu(F) \leqq \mu(E)). \tag{2}$$

We shall delete the H and merely write stable (resp. semi-stable) when (2) holds. If (2) is false, we call E *unstable*. Observe that these definitions make sense in the L-category for L-bundles.

One sees rather simply that if $f: E \to E'$ is a homomorphism of maximal rank between vector bundles of the *same* rank, then $\deg(E) \leqq \deg(E')$. (Cf. [12].) Hence, we find that if E, E' have the same rank, and f is a maximal rank map from E to E', then $\mu(E) \leqq \mu(E')$.

There are several properties of the function μ which are extremely useful. They are all trivial to prove; so, we shall just state them and omit the proofs.

(A) *If $0 \to V' \to V \to V'' \to 0$ is an exact sequence of vector bundles* (*or L-vector bundles*) *on X, then*

$$\mu(V) \leqq \max\{\mu(V'), \mu(V'')\} \tag{3}$$

$$\mu(V) \geqq \min\{\mu(V'), \mu(V'')\} \tag{4}$$

and

(5) *Equality holds in* (3) *iff equality holds in* (4) *iff* $\mu(V') = \mu(V'')$.

(B) *More generally, if* $V \supset V_{\ell-1} \supset V_{\ell-2} \supset \cdots \supset V_1 \supset (0)$ *is a flag of subbundles* (*or L-subbundles*) *of* V, *then*

$$\mu(V) \leqq \max_i \{\mu(V_{i+1}/V_i)\} \tag{6}$$

$$\mu(V) \geqq \min_i \{\mu(V_{i+1}/V_i)\}, \tag{7}$$

and

(8) *Equality holds in* (6) *iff it holds in* (7) *iff* $\mu(V_{i+1}/V_i) = \mu(V_{j+1}/V_j)$ *for all* i *and* j.

(C) *If* V, W *are bundles* (*or L-bundles*) *on* X, *we have*

$$\mu(V \otimes W) = \mu(V) + \mu(W). \tag{9}$$

Note. Harder and Narasimhan discuss these properties in their article [10].

PROPOSITION L3: *Let* V, W *be semi-stable vector bundles of the same rank and assume* $\mu(V) = \mu(W)$. (V, W *may also be L-vector bundles.*) *Let* $f: V \to W$ *be a non-zero homomorphism and assume one of* V *or* W *is stable. Then* f *is an L-isomorphism and the other bundle is stable.*

PROOF: This was proved by Narasimhan and Seshadri for curves [15] and by Langton [12] in the general case—we give a slightly different proof. In the first place, when f has maximal rank (say n), then $\Lambda^n f: \Lambda^n V \to \Lambda^n W$ is an L-isomorphism of L-line bundles because $\deg \Lambda^n V = \deg \Lambda^n W$. It follows instantly that f is an L-isomorphism of V to W. In the general case, make the canonical factorization of f:

$$\begin{array}{ccccccccc}
0 \to & V_1 \to & V & \to & V_2 & \to 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & \\
0 \leftarrow & W_2 \leftarrow & W & \leftarrow & W_1 & \leftarrow 0.
\end{array}$$

Since V, W are semi-stable, we deduce from property (A) (3, 4) that $\mu(V) \leqq \mu(V_2)$; $\mu(W_1) \leqq \mu(W)$. On the other hand, as φ has maximal rank, our previous remarks show that $\mu(V_2) \leqq \mu(W_1)$. This yields the inequalities

$$\mu(V) \leqq \mu(V_2) \leqq \mu(W_1) \leqq \mu(W) = \mu(V)$$

all of which must therefore be equalities. However, as one of V, W is stable, we necessarily have either $V = V_2$ or $W_1 = W$. Either case implies the other as $\mathrm{rk}\, V = \mathrm{rk}\, W$, and we are done.

The same argument yields the

COROLLARY: *If V, W are stable vector bundles such that $\mu(V) = \mu(W)$ and if $f: V \to W$ is a non-zero homomorphism, then*

(1) $\mathrm{rk}\, V = \mathrm{rk}\, W$, *and*

(2) *f is an L-isomorphism.*

The next proposition is very important for what follows, it is a direct generalization of the results of Narasimhan and Seshadri, [15].

PROPOSITION 4: *Let V, W be semi-stable bundles on X and assume $\mu(V) > \mu(W)$. Then*

$$H^0(X, \mathrm{Hom}\,(V, W)) = (0),$$

that is, there does not exist any non-zero homomorphism from V to W.

PROOF: If a non-zero homomorphism $f: V \to W$ were to exist, we could factor it canonically and obtain

$$\begin{array}{ccccccccc} 0 & \to & V_1 & \to & V & \to & V_2 & \to & 0 \\ & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \leftarrow & W_2 & \leftarrow & W & \leftarrow & W_1 & \leftarrow & 0. \end{array}$$

Since φ has maximal rank, $\mu(V_2) \leqq \mu(W_1)$. On the other hand, the semi-stability of V and W shows that $\mu(V) \leqq \mu(V_2)$, $\mu(W_1) \leqq \mu(W)$; hence, $\mu(V) \leqq \mu(W)$, a contradiction. Q.E.D.

REMARKS: (1) One sees easily that stability and semi-stability are L-concepts.

(2) If $F_1, \ldots, F_r$ are semi-stable bundles of the same slope, then their direct sum is semi-stable and of the same slope. Conversely, if a semi-stable bundle, G, is a direct sum of bundles $F_1, \ldots, F_r$, then each F_j is semi-stable and $\mu(F_j) = \mu(G)$ for every j.

§3. Harder-Narasimhan flags and polygons

DEFINITION: Let E be a vector bundle on X. A *Harder-Narasimhan Flag for E* (HNF *for E*) is a flag

$$E = E_s > E_{s-1} > E_{s-2} > \cdots > E_1 > (0)$$

of subbundles of E having the following two properties:

(1) $\mu(E_{j+1}/E_j) < \mu(E_j/E_{j-1})$, $1 \leqq j \leqq s-1$,

(2) E_j/E_{j-1} is semi-stable, $1 \leqq j \leqq s$.

Several properties of an HNF for E follow immediately from the definition. Here are some of these properties:

PROPOSITION 5: *If $E = E_s > E_{s-1} > \cdots > E_1 > (0)$ is an HNF for E, then*

(a) $\mu(E_j/E_i) > \mu(E_k/E_i)$ *for* $i < j < k$,

(b) $\mu(E_k/E_i) > \mu(E_k/E_j)$ *for* $i < j < k$, *and*

(c) $\mu(E_1) > \mu(E_2) > \mu(E_3) > \cdots > \mu(E_s) = \mu(E)$.

PROOF: Consider the exact sequences

$$0 \to E_{i+1}/E_i \to E_{i+2}/E_i \to E_{i+2}/E_{i+1} \to 0$$
$$0 \to E_{i+2}/E_i \to E_{i+3}/E_i \to E_{i+3}/E_{i+2} \to 0$$
$$0 \to E_{j-1}/E_i \to E_j/E_i \to E_j/E_{j-1} \to 0.$$

By (1) of the definition of HNF and by the properties of μ, we obtain

$$\mu(E_{i+1}/E_i) > \mu(E_{i+2}/E_i) > \mu(E_{i+2}/E_{i+1})$$
$$\mu(E_{i+2}/E_i) > \mu(E_{i+3}/E_i) > \mu(E_{i+3}/E_{i+2}), \textit{ etc.}$$

Proceeding in this way stepwise, by induction one proves (a).

The exact sequence

$$0 \to E_j/E_i \to E_k/E_i \to E_k/E_j \to 0, \quad i < j < k,$$

and property (a) now imply, that (b) holds. Finally, one proves (c) by an elementary induction using the exact sequences

$$0 \to E_j \to E_{j+1} \to E_{j+1}/E_j \to 0. \qquad \text{Q.E.D.}$$

We now wish to prove the existence and uniqueness of HNF's for a given vector bundle E. This was done for the case: $X =$ a curve, by Harder and Narasimhan [10]; our proof will be different and we will give essentially two proofs of uniqueness. For the first of these proofs we need

LEMMA 1: *Let E be a vector bundle on X and let $H \subseteq E$ be a subbundle. Assume that*

(1) $\mu(H) > \mu(E)$ *(so that E is unstable) and*

(2) *If F is any subbundle of E whose rank is larger than* $\mathrm{rk}(H)$, *then* $\mu(F) \leqq \mu(E)$.

Under these assumptions, E/H is semi-stable.

PROOF: Suppose that E/H were not semi-stable. Then there would exist a subbundle F/H of E/H with $\mu(F/H) > \mu(E/H)$. Since the rank of F is larger than the rank of H, we find that $\mu(F) \leqq \mu(E)$. The exact sequences

$$0 \to F/H \to E/H \to E/F \to 0$$

$$0 \to F \to E \to E/F \to 0$$

show that $\mu(E/H) > \mu(E/F) \geqq \mu(E)$. However, $\mu(H) > \mu(E)$ and we arrive at a contradiction of equation (4) for the exact sequence

$$0 \to H \to E \to E/H \to 0. \qquad \text{Q.E.D.}$$

LEMMA 2: *Let E be a vector bundle over X, then the numbers $\mu(Q)$ as Q ranges over all subbundles of E are bounded above.*

PROOF: This is a variation on a classic argument of Grothendieck [8]. In the first place, we may assume E is an L-bundle as μ depends on a bundle only up to L-isomorphism. Secondly, if we show that the degrees of the L-subline bundles of the finitely many L-bundles E, $\Lambda^2 E, \ldots, \Lambda^n E$ ($n = \operatorname{rk} E$) are bounded above we will be done; for if Q is an L-subbundle of E, then $\Lambda^r Q$ is an L-subline bundle of $\Lambda^r E$ ($r = \operatorname{rk} Q$) and $\mu(Q) \leqq \deg Q = \deg \Lambda^r Q$. But, Grothendieck showed precisely that the degrees of the subline bundles of a bundle are bounded above; hence, the proof is complete. Q.E.D.

PROPOSITION 6: *Let E be an unstable vector bundle on X and let*

$\mathcal{S} = \{G \mid (1)$ *G is a semi-stable subbundle of E,*
(2) *$\mu(G)$ is maximal among all semi-stable subbundles of E,*
(3) *Among the semi-stable subbundles of maximum μ, G has maximal rank$\}$.*

Then

(a) *$\mathcal{S} \neq \emptyset$, i.e., such G exist,*
(b) *$\mu(G) > \mu(E)$,*
(c) *$\mathcal{S} = \{G\}$, i.e., there is only one such G.*

PROOF: By Lemma 2, there certainly exist bundles Q of maximal μ. Clearly, any subbundle of maximal μ must be semi-stable. Hence, the set

$\mathcal{S}_0 = \{Q \mid (1)$ Q is a semi-stable subbundle of E

(2) $\mu(Q)$ is maximal among all subbundles of E

(3) $\mu(Q) > \mu(E)\}$

is non-empty ((3) holds as E is unstable). In $\mathscr{S}_0$, pick any element of maximal rank; such an element lies in $\mathscr{S}$, so (a) and (b) are proved.

We will give two proofs of the uniqueness statement (c). The first is by induction on rk E. If rk $E = 1$, the hypothesis fails; so, assume the uniqueness for all unstable bundles of rank $< n$. Let E have rank n, and among all subbundles, H, of E with $\mu(H) > \mu(E)$ pick one (call it H, again) of maximal rank. Of course, $\operatorname{rk}(H) \geqq \operatorname{rk}(G)$ for any $G \in \mathscr{S}$. Now if $\operatorname{rk}(F) > \operatorname{rk}(H)$, we have $\mu(F) \leqq \mu(E)$; so, Lemma 1 shows that E/H is semi-stable.

Let G and G' belong to $\mathscr{S}$, and observe that $\mu(G) = \mu(G') > \mu(E) > \mu(E/H)$. By Proposition 4, the composed maps

$$G \to E \to E/H, \quad G' \to E \to E/H$$

are both zero. This means both G and G' are subbundles of H. If H is semi-stable, it follows instantly by the choice of H, G, G' that $\operatorname{rk}(H) = \operatorname{rk} G = \operatorname{rk} G'$ and that $G = H = G'$. If H is unstable, the induction hypothesis applies to show that $G = G'$.

The second proof of uniqueness follows immediately from the next lemma.

LEMMA 3: *Let E be a vector bundle on X and let G be an element of the set $\mathscr{S}$ introduced above. If F is a subbundle of E with $\mu(F) = \mu(G)$, then $F \subseteq G$.*

PROOF: Consider the bundles $F \vee G$, $F \cap G$ as defined in [12]. Langton showed that

$$\deg F + \deg G \leqq \deg (F \cap G) + \deg (F \vee G).$$

Let $d = \deg G$, $r = \operatorname{rk} G$, $s = \operatorname{rk} F$, and $\rho = \operatorname{rk}(F \cap G)$. Since $\mu(F) = \mu(G)$, we find $\deg F = (s/r)d$. Also, $\mu(F \cap G) \leqq \mu(G)$, and so $\deg (F \cap G) \leqq (\rho/r)d$. Hence,

$$\begin{aligned} \deg (F \vee G) &\geqq \frac{d}{r}(s + r - \rho) \\ &\geqq \mu(G) \operatorname{rk}(F \vee G). \end{aligned}$$

It follows that $\mu(F \vee G) = \mu(G)$ and $\operatorname{rk}(F \vee G) = \operatorname{rk} G$. Hence, $G = F \vee G$; that is, $F \subseteq G$. Q.E.D.

THEOREM 1: *Every vector bundle, E, on X possesses a unique Harder-Narasimhan Flag.*

PROOF: *Existence*. Use induction on the rank of E. If $\mathrm{rk}(E) = 1$ or if E is semi-stable, the existence is trivial. Thus, suppose E has rank $n > 1$ and is unstable. Let E_1 be the unique subbundle guaranteed to exist by Proposition 6. According to the induction hypothesis, the bundle E/E_1 possesses an HNF, say

$$E/E_1 = \bar{E}_t > \bar{E}_{t-1} > \cdots > \bar{E}_1 > (0). \tag{**}$$

This flag lifts to a flag

$$E = E_{t+1} > E_t > \cdots > E_2 > E_1 > (0) \tag{***}$$

in which $E_{j+1}/E_1 = \bar{E}_j$, for $1 \leqq j \leqq t$. It follows that E_{j+1}/E_j is semi-stable for $j = 1, 2, \ldots, t$, and that

$$\mu(E_{j+1}/E_j) < \mu(E_j/E_{j-1}), \quad 2 \leqq j \leqq t.$$

Since E_1 is semi-stable, all we must prove is that $\mu(E_2/E_1) < \mu(E_1)$. Now both E_1 and E_2 are subbundles of E; so, Proposition 6 shows that $\mu(E_1) \geqq \mu(E_2)$. Next, E_2 cannot be semi-stable, for if it were, we would have $\mu(E_1) \leqq \mu(E_2)$. This would contradict Proposition 6, because the rank of E_2 is strictly larger than that of E_1. As E_2 is unstable, it possesses a subbundle, H, with $\mu(H) > \mu(E_2)$. By Proposition 6 again, $\mu(E_1) \geqq \mu(H) > \mu(E_2)$, as required.

Uniqueness. Once again we proceed by induction on the rank of E. If the rank of E is 1, the result is trivial; and more generally, if E is semi-stable, then Proposition 5(c) yields the uniqueness immediately. Consequently, assume E is unstable of rank n, and let

$$E = E_s > E_{s-1} > \cdots > E_1 > (0)$$
$$E = E'_t > E'_{t-1} > \cdots > E'_1 > (0)$$

be two HNF's for E. Let G be the unique semi-stable subbundle of E given by Proposition 6, and let j be the smallest integer such that the inclusion $G \hookrightarrow E$ factors through E_j. Then we obtain a *non-zero* homomorphism $G \to E_j \to E_j/E_{j-1}$, and as E_j/E_{j-1} is semi-stable, Proposition 4 implies that $\mu(G) \leqq \mu(E_j/E_{j-1})$. However, by Proposition 5,

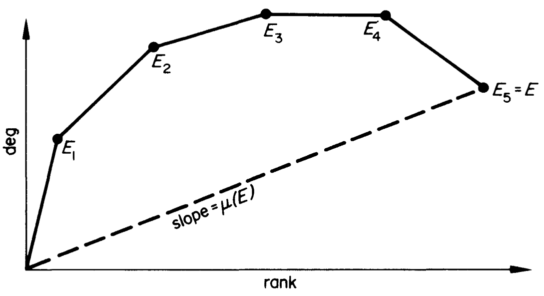
$$\mu(G) \geqq \mu(E_1) > \cdots > \mu(E_j) > \mu(E_j/E_{j-1})$$

which is a contradiction unless $j = 1$. In this case, G is contained in E_1 which is semi-stable; hence, $\mu(G) = \mu(E_1)$. By Proposition 6, G and E_1 have the same rank; therefore, $G = E_1$. In a similar way, $G = E'_1$. By considering $E/E_1 = E/E'_1$, we reduce the rank of the bundle under consideration, and the induction hypothesis completes the proof. Q.E.D.

Since an HNF for E is unique, we may speak of *the* HNF for E and write HNF(E) for this. If F is a vector bundle, we may associate to it the point $p(F) = (r, d)$ (where $r = \operatorname{rk} F$ and $d = \deg F$), in the plane with coordinates rank and degree. Of course, the slope of F is then merely the slope of the line joining the origin and $p(F)$. Now if

$$E = E_t > E_{t-1} > \cdots > E_1 > (0)$$

is the HNF(E), we may consider the points $p(E_1), p(E_2), \ldots, p(E_t) = p(E)$ in the plane and connect them successively by line segments. The result is a polygon in the plane which we call the *Harder-Narasimhan Polygon of E*, HNP(E). Observe that the slope of the bundle E_j/E_{j-1}, $\mu(E_j/E_{j-1})$, is precisely the slope of the line segment joining $p(E_{j-1})$ with $p(E_j)$. Hence, from the definition of HNF's, we find that the HNP(E) is a *convex* polygon. A typical HNP is sketched below.



It is clear what we mean by saying that a point in the plane lies on or below the HNP(E). *Hence, if* $\mu(F) \leqq \mu(E)$, *then* F (*that is*, $p(F)$) *certainly lies below the* HNP(E). Secondly, observe that if $F \supseteq E_1$, then the coordinates of F/E_1 are exactly the coordinates of $p(F)$ referred to the new origin $p(E_1)$. *Hence, F lies on or below* HNP(E) *if and only if* F/E_1 *lies on or below* HNP(E/E_1)—because the E_j/E_1 are the Harder-Narasimhan subbundles for E/E_1.

THEOREM 2: *Let E be a vector bundle on X, and let F be a subbundle of E. Then F lies on or below the* HNP(E).

PROOF: We use induction on the length of the HNF(E). If $E = E_1$, then $\mu(F) \leqq \mu(E)$ and our first observation above shows that F lies on or below the HNP(E). Let E have HN length $= n$, and let $F \subseteq E$ be a given subbundle. Set $r = \operatorname{rk} E_1$, $s = \operatorname{rk} F$, $d = \deg E_1$ and $\delta = \deg F$. If $\mu(F) = \mu(E_1)$, then the basic properties of E_1 (or Lemma 3) show that $\operatorname{rk} F \leqq \operatorname{rk} E_1$; hence, F lies on or below $\operatorname{HNP}(E)$.

Thus, we may and do assume throughout the rest of the proof that $\mu(F) < \mu(E)$. Let $\rho = \operatorname{rk}(F \cap E_1)$, and consider the bundle $E_1 \vee F$, [12]. Since $\deg(E_1 \cap F) \leqq (d/r)\rho$, and since

$$\deg(E_1 \vee F) + \deg(E_1 \cap F) \geqq \deg E_1 + \deg F,$$

we obtain

$$\deg(E_1 \vee F) \geqq \frac{d}{r}(r-\rho) + \frac{\delta}{s} \cdot s. \tag{$\dagger$}$$

(a) If $r = \rho$, then $F \supseteq E_1$; and F/E_1 being a subbundle of E/E_1, the induction hypothesis implies that F/E_1 lies on or below $\operatorname{HNP}(E/E_1)$. Our remarks above show that F lies on or below $\operatorname{HNP}(E)$.

(b) We now have the interesting case: $\rho < r$ and $\mu(F) < \mu(E_1)$. Since $(\delta/s) < (d/r)$ and $r - \rho > 0$, equation (†) yields

$$\deg(E_1 \vee F) > \frac{\delta}{s}(r-\rho) + \frac{\delta}{s} \cdot s;$$

hence,

$$\mu(E_1 \vee F) > \mu(F). \tag{$\dagger\dagger$}$$

Now $E_1 \vee F \supseteq E_1$; so, by the induction hypothesis for E/E_1 and by the remarks above, we find that $E_1 \vee F$ lies on or below $\operatorname{HNP}(E)$. But as $\operatorname{rk} F \leqq \operatorname{rk}(E_1 \vee F)$, (††) implies that F lies below $E_1 \vee F$. Thus, F lies below $\operatorname{HNP}(E)$, as required. Q.E.D.

COROLLARY: *Any polygon whose vertices are subbundles of a fixed vector bundle, E, on X is dominated by the* $\operatorname{HNP}(E)$. (Maximal Property of $\operatorname{HNP}(E)$.)

PROOF: By Theorem 2, all the vertices lie on or below the $\operatorname{HNP}(E)$, and $\operatorname{HNP}(E)$ is convex; so, we are done.

REMARK: Theorem 2 and its Corollary hold for subsheaves F of E, not just for subbundles.

We can refine the HNF of a vector bundle E by decomposing the semi-stable factors into flags whose factors are actually stable. Indeed we have

PROPOSITION 7: *Let E be a semi-stable bundle on X, then E possesses a flag*

$$E = E_v > E_{v-1} > E_{v-2} > \cdots > E_1 > (0)$$

for which

(a) $\mu(E) = \mu(E_j/E_{j-1})$, $1 \leq j \leq v$,
(b) E_j/E_{j-1} *is stable*, $1 \leq j \leq v$, *and*
(c) $\mathrm{gr}\,(E) = \coprod_{j=1}^{r} E_j/E_{j-1}$ *is unique up to L-isomorphism.*

PROOF: *Existence.* Let H be a proper subbundle of E with $\mu(H) = \mu(E)$ and having maximal rank with this property. (If no such H exists, then E is already stable.) Observe that H is automatically semi-stable; so the induction hypothesis gives the required flag in H. We will be done when we show E/H is stable. Let $F/H \hookrightarrow E/H$ be a proper subbundle. Since $F > H$, we have $\mu(F) < \mu(E)$; thus, $\mu(E) < \mu(E/F)$. But $\mu(E/H) = \mu(E)$, consequently $\mu(F/H) < \mu(E/H)$, as required.

Uniqueness of the graded object. We use induction on $\mathrm{rk}\,(E)$, the result being trivial for $\mathrm{rk}\,(E) = 1$. Let

$$E > E_{r-1} > \cdots > E_1 > (0), \text{ and } E > E'_{s-1} > \cdots > E'_1 > (0),$$

be two flags satisfying (a) and (b). Pick t minimal such that $E_1 \subseteq E'_t$ (t may very well be s). By the Corollary of Proposition 3, we find that E_1 is L-isomorphic to E'_t/E'_{t-1}, and hence that E'_t is L-isomorphic to $E_1 \oplus E'_{t-1}$. Since $E_1 \subseteq E'_t$, this yields the L-exact sequence

$$(0) \to E'_{t-1} \to E/E_1 \to E/E'_t \to (0).$$

If Z_j denotes the inverse image of E'_j/E'_t in E/E_1, then

$$Z_j/Z_{j-1} \cong E'_j/E'_{j-1} = \mathrm{gr}\,(E)'_j,$$

and we obtain the filtrations

$$E/E_1 > Z_{s-1} > \cdots > Z_{t+1} > E'_{t-1} > E'_{t-2} > \cdots > E'_1 > (0),$$
$$E/E_1 > E_{r-1}/E_1 > \cdots > E_2/E_1 > (0).$$

The induction hypothesis and the isomorphism $E_1 \cong E'_t/E'_{t-1}$ now complete the proof. Q.E.D.

A flag obtained by interpolating the stable factors in the HNF will be called a *complete* HNF. The polygon corresponding to a complete HNF is the same as the HNP, we have merely inserted extra vertices along the original edges.

When $X = \boldsymbol{P}^1$ (the projective line) a complete HNF corresponds exactly to Grothendieck's decomposition [8] of a bundle into a direct sum of line bundles.

One more remark: By following an argument of Atiyah [1], we can prove:

PROPOSITION 8: *Let E be a vector bundle over the curve X, and let K denote the canonical line bundle on X. If E is indecomposable, and if*

$$E > E_{r-1} > \cdots > E_1 > (0)$$

is the HNF(E), *then*

$$\mu(E_{i+1}/E_i) \geqq \mu(E_{j+1}/E_j) - (i-j)\mu(K)$$

for $0 \leqq j \leqq i \leqq r-1$.

One proves this by following Atiyah's argument word for word (except that Proposition 4 is used) for the cases $j = 0$, $j = i - 1$. The general case then follows trivially.

§4. Algebraic families of vector bundles

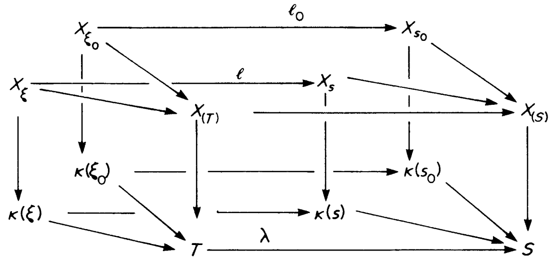
Let X be, as above, a non-singular projective variety over an algebraically closed field, k. Again, H will denote a very ample divisor class on X. Let S be a scheme over k; we shall usually assume that S is connected or, even better, irreducible. We consider Xx_kS and let p_1 (resp. p_2) be the projection onto X (resp. S). By [4], $H \otimes_k \mathcal{O}_S = p_1^*H$ is very ample for Xx_kS over S. If $s \in S$ is given, it follows that $H_s = i_s^* p_1^* H$ is very ample for X_s over $\kappa(s)$, where X_s is the fibre of Xx_kS over s and i_s is the morphism of the fibre X_s into the scheme Xx_kS.

By an *algebraic family of vector bundles on X parametrized by S*, we mean a vector bundle, E, on Xx_kS which is *flat* over S. If E is an algebraic family of vector bundles (parametrized by S) on X, and if $s \in S$, then we let E_s denote the sheaf i_s^*E on the fibre X_s. As part of the definition, we assume that E_s is an L-bundle on X_s, and that this is still true even if E is only an L-bundle on Xx_kS. We have observed that slopes are defined for L-bundles and that they are invariant under L-isomorphism; so we define the slope of E_s, $\mu(E_s)$, to be its slope with respect to the very ample sheaf H_s.

Now let s, s_0 be points of S with s_0 a specialization of s, and let T be the spectrum of a discrete valuation ring (DVR) which "covers" s and s_0 in the sense of [5]. Here, if ξ (resp. ξ_0) is the general (resp. special) point of T, there is a morphism $T \to S$ such that $\xi \mapsto s$ and $\xi_0 \mapsto s_0$. Let $\lambda : T \to S$ be the given morphism and let ℓ (resp. ℓ_0) be the induced morphisms

$$\ell : X_\xi \to X_s \text{ (resp. } \ell_0 : X_{\xi_0} \to X_{s_0}),$$

so that we have the commutative diagram



in which $X_{(S)}$ is Xx_kS, and $X_{(T)}$ is Xx_kT. Then $H_\xi = \ell^*H_s$ is very ample for X_ξ; similarly, H_{ξ_0} is very ample for X_{ξ_0}. Moreover, we have

$$c_1(E_\xi) = c_1(\ell^*E_s) = \ell^*(c_1(E_s)),$$

so that

$$\deg(E_\xi) = (\ell^*c_1(E_s)\cdot(\ell^*H_s)^{n-1}).$$

Since these intersection numbers are given by Hilbert Polynomials (which are in turn certain Euler Characteristics) [12], Proposition 7.9.7 of EGA 3, [6], shows that

$$\deg(E_\xi) = \deg(E_s);$$

hence, that $\mu(E_\xi) = \mu(E_s)$ and similarly $\mu(E_{\xi_0}) = \mu(E_{s_0})$. Observe, moreover, that in view of the invariance of Hilbert Polynomials in flat families, we have

$$\deg(E_s) = \deg(E_{s_0}); \quad \mu(E_s) = \mu(E_{s_0}).$$

Given an algebraic family of vector bundles on X, parametrized by S, we have the induced L-bundles E_s on each fibre X_s. Since HNF's exist and are unique for L-bundles as well, we may examine the HNF for E_s:

$$E_s = E_{s,r} > E_{s,r-1} > \cdots > E_{s,1} > (0).$$

We set

$$(10)\qquad \begin{cases} \rho_j(E,s) = \mathrm{rk}\,(E_{s,j}), \\ d_j(E,s) = \deg(E_{s,j}), \text{ and} \\ \mu_j(E,s) = \mu(E_{s,j}/E_{s,j-1}) = \dfrac{d_j(E,s)-d_{j-1}(E,s)}{\rho_j(E,s)-\rho_{j-1}(E,s)}. \end{cases}$$

Of course,

$$\mu_1(E,s) > \mu_2(E,s) > \cdots,$$

and

$$\mu_m(E, s) = -\infty \text{ for } m \gg 0 \text{ (by convention).}$$

For each $s \in S$ we form the HNP of E_s (from the above vertices $\langle \rho_j(E, s), d_j(E, s) \rangle$) and we denote it by HNP(E, s). Since convex polygons have an obvious partial ordering, we have a mapping from S into the partially ordered set of convex polygons with lattice point vertices.

If s and $s_0 \in S$ are given with s_0 a specialization of s, and if T is the spectrum of a DVR covering s and s_0, then by [12, Prop. 3] we know that E_ξ (resp. E_{ξ_0}) is semi-stable if and only if E_s (resp. E_{s_0}) is semi-stable. Since the μ's remain invariant under passage to T, it follows that the HNF for E_s pulls back to the HNF for E_ξ, and similarly for s_0 and ξ_0. (Remark: One needs to observe that inverse image preserves L-exact sequences of L-bundles.) This proves

Lemma 4 (Reduction Lemma): *In studying the behavior of* HNP(E, s) *and the functions* $\mu_j(E, s)$ *under specialization, we may assume S is the spectrum of a discrete valuation ring.*

Consider Grothendieck's quotient functor *Quot* $(E/Xx_kS/S)$, which we shall abbreviate $Q(E, X, S)$. Recall that for S' over S, the value of $Q(E, X, S)$ on S' is the set of all quasi-coherent quotients of $E \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ which are *flat* over S'. We shall always stay inside the category of locally Noetherian schemes (for S and S'), so the quotients we get as well as the subsheaves of $E \otimes_{\mathcal{O}_S} \mathcal{O}_S$ we obtain as kernels will always be coherent. As usual, *Grass* (E/Xx_kS) will denote the subfunctor of $Q(E, X, S)$ consisting of those quotients which are locally free. If U is an open subset of Xx_kS, then we have a commutative diagram

$$\begin{array}{ccc} \textit{Grass}\ (E/Xx_kS) & \overset{i}{\hookrightarrow} & Q(E, X, S)(S) \\ \downarrow \pi & & \downarrow \pi \\ \textit{Grass}\ (E \restriction U/U) & \overset{i_U}{\hookrightarrow} & Q(E \restriction U/U/S)(S). \end{array}$$

Now if $\sigma \in Q(E, X, S)(S)$, and if for some open U of Xx_kS containing all points of codimension $\leqq 1$, the element $\pi(\sigma)$ lies in the image of i_U, then the quotient determined by σ is an L-bundle, and σ determines an L-subbundle of E over Xx_kS. If σ is given and the corresponding quotient, E''_σ, of E is torsion-free, then σ determines a subbundle of E over Xx_kS.

PROPOSITION 9: *Let S be* Spec A *where A is a* DVR, *and let E be an algebraic family of vector bundles on X parametrized by S. Let s (resp. s_0) be the general (resp. special) point of S and let*

$$E_s > \mathscr{E}^{(r)} > \cdots > \mathscr{E}^{(1)} > (0)$$

be a given flag of subbundles of the vector bundle E_s. Then there exists a unique flag of subbundles of E

$$E > E^{(r)} > \cdots > E^{(1)} > (0)$$

which induces the given flag on the generic fibre. That is,

$$E^{(j)} \restriction X_s = \mathscr{E}^{(j)}, \quad 1 \leqq j \lneqq r.$$

PROOF: The essential case occurs when $r = 1$, and we shall treat this case first. The flag $E_s > \mathscr{E}^{(1)} > (0)$ gives rise to a point σ of $Q(E, X, S)$ with values in Spec $\kappa(s)$. By Lemma 3.7 of [9], this section can be extended to a global section, call it σ again, of $Q(E, X, S)$ over S. The global section corresponds to a coherent quotient $E/E^{(1)}$ of E over Xx_kS, which is flat over S. Indeed there is a unique way of making this extension.[1] Since $E^{(1)}$ is already torsion-free, all we need show is that $E/E^{(1)}$ is torsion-free. Let $\tau(E/E^{(1)})$ be the torsion subsheaf of $E/E^{(1)}$ and let F be the subsheaf of E such that $F/E^{(1)}$ is isomorphic to $\tau(E/E^{(1)})$. Then F is a subbundle of E and E/F is flat over S. Yet at any point, x, of the generic fibre X_s, we have the inclusion

$$(F/E^{(1)})_x \hookrightarrow (E_s)_x/\mathscr{E}^{(1)}_x$$

and the right-hand side is torsion-free by assumption. Hence $F \restriction X_s$ is $\mathscr{E}^{(1)}$, and the uniqueness shows that $\tau(E/E^{(1)}) = (0)$.

The general case is by induction on r. We obtain the subbundle $E^{(r)}$ which induces $\mathscr{E}^{(r)}$ by the above, and we use induction to obtain the smaller flag $E^{(r)} > E^{(r-1)} > \cdots > E^{(1)} > (0)$ in $E^{(r)}$. Now splice the two flags together to obtain the required flag in E. Q.E.D.

[1] The uniqueness is a standard argument: We may assume $E^{(1)}$ is the largest subsheaf of E inducing $\mathscr{E}^{(1)}$. If $\tilde{E}^{(1)}$ is another such subsheaf with $E/\tilde{E}^{(1)}$ flat over S, then the exact sequence

$$0 \to E^{(1)}/\tilde{E}^{(1)} \to E/\tilde{E}^{(1)} \to E/E^{(1)} \to 0$$

shows that $E^{(1)}/\tilde{E}^{(1)}$ is flat, hence torsion-free, over S. However, at the generic point $s \in S$, the sheaf $E^{(1)}/\tilde{E}^{(1)}$ must vanish; so, $E^{(1)}/\tilde{E}^{(1)}$ is a torsion sheaf over S. Therefore, $E^{(1)}/\tilde{E}^{(1)} = (0)$, as contended.

THEOREM 3: *Let E be an algebraic family of vector bundles on X, parametrized by the scheme S. Let s, $s_0 \in S$ with s_0 a specialization of s. Then* $\mathrm{HNP}(E, s_0) \geqq \mathrm{HNP}(E, s)$ *(in the partial ordering of convex polygons); that is, the Harder-Narasimhan Polygon rises under specialization. In particular,* $\mu_1(E, s_0) \geqq \mu_1(E, s)$.

PROOF: By the reduction lemma, we may and do assume that $S = \operatorname{Spec} A$, with A a DVR, and with s as generic point and s_0 as closed point. Let

$$E_s > E_{s,r-1} > \cdots > E_{s,1} > (0)$$

be the HNF for E_s. According to Proposition 9, the above flag can be extended to the algebraic family, E, and it induces a flag on E_{s_0}:

$$(*) \qquad E_{s_0} > E_{s_0}^{(r-1)} > \cdots > E_{s_0}^{(1)} > (0).$$

By our remarks at the beginning of this section, the polygon corresponding to (*) is precisely the $\mathrm{HNP}(E, s)$. However, by the corollary of Theorem 2, the polygon of the flag (*) is dominated by the $\mathrm{HNP}(E, s_0)$. Therefore, $\mathrm{HNP}(E, s_0) \geqq \mathrm{HNP}(E, s)$, as required. Q.E.D.

§5. Constructibility of HNP(E)

We have shown that the Harder-Narasimhan polygon rises under specialization. Here, we prove the constructibility of these polygons as functions on S. Throughout this section S will be a locally Noetherian scheme.

LEMMA 5: *If E is an algebraic family of vector bundles on X over S, if S is irreducible, and if $s \in S$ is the generic point, then every flag*

$$E(s) > E_{v-1}(s) > \cdots > E_1(s) > (0)$$

of bundles on the generic fibre, X_s, may be extended to a flag on Xx_kU, where U is an open subset of S containing s.

PROOF: As in Prop. 9, the essential case is when $v = 2$; we shall leave the induction step to the reader. We know (by the remarks preceding Prop. 9) that to get the subbundle E_1 of E over Xx_kU, it will suffice to produce a section

$$\sigma \in Q(E, X, S)(U), \quad s \in U \subseteq S$$

such that there is an open set W of Xx_kU having the property that the corresponding sheaf $\mathscr{F}_\sigma$, when restricted to W, is locally free and all points of codim $\leqq 1$ lie in W.

Now, $E_1(s)$ corresponds to a section σ of $Q(E \upharpoonright X_s/X_s/S)$ over Spec $\kappa(s)$, such that in the diagram

$$\begin{array}{ccc} & & Q(E \upharpoonright X_s/X_s/S)(\kappa(s)) \\ & & \big\downarrow \text{restr.} \\ \textit{Grass}\,(E \upharpoonright U_s/U_s) & \hookrightarrow & Q(E \upharpoonright U_s/U_s/S)(\kappa(s)) \end{array}$$

restr. $(\sigma) \in \mathrm{Grass}\,(E \upharpoonright U_s/U_s)$. Here, U_s is open in X_s and contains all points of codim ≤ 1. As σ is a rational section of the scheme $Q(E, X, S)$ over S, it extends to a section over some open set, V, containing s. Hence, there exists a subsheaf E_1 of E over Xx_kV such that E/E_1 is coherent and flat over V. Of course, E_1 is torsion-free.

Write $F = E/E_1$ on Xx_kV and apply [3, Prop. 3.3.1] to F and $\mathcal{O}_V$. We obtain

$$\mathrm{Ass}\,(F) = \bigcup \{\mathrm{Ass}\,(F_t) | t \in \mathrm{Ass}\,(V)\}.$$

Since V is locally noetherian, there is an open set $U \subseteq V$ such that $\mathrm{Ass}\,(V) \cap U$ consists only of the generic point s of S ([3, Prop. 3.1.6]). However, F_s is torsion-free; and so $\mathrm{Ass}\,(F)$, for F over Xx_kU, consists entirely of the generic point of Xx_kU. Thus, F over Xx_kU is torsion-free, and E_1 is a subbundle over U. Q.E.D.

S. Kleiman[1] has proved the following theorem.

THEOREM (Kleiman): *Let $X \to S$ be a projective morphism of schemes with geometrically integral fibres and assume S is locally Noetherian. If F is a coherent $\mathcal{O}_X$-module flat over S and if $s \in S$, set*

$$\delta(F(s)) = \sup\{\mathrm{rk}\,(F(s)) \deg G - \deg\,(F(s))\,\mathrm{rk}\,(G) | G \text{ is coherent and } 0 < G < F(s)\}.$$

Then, for every integer p, the set

$$U(p) = \{s \in S | \delta(F(s)) < p\}$$

is open in S.

Now a simple argument (left to the reader) using the fact that torsion sheaves have non-negative degree shows that

[1] Private communication.

LEMMA 6: *Let F be an algebraic family of vector bundles parametized by S. Then*

(1) *$F(s)$ is semi-stable iff $\delta(F(s))<1$, and*

(2) *$F(s)$ is stable iff $\delta(F(s))<0$.*

Hence, the sets

$$U_\sigma=\{s\in S|F(s) \text{ is stable}\} \text{ and } U_{\sigma\sigma}=\{s\in S|F(s) \text{ is semi-stable}\}$$

are open in S.[1]

PROPOSITION 10: *Let X, S be an irreducible, non-singular, projective k-variety and a locally noetherian scheme over k respectively, and let E be an algebraic family of vector bundles on X parametrized by S. Then the function*

$$s\mapsto \mathrm{HNP}(E,s)$$

is constructible.

PROOF: Let $\mathscr{E}$ be the pull back of E to the generic fibre X_s of Xx_kS. (We may assume S is irreducible.) Form the HNF of $\mathscr{E}$

$$\mathscr{E}>\mathscr{E}_{v-1}>\cdots>\mathscr{E}_1>(0)$$

and use Lemma 5 to extend this to a flag

$$E>E_{v-1}>\cdots>E_1>(0) \tag{*}$$

over Xx_kU. The bundles $(E_j/E_{j-1})_s=\mathscr{E}_j/\mathscr{E}_{j-1}$ are semi-stable for $1\le j\le v$; hence, by repeated application of Lemma 6, we see that there is an open set, call it U again, containing s, such that for every $t\in U$, the bundles $(E_j/E_{j-1})_t$ are semi-stable, $1\le j\le v$.[2]

Now, for any t, we have

$$\mu((E_j/E_{j-1})_t)=\mu(\mathscr{E}_j/\mathscr{E}_{j-1}),$$

and it follows that for each $t\in U$ the flag induced at t by (*) satisfies the two properties of HNF's. By the uniqueness of HNF's, each of these induced flags is the HNF for E at the corresponding fibre. However, all these flags have the same HNP; therefore, $\mathrm{HNP}(E,t)$ is contant on U. Q.E.D.

COROLLARY: *Under the hypothesis of Prop.* 10, *the function* $s\mapsto\mathrm{HNP}(E,s)$ *is upper semi-continuous.*

[1] M. Maruyama [13] has also proved that stability and semi-stability are open conditions in an algebraic family of sheaves.

[2] $(E_jE_{j-1})_t$ is torsion-free in an open set containing s by [7, Theorem 12.2.1(i)].

PROOF: Conjoin Theorem 3 and Proposition 10.

One can view the corollary more geometrically as follows: Let P be a fixed polygon which is convex and has vertices at lattice points. Then *the following three conditions are equivalent* (when S is Noetherian):

(a) $\{s \in S | \mathrm{HNP}(E, s) \geq P\}$ *is closed*

(b) $\{s \in S | \mathrm{HNP}(E, s) = P\}$ *is locally closed*

(c) $\{s \in S | \mathrm{HNP}(E, s) = P\}$ *is constructible.*

(By Prop. 11 below (a) $\Rightarrow$ (b); clearly (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) by the proof of the corollary.) Hence, *the vector bundle E stratifies S into closed sets, namely the sets, $S_P(E)$, of* (a).

PROPOSITION 11: *If S is Noetherian, the stratification of S induced by the vector bundle E is finite.*

PROOF: We first show that the numbers $\mu(F)$ as F ranges over all subbundles of the vector bundles E_s for all $s \in S$ are uniformly bounded above. Since S is Noetherian it has only finitely many irreducible components, and so we may assume S is irreducible. Let s be the generic point of S. Recall (Lemma 2) that a flag of line bundles in each of the bundles $\Lambda^m E_s$ $(1 \leq m \leq \mathrm{rk}\, E)$ serves to bound the numbers $\mu(F)$ as F ranges over subbundles of E_s. By Lemma 5, these flags extend over an open set of S; hence the $\mu(F)$ are uniformly bounded as F ranges over all subbundles of E_t with t lying in an open subset, U, of S. The complement, Z, of U in S has lower dimension; so an obvious inductive argument completes the uniform boundedness.

Let P_0 be the HNP of E_s, where s is generic in S. By Theorem 3, $\mathrm{HNP}(E, t) \geq P_0$ for all $t \in S$. Let λ be the largest of the $\mu(F)$ as above, and let P_1 be the polygon starting at $(0, 0)$, consisting of a straight line of slope λ, and ending at the line $x = \mathrm{rk}\, E$. By the properties of HNP's and by the choice of λ, we have $P_1 \geq \mathrm{HNP}(E, t)$ for all $t \in S$. As all HNP's have lattice point vertices and as there are only finitely many lattice points in the region bounded by P_0, P_1, $x = 0$, and $x = \mathrm{rk}\,(E)$, we are done. Q.E.D.

LEMMA 7: *If S is Noetherian and irreducible, if E, X are as above, and if P_0 is the* HNP *of E_s for the generic point $s \in S$, then the set*

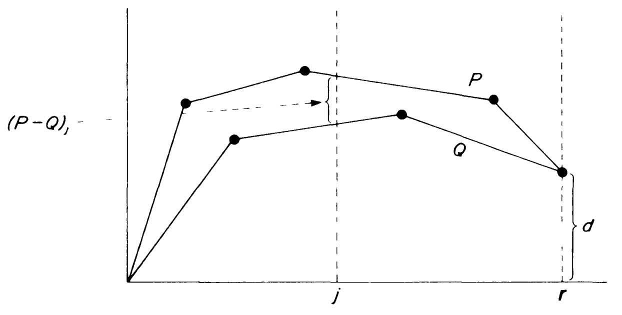
$$S^*_{P_0}(E) = \{t \in S | \mathrm{HNP}(E, t) = (P_0\}$$

is open in S.

PROOF: By the proof of Prop. 10, our set contains an open set ($\neq \emptyset$); on the other hand our set is locally closed by the above remarks. Hence, $S^*_{P_0}(E)$ is open. Q.E.D.

§6. Connection with algebraic cycles on S

Throughout this section we will assume that S is Noetherian. Suppose that P and Q are convex polygons which begin together (say at $(0, 0)$) and which end together (say at (r, d)). Assume both have vertices at lattice points. If $P \geq Q$ in the obvious ordering of convex polygons, we shall define $P - Q$ as an ordered $r - 1$ tuple of rational numbers. Namely, the j^{th} component of $P - Q$ $(1 \leq j \leq r - 1)$ is the difference of the ordinates of P and Q at the abscissa $x = j$.



Clearly, by the convexity of P and Q, we have $P \geq Q$ if and only if $P - Q$ is a vector of *non-negative* rational numbers, and $P = Q$ iff $(P - Q)_j = 0$, for all j.

LEMMA 8: *With the notations above, if both polygons end at abscissa r, then $r!(P - Q)_j$ is an integer for all j.*

PROOF: We simply observe that the only denominators introduced into $(P - Q)_j$ are integers between 1 and r (explicit computation). Hence, $r!(P - Q)_j \in \boldsymbol{Z}$. Q.E.D.

Our application is to the case: $P = \mathrm{HNP}(E, s_0)$, $Q = \mathrm{HNP}(E, s)$, where E is a vector bundle on Xx_kS (flat over S) and s_0 is a specialization of s. By an inductive procedure (inducing on dim S), we shall associate to E (on Xx_kS) algebraic cycles (with positive coefficients) of S.

Assume first that S is irreducible. If $\dim S = 0$, we associate to E the empty cycle. If $\dim S = 1$, let s be the generic point of S, then by Lemma 7, there is an open set, U, of S consisting of points $t \in S$ for which $\mathrm{HNP}(E, t) = \mathrm{HNP}(E, s)$. Let $S - U = \{t_1, \ldots, t_q\}$, and associate with t_i the $r-1$ non-negative cycles

$$r!(\mathrm{HNP}(E, t_i) - \mathrm{HNP}(E, s))_j t_i, \quad 1 \leq j \leq r-1.$$

The cycles we then associate to E are:

$$\sum_{i=1}^{q} r!(\mathrm{HNP}(E, t_i) - \mathrm{HNP}(E, s))_j t_i, \quad 1 \leq j \leq r-1.$$

If S is reducible, we merely restrict E to Xx_kS_m for $m = 1, \ldots, p$, where the S_m are the irreducible components of S, and proceed as above for each S_m.

Now assume we have constructed algebraic cycles of S (associated to E) for all S of dimension $< b$. Let S be b-dimensional, Noetherian and irreducible; then by Lemma 7, the set

$$\{t \in S \mid \mathrm{HNP}(E, t) = \mathrm{HNP}(E, s),\ s \text{ generic in } S\}$$

is open. Call this set U, and let $S_1, S_2, \ldots, S_n$ be the irreducible components of $S - U$. If s_i is the generic point of S_i, we form the $r-1$ algebraic cycles

$$\sum_{i=1}^{n} r!(\mathrm{HNP}(E, s_i) - \mathrm{HNP}(E, s))_j S_i, \quad 1 \leq j \leq r-1,$$

where, as usual, s is generic for S. These cycles are what we get "at the first step." We now restrict E to Xx_kS_i for $i = 1, \ldots, n$ and repeat the process there. The induction hypothesis yields algebraic cycles of the S_i; hence, of S. Upon putting these together with the cycles defined explicitly in the first step, we obtain $r-1$ cycles with non-negative coefficients. If S is reducible, let $S_1, \ldots, S_q$ be its irreducible components. Restrict E to Xx_kS_i, for $i = 1, 2, \ldots, q$ and apply the above procedure to these S_i. The cycles we get are considered as cycles on S.

If $\mathscr{A}(S)$ denotes the semi-group of non-negative cycles on S, then we have constructed a map

$$\Theta : \mathrm{Vect}_k^r(Xx_kS) \to \mathscr{A}(S)^{r-1},$$

where $\mathrm{Vect}_k^r(Xx_kS)$ denotes the families of rank r vector bundles on X parametrized by S. The set $\mathrm{Vect}_k^r(Xx_kS)$ breaks up into a disjoint union

$$\mathrm{Vect}_k^r(Xx_kS) = \coprod_{P_1, \ldots, P_q} \mathrm{Vect}_k^{P_1, \ldots, P_q}(Xx_kS),$$

in which $P_1, \ldots, P_q$ are convex polygons starting at $(0, 0)$, ending at (d, r), lying above (or on) the line through $(0, 0)$ and (r, d), and where S is reducible with $S_1, \ldots, S_q$ as components, s_i is the generic point of S_i, and $P_i = \mathrm{HNP}(E, s_i)$. Of course, d (the degree of E) is arbitrary.

Now given two families E, F of vector bundles on X parametrized by S, we shall say that *E is Harder-Narasimhan equivalent to F* if and only if

$$\mathrm{HNP}(E, s) = \mathrm{HNP}(F, s), \quad \text{for every } s \in S.$$

PROPOSITION 12: *Let S be Noetherian with irreducible components $S_1, \ldots, S_q$ and let $P_1, \ldots, P_q$ be convex polygons beginning at $(0, 0)$ and ending at (r, d). If E, F are families of vector bundles on X lying in $\mathrm{Vect}_k^{P_1, \ldots, P_q}(Xx_kS)$, then E is HN equivalent to F if and only if $\Theta(E) = \Theta(F)$ in $\mathscr{A}(S)^{r-1}$. Hence, the fibres of Θ in each piece of $\mathrm{Vect}_k(Xx_kS)$ are the HN equivalence classes.*

PROOF: We may obviously assume S is irreducible, and we will prove both necessity and sufficiency by induction on $\dim S$.

Necessity. The statement is vacuous for $\dim S = 0$; so, we look at S of dimension p. If s is generic for S, then $\mathrm{HNP}(E, s) = P = \mathrm{HNP}(F, s)$, $(P = P_1)$. Write $S - U = \cup S_i$ as in the construction of Θ, and let s_i be generic in S_i. Since the cycle $\Theta(E)$ has S_i components given by $r!(\mathrm{HNP}(E, s_i) - \mathrm{HNP}(E, s))_j S_i$, we find that $\Theta(E)$ and $\Theta(F)$ have the same S_i components, $i = 1, 2, \ldots, q$. Therefore, $E \restriction Xx_kS_i$, $F \restriction Xx_kS_i$ belong to the same piece of $\mathrm{Vect}(Xx_kS_i)$. Since these restricted bundles are *HN* equivalent and since $\dim S_i < p$, the induction hypothesis shows that all other components of $\Theta(E)$ and $\Theta(F)$ agree. Therefore $\Theta(E) = \Theta(F)$.

Sufficiency. If $\dim S = 0$, then S is one point, and we are given $\mathrm{HNP}(E, s) = \mathrm{HNP}(F, s) = P$ at this point. So, assume $\dim S = p$, and observe that if s is generic for S, we know that $\mathrm{HNP}(E, s) = P = \mathrm{HNP}(F, s)$. The last relation holds on the open set $U \cap V$ where $\mathrm{HNP}(E, t) = P$ and $\mathrm{HNP}(F, t) = P$. (Cf. the construction of $\Theta(E)$, and apply this construction to E and F.) Suppose $t \in U$ but $t \notin V$. Then if $S - V = \cup T_i$, the cycle $\Theta(F)$ contains each T_i with a positive coefficient. Let $t \in T_i$ (say). Since $\Theta(F) = \Theta(E)$, the cycle $\Theta(E)$ contains T_i with a positive coefficient. Now $T_i \neq S_\ell$ for any ℓ as $t \notin S_\ell$ for any ℓ; hence the piece of $\Theta(E)$ equaling T_i (with positive coefficient) arises from $E \restriction Xx_kS_\ell$ for some ℓ. That is, $T_i \subseteq S_\ell$ for some ℓ; hence, $t \in S_\ell$ for some ℓ, a contradiction. This shows $U \subseteq V$, and by symmetry, $U = V$.

We have shown that the irreducible components T_i, S_ℓ are pairwise identical; label them so that $T_1 = S_1$, *etc.* Since $\Theta(E) = \Theta(F)$, we have

$$r!(\mathrm{HNP}(E, s) - \mathrm{HNP}(E, s_i))_j S_i = r!(\mathrm{HNP}(F, s) - \mathrm{HNP}(F, s_i))_j T_i$$

for $1 \leq j \leq r-1$ and $i = 1, 2, \ldots, q$. We deduce that $\mathrm{HNP}(E, s_i) = \mathrm{HNP}(F, s_i)$, $i = 1, \ldots, q$. (Here, of course, s_i is generic in $S_i = T_i$.) It follows that $E \restriction Xx_kS_i$ is in the same piece as $F \restriction Xx_kS_i$, for all i. Hence, as $\Theta(E \restriction Xx_k \cup S_i) = \Theta(F \restriction Xx_k \cup S_i)$, the induction hypothesis shows that E and F are HN equivalent. Q.E.D.

In another paper, we shall apply Prop. 12 to the study of vector bundles on ruled surfaces.

REFERENCES

[1] M. F. ATIYAH: Vector Bundles over an Elliptic Curve. *Proc. Lond. Math. Soc. 7* (1957) 414–452.

[2] D. GIESEKER: On the Moduli of Vector Bundles on an Algebraic Surface, preprint.

[3] A. GROTHENDIECK and J. DIEUDONNÉ: Éléments de Géométrie Algébrique, 4. *Pub. Math. I.H.E.S., 24,* 1965.

[4] A. GROTHENDIECK and J. DIEUDONNÉ: Éléments de Géométrie Algébrique, 2. *Pub. Math. I.H.E.S., 8,* 1961.

[5] A. GROTHENDIECK and J. DIEUDONNÉ: Éléments de Géométrie Algébrique, 2.

[6] A. GROTHENDIECK and J. DIEUDONNÉ: Éléments de Géométrie Algébrique, 3. *Pub. Math. I.H.E.S., 17,* 1963.

[7] A. GROTHENDIECK and J. DIEUDONNÉ: Éléments de Géométrie Algébrique, 4, *Pub. Math. I.H.E.S., 28,* 1966.

[8] A. GROTHENDIECK: Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Am. J. Math. 79* (1957) 121–138.

[9] A. GROTHENDIECK: *Techniques de Construction en Géométrie Algébrique IV.* Sem. Bourb. 221 (1961).

[10] G. HARDER and M. NARASIMHAN: On the Cohomology Groups of Moduli Spaces of Vector Bundles on Curves. *Math. Ann. 212* (1975) 215–248.

[11] S. KLEIMAN: Toward a Numerical Theory of Ampleness. *Ann. of Math. 84* (1966) 292–343.

[12] S. LANGTON: Valuative Criteria for Vector Bundles. *Ann. of Math. 101* (1975) 88–111.

[13] M. MARUYAMA: Openness of a family of torsion-free sheaves, preprint.

[14] D. MUMFORD: *Projective Invariants of Projective Structures and Applications.* Proc. Stockholm Cong., 1962, 526–530.

[15] M. NARASIMHAN and C. S. SESHADRI: Stable and Unitary Vector Bundles on a Compact Riemann Surface. *Ann. of Math. 82* (1965) 540–567.

[16] F. TAKEMOTO: Stable Vector Bundles on an Algebraic Surface. *Nagoya Math J.* (1972).

[17] A. TJURIN: Classification of Vector Fiberings over an Algebraic Curve of Arbitrary Genus. *Amer. Math. Soc. Transl.* (2) *63* (1967).

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