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THE ABSOLUTE CONTINUITY OF A LIMIT LAW FOR SYLVESTER SERIES*

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Every real number $x \in (0, 1]$ has a unique representation in the form

$$x = 1/d_1 + 1/d_2 + \cdots + 1/d_n + \cdots \tag{1}$$

where the $d_j = d_j(x)$ are positive integers satisfying

$$d_1 \geq 2 \quad \text{and} \quad d_{j+1} \geq d_j^2 - d_j + 1,\ j \geq 1. \tag{2}$$

The representation (1) and (2) is called the Sylvester expansion of x. It was shown in [1] that, as $n \to +\infty$,

$$\lim 2^{-n} \log d_n(x) = \beta(x) \tag{3}$$

exists and is positive. The behaviour of $\beta(x)$ drew much attention in the literature; see, in particular, Vervaat [5], pp. 148–151, who particularly stresses the need for solving the following problem (correct the typing error and thus $\log d_n$ is to replace d_n in part (b) of his statement). Put

$$F(z) = P(\beta(x) < z), \tag{4}$$

where P stands for Lebesgue measure. The question is whether the function $F(z)$ is continuous for all real z. In the present note we settle this question by proving the following result.

THEOREM. *The distribution function $F(z)$ of* (4) *is absolutely continuous. Its derivative $f(z)$ is continuous for all z.*

In the proof, we need the following lemmas.

* This research was done while the author was on Research and Study Leave from Temple University and, as a Fellow of the Humboldt Foundation, he was at the Goethe University, Frankfurt am Main.

LEMMA 1: *Let $\varphi(t)$ be the characteristic function of $F(z)$. Assume that $|\varphi(t)|$ is integrable over the whole real line. Then the derivative $f(z)$ of $F(z)$ exists and is continuous for all z.*

Its proof can be found in [4], p. 267.

The following lemma was obtained in [2], p. 188.

LEMMA 2: *$\beta(x)$ of* (3) *has the form*

$$\beta(x) = \frac{1}{2}\log d_1(x) + \frac{1}{2}\sum_{n=1}^{+\infty} 2^{-n} \log \frac{d_{n+1}}{d_n^2}.$$

Our last lemma was independently discovered by the present author [3], p. 138 and by Vervaat [5], p. 111.

LEMMA 3: *Put in* (1)

(5) $$x_n = 1/d_n + 1/d_{n+1} + \cdots.$$

Then $y_n = d_n(d_n - 1)x_{n+1}$ is uniformly distributed on the interval $(0, 1)$. *Furthermore, y_n is stochastically independent of $d_1, d_2, \ldots, d_n$.*

PROOF OF THE THEOREM: Put

$$\beta_N(x) = \frac{1}{2}\log d_1(x) + \frac{1}{2}\sum_{n=1}^{N} 2^{-n} \log \{d_{n+1}(x)/d_n^2(x)\}$$

and

$$\varphi_N(t) = \int_0^1 \exp\{it\beta_N(x)\}\, dx,\ t \text{ real}.$$

By Lemma 2,

(6) $$\varphi(t) = \lim \varphi_N(t) \quad (N \to +\infty).$$

Our aim now is to get a recursive formula for $\varphi_N(t)$. This is done by plugging in y_N into the formula for $\beta_N(x)$ and we then apply Lemma 3. We write

(7) $$\frac{d_{N+1}}{d_N^2} = \frac{d_{N+1}x_{N+1}}{y_N} \cdot \frac{d_N - 1}{d_N}.$$

Since, by (2) and (5),

$$1 < d_{N+1}x_{N+1} \leq d_{N+1}/(d_{N+1} - 1),$$

(7) reduces to

$$d_{N+1}/d_N^2 = (1/y_N)(1 + \vartheta/d_N), \quad |\vartheta| \leq 1.$$

Hence,

(8) $$\beta_N(x) = \beta_{N-1}(x) - 2^{-N-1} \log y_N + 2^{-N-1} \log(1 + \vartheta/d_N),$$

where $|\vartheta| \leq 1$. Before we turn to $\varphi_N(t)$, we estimate the last term of (8), in order to make it independent of x. First define the numbers D_n by

$$D_1 = 2,\ D_{n+1} = D_n^2 - D_n + 1, \quad n \geq 1.$$

In view of (2), $d_n(x) \geq D_n$ for all n and all x. Thus

(9) $$|\log(1 + \vartheta/d_N)| \leq 2/D_N.$$

We now appeal to Lemma 3. It says that

(10) $$\int_0^1 \exp\{it \log(1/y_N(x))\}\, dx = 1/(1 - it).$$

Also, since $\beta_{N-1}(x)$ depends on $d_1, d_2, \ldots, d_N$ only, $\beta_{N-1}(x)$ and $y_N(x)$ are independent. Therefore, as is well known from the elements of probability theory, (8)–(10) yield

(11) $$\varphi_N(t) = \varphi_{N-1}(t)(1 - it/2^{N+1})^{-1}(1 + \vartheta_N t/2^{N+1} D_N),$$

where $|\vartheta_N| \leq 2$. We thus have from (6) and (11) that, for an arbitrary integer $M \geq 1$ and with a suitable sequence ϑ_N of numbers which satisfy $|\vartheta_N| \leq 2$,

$$\varphi(t) = \varphi_M(t) \prod_{N=M+2}^{+\infty} \{(1 - it/2^N)^{-1}(1 + \vartheta_{N-1} t/2^N D_{N-1})\}.$$

From the above formula, one can easily estimate the tails of $\varphi(t)$ by making use of the fact that $\log D_N > 2^{N-3}$ and that $|\varphi_M(t)| \leq 1$. Combining these estimates with the fact that, as any characteristic function, $\varphi(t)$ is continuous, we obtain that $|\varphi(t)|$ is integrable on the whole real line. Lemma 1 therefore completes the proof.

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