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Let G be a finitely generated group, let $\mathfrak{E}(G)$ be the compact space of its ends. The group G acts on $\mathfrak{E}(G)$. If the cardinality of $\mathfrak{E}(G)$ is infinite Freudenthal [5] showed that there is at most one fixed point in $\mathfrak{E}(G)$. He left open the following

QUESTION: Are there finitely generated groups G with a fixed point in $\mathfrak{E}(G)$ and $\mathfrak{E}(G)$ infinite? The answer is *no.*

This follows from Stallings's (cf. [7]) structure theorem on finitely generated groups with infinitely many ends. Together with Freudenthal's result we have a more precise statement. A G-space X is called *minimal* if any closed G-stable subset of X is empty or equal to X, equivalently: if the orbit of any point $x \in X$ is dense in X.

THEOREM 1: *Let G be a finitely generated group. The space $\mathfrak{E}(G)$ of ends of G is a minimal G-space, except when $\mathfrak{E}(G)$ consists of two points which are fixed.*

One can ask the corresponding question for locally compact groups. Here the answer is yes. The structure of the occurring groups is completely described in terms of generators and relations (s. Theorem 2 below). This is done in §2.

In §3 consequences for almost proper transformation groups are given. An action of a locally compact topological group G on a locally compact topological space is called almost proper if – roughly – it is proper except for a 0-dimensional subset. Let R_0 be the set of limit points of such an action: R_0 is – roughly – the set of non proper points. As a result of §3 together with known results [1, 2] we get a complete classification of the sets R_0 of limit points. Also the set R_0 determines the structure of the group G, except if Card $R_0 = 1$ (Theorem 4).

2

Stallings's structure theorem on finitely generated groups with infinitely many ends [7] and its generalization to locally compact topological groups [2] yield two types of groups: amalgamated free products and HNN-extensions. In this paragraph we shall discuss these two types of groups separately with respect to Freudenthal's question.

2.1. First some notation. Let A and B be subsets of a topological space X. We say A is *almost contained* in B and write $A \overset{a}{\subset} B$ if there is a compact subset K of X such that $A \subset B \cup K$. The two sets A and B are called *almost equal*: $A \overset{a}{=} B$ if $A \overset{a}{\subset} B$ and $B \overset{a}{\subset} A$.

A subset E of a locally compact topological group G is called *almost invariant* if for any non empty compact subset W of G we have $E.W \overset{a}{=} E$. The complement of a subset E of G is denoted E^*.

The Boolean algebra $\mathfrak{A}$ of all almost invariant subsets of G contains the ideal $\mathfrak{K}$ of all subsets with compact closure. The maximal (= prime) ideal space Spec $\mathfrak{A}/\mathfrak{K}$ of the Boolean algebra $\mathfrak{A}/\mathfrak{K}$ is the space $\mathfrak{E}_{\text{top gr}}(G)$ of *ends of the topological group* G, as defined by Specker [6]. For a finitely generated discrete group G this is equivalent with Freudenthal's definition of the space of ends of the group G. In general the space of ends $\mathfrak{E}_{\text{space}}(G)$ of the topological *space* G is of course different from $\mathfrak{E}_{\text{top gr}}(G)$. We consider $\mathfrak{E}_{\text{top gr}}(G)$ as a G-space under the action induced by left translations.

2.2. The first type of groups occurring in the structure theorem is an *amalgamated free product*: Let G_1 and G_2 be two locally compact (always: Hausdorff) topological groups both containing the same open compact subgroup K. Then the amalgamated free product $G = G_1 *_K G_2$ has a unique topology such that G is a topological group and K is an open compact subgroup.

2.3. LEMMA: *Suppose $K \neq G_1$ and $K \neq G_2$. Then there is an almost invariant subset E of $G = G_1 *_K G_2$ and elements $g_1, g_2 \in G$ such that $g_1E \subset E^*$, $g_2E^* \subset E$.* So *$\mathfrak{E}_{\text{top gr}}(G)$ has no fixed point.*

PROOF: Let E_i be the set of words in $G_1 *_K G_2$ starting with an element of G_i, i.e.

$$E_1 = \{g_1 \dots g_n;\, n \geq 1,\, g_1, g_3, \dots \in G_1 \setminus K,\, g_2, g_4, \dots \in G_2 \setminus K\}$$

and

$$E_2 = \{g_1 \dots g_n;\, n \geq 1,\, g_1, g_3, \dots \in G_2 \setminus K,\, g_2, g_4, \dots \in G_2 \setminus K\}.$$

We have $E_1^* = E_2 \cup K$. For $g_1 \in G_1 \setminus K$ we have $g_1 E_1^* \subset E_1$ and for $g_2 \in G_2 \setminus K$ we have $g_2 E_1 \subset E_2 \subset E_1^*$. So it remains to prove that E_1^* – or equivalently E_1 – is almost invariant. The following relations are obvious: $E_1 \cdot g_1 = E_1 \cup K$ for $g_1 \in G_1 \setminus K$, $E_1 \cdot g_2 = E_1$ for $g_2 \in G_2$. Since $G_1 \cup G_2$ generates G the sequence $(G_1 \cdot G_2)^l$, $l \in \mathbb{N}$ of open subsets of G covers G. So any non empty compact subset W of G is contained in some $(G_1 \cdot G_2)^l$, which implies $E_1 \subseteq E_1 \cdot W \overset{a}{\subseteq} E_1$. So E_1 is almost invariant.

2.4. The second type of groups occurring in the structure theorem is an HNN extension as follows: Let K be a compact open subgroup of a (locally compact) topological group G_1, let $\alpha : K \to G_1$ be a continuous open injective homomorphism. Then the HNN extension $G = G_1 *_\alpha$ defined by the presentation $\langle G_1, x;\, x^{-1} \cdot k \cdot x = \alpha(k)$ for $k \in K \rangle$ has a unique topology such that G is a topological group and K is a compact open subgroup thereof.

2.5. Lemma: *Suppose $K \neq G_1$ and $\alpha(K) \neq G_1$. Then there are three disjoint almost invariant subsets E_1, E_2, E_3 of $G = G_1 *_\alpha$ and elements g_1, g_2, g_3 of G such that $E_1 \cup E_2 \cup E_3 = G$ and $g_1 E_1 \subset E_2$, $g_2 E_2 \subset E_3$, $g_3 E_3 \subset E_1$. So $\mathfrak{E}_{\mathrm{top\,gr}}(G)$ has no fixed point.*

Note that the two cases not covered by 2.5.: (1) $K = G_1$, (2) $\alpha(K) = G_1$ coincide if G is discrete, whereas this is not the case in general. This is exactly the reason why the answer to Freudenthal's question is no for discrete groups and yes in general.

Proof: Let $T \subset G_1$ be a set of representatives for G_1/K and $T' \subset G_1$ a set of representatives for $G_1/\alpha(K)$, both containing 1. Then every element of $G_1 *_\alpha$ can be uniquely written as $g_1 x^{\epsilon_1} \dots g_n x^{\epsilon_n} g_{n+1}$ where $n \geq 0$, $\epsilon_i = \pm 1$, $g_i \neq 1$ if $\epsilon_i + \epsilon_{i-1} = 0$, and, for $i \leq n$, $g_i \in T$ if $\epsilon_i = +1$, while $g_i \in T'$ if $\epsilon_i = -1$ [4 p. 41]. Let

$$E_1 = \{u \in G_1 *_\alpha \text{ with } \epsilon_1 = 1 \text{ and } g_1 = e,\, u \text{ in the normal form}\}$$

$$E_2 = \{u \in G_1 *_\alpha \text{ with } \epsilon_1 = -1 \text{ and } g_1 = e,\, u \text{ in the normal form}\}$$

E_3 the rest: $E_3 = G \setminus (E_1 \cup E_2) = \{u;\, u \in G_1$ or $\epsilon_1 = 1$ and $g_1 \neq 1$ or

$\epsilon_1 = -1$ and $g_1 = 1\}$. Then we have

$$t'E_2 \subset E_3 \text{ for } t' \in T, t' \neq 1$$
$$xE_3 \subset E_1$$

and $x^{-1}tE_1 \subset E_2$ for $t \in T$, $t \neq 1$.

So it remains to prove that the three sets E_i are almost invariant. For E_1 we have $E_1 \cdot g = E_1$ for $g \in G_1$ and $E_1 \cdot x^{-1} = E_1 \cup K$. This implies as above that E_1 is almost invariant, similarly for E_2. Then also E_3 is almost invariant, since the set of almost invariant subsets of G form a Boolean algebra with unit.

2.6. Proof of theorem 1: The cardinality of $\mathfrak{E}(G)$ is 0, 1, 2 or infinite [5]. The theorem is trivial for card $\mathfrak{E}(G) \leq 2$. If G has infinitely many ends Stallings' structure theorem [7, cf. 4] says that G is of either of the following two types: $G = G_1 *_K G_2$ where K is a finite subgroup of both G_1 and G_2, $K \neq G_1$, $K \neq G_2$ and card$(G_1/K) > 2$ or card$(G_2/K) > 2$. Or $G = G_1 *_\alpha$ where K is a finite subgroup of G and $\alpha: K \to G_1$ is a monomorphism and $K \neq G_1$, which implies $\alpha(K) \neq G_1$. So the two lemmas 2.3. and 2.5. imply that $\mathfrak{E}(G)$ has no fixed point. A theorem of Freudenthal's then implies the theorem: $\mathfrak{E}(G)$ is minimal or contains a fixed point [5].

Almost the same proof yields

Theorem 2: *Let G be a compactly generated locally compact topological group. The space $\mathfrak{E}_{\mathrm{top\,gr}}(G)$ of ends of the topological group G is a minimal G-space except in the following two cases*: (1) *$\mathfrak{E}(G)$ consists of two fixed points.* (2) *$G = K *_\alpha$ where $\alpha: K \to K$ is an injective continuous open homomorphism which is not surjective.*

Proof: Set $R := \mathfrak{E}_{\mathrm{top\,gr}}(G)$. We have card$(R) \leq 2$ or infinite [2]. If card R is infinite G is either an amalgamated free product $G = G_1 *_K G_2$, K compact open in G_1 and G_2, $K \neq G_1$, $K \neq G_2$ and card$(G_1/K) > 2$ or card$(G_2/K) > 2$, or G is an HNN-extension $G = G_1 *_\alpha$ where K is a compact open subgroup of G_1, $\alpha: K \to G_1$ is an injective open continuous homomorphism but not an isomorphism [2]. So the lemmas 2.3. and 2.5. imply that there is no fixed point in R, except if $G_1 = K$ or $\alpha(K) = G_1$. Since $G_1 *_\alpha$ and $G_1 *_{\alpha^{-1}}$ are isomorphic we obtain $G = K *_\alpha$ in the exceptional case. Again we have that R is either a minimal G-space or contains a fixed point [2].

One can conclude theorems 1 and 2 from Oxley's proof of Stallings' structure theorem: If $\mathfrak{E}_{\mathrm{top\,gr}}(G)$ is infinite and contains a fixed point then in the notation of [4 p. 51]: $G_1 = G_2 = K = H$, $P \neq \emptyset$. The proof is applicable for compactly generated locally compact topological groups [2, §5].

3

3.1. Let G be a locally compact topological group acting continuously on a locally compact topological space X. The action is called *proper* if for any compact subset K of X the subset $\{g \in G;\ gK \cap K \neq \emptyset\}$ of G has compact closure.

3.2. Definition: A locally compact G-space Y is called *almost proper* if Y contains an open dense G-stable subspace X with the following properties

(AP 1) $R := Y \setminus X$ is 0-dimensional, i.e. the topology of R has a base of open closed sets.

(AP 2) The induced action of G on X is proper.

(AP 3′) X is connected, locally connected and σ-compact.

This definition is slightly stronger than that of [3]. The two definitions coincide if Y is locally connected and separable. A standard example of an almost proper G-space is the end point compactification $\hat{X}$ of a proper G-space X satisfying (AP 3′).

A point y of a G-space Y is called a *limit point* if there is a point $z \in Y$, a filter $\mathfrak{F}$ on G that does not converge in G such that $\mathfrak{F}(z) = \{F(z); F \in \mathfrak{F}\}$ converges to y.

3.3. Theorem 3: *Let Y be a compact almost proper G-space. If the set R_0 of limit points is infinite and contains a fixed point, then $G = K *_\alpha$ where K is a compact group and $\alpha: K \to K$ is a continuous open injective non-surjective homomorphism.*

The conserve is part of remark 3.5.

Proof: It was shown in [3] that the concept of limit point adopted here is the same as that of [1, 2]. If R_0 is infinite we have the same structure theorem as in the proof of theorem 2. Conversely: For any such group the space $\mathfrak{E}_{\text{top gr}}(G)$ is infinite. Suppose G has not the exceptional structure described in theorem 3. Then any compactly generated subgroup of G is contained in a subgroup of the inductive system S of all subgroups H of G with $\mathfrak{E}_{\text{top gr}}(H)$ an infinite minimal H-space.

To any almost proper action of a locally compact group G belongs a Specker compactification $\hat{G}$ of G; $\hat{G} = G \cup R_0$ is topologized such that for any $x \in X$ the mapping φ_x of G to the orbit Gx extends to a continuous mapping $\hat{\varphi}_x: \hat{G} \to \overline{Gx}^Y = Gx \cup R_0$ with $\hat{\varphi}_x | R_0 = id_{R_0}$ (s. [1]).

It is known [2, Lemma 7.1.] that $\cup_{H \in S}(\bar{H} \setminus H)$ is dense in R_0 where the bar denotes closure in $\hat{G}$. Since $\bar{H}$ is a Specker compactification of H we have card$(\bar{H} \setminus H) \leq 2$ or infinite. Since R_0 is infinite, $\bar{H} \setminus H$ is infinite for some $H \in S$. There is a unique H-mapping from the universal Specker compactification $H \cup \mathfrak{E}_{\text{top gr}}(H) \to \bar{H}$ extending the identity on H. Since H is dense in $\bar{H}$ the induced H-mapping $\mathfrak{E}_{\text{top gr}}(H) \to \bar{H} \setminus H$ is surjective and $\bar{H} \setminus H$ is a minimal H-space.

On the other hand we may consider Y as an almost proper H-space. The corresponding Specker compactification of H is $\bar{H}$. So the set $R_0(H)$ of H-limit points in Y is an infinite minimal H-space as it is H-homeomorphic to $\bar{H} \setminus H$.

Now suppose Y contains a G-fixed point y_0, this point is of course H-fixed. But for any locally compact non compact group H and any compact almost proper G-space, any point $y \in Y$ is H-fixed or the closure of its orbit contains $R_0(H)$ [1, 4.11.6.]. So $y_0 \in R_0(H)$ which contradicts the fact that $R_0(H)$ is an infinite minimal H-space.

The theorem just cited actually shows that $R_0(G)$ is a minimal G-space.

Together with results from [1, 2] we obtain the following complete classification of sets of limit points:

3.4. Theorem 4: *Let Y be an almost proper G-space. Let R_0 be its set of limit points. G acts properly on the open G-subspace $Y \setminus R_0$*

(C) *Suppose Y is compact. Then R_0 is one of the following spaces*

(C.0) *R_0 is empty. This is equivalent to G being compact.*

(C.1) *R_0 consists of one point.*

(C.2) *R_0 consists of two points. Then the orbit space $G \backslash (Y \setminus R_0)$ is compact. There are two subcases:*

(C.2.a) *G acts trivially on R_0. Then G has a – unique maximal – compact normal subgroup K such that G/K is topologically isomorphic to $\mathbb{R}$ or $\mathbb{Z}$.*

(C.2.b) *G acts non trivially on R_0. Then the kernel of this action $H = \{g \in G; g(y) = y \text{ for } y \in R_0\}$ has index 2 in G and H has the structure of* (C.2.a). *If K is the maximal compact normal subgroup of H, G/K is the split extension of H/K with G/H, G/H acting non trivially on H/K.*

(C.3) *R_0 is a Cantor discontinuum.*

(C.3.a) *R_0 is a minimal G-space. Then*

(α) *G is an amalgamated free product $G = G_1 *_K G_2$, where K is an open subgroup of both G_1 and G_2, $K \neq G_1$, $K \neq G_2$ and not $card(G_1/K) = card(G_2/K) = 2$ or*

(β) *G is an HNN extension $G = G_1 *_\alpha$ where K is an open compact subgroup of G_1 and $\alpha\colon K \to G_1$ is a continuous open injective homomorphism and $K \neq G_1$ and $\alpha(K) \neq G_1$*

(C.3.b) *R_0 contains exactly one fixed point. The closure of the orbit of all other points of R_0 is R_0. Then $G = K *_\alpha$ where K is a compact group and $\alpha\colon K \to K$ is an injective open continuous non surjective homomorphism.*

(N) *If Y is non compact the one point compactification $Y^\cdot$ of Y is a compact almost proper G-space and the added point $\infty = Y^\cdot \setminus Y$ is a fixed point. So we have the following cases*

(N0) *R_0 is empty. The action on Y is proper.*

(N1) *R_0 consists of one point. This corresponds to* (C.2.a) *for $Y^\cdot$.*

(N2) *R_0 is a Cantor discontinuum minus one point. This corresponds to* (C.3.b) *for $Y^\cdot$.*

3.5. Remark: All these cases occur. More precisely: For any compactly generated group as mentioned in the theorem under (C.n.x), $n \neq 1$, there is an almost proper G-space such that the space R_0 of limit points has the properties mentioned under (C.n.x), actually there is one with $R_0 = \mathfrak{E}_{\text{top gr}}(G)$.

Proof of 3.5: For $n = 0$: G compact, take Y = one point. For $n = 2$ take $X = \mathbb{R}$. In case (C.2.a) let $G/K = \mathbb{Z}$ or $\mathbb{R}$ act on $\mathbb{R}$ by translations. In case (C.2.b) let H/K act on $\mathbb{R}$ by translations and for an element $g \in G/K \setminus H/K$ of order two define $g(x) = -x$ for $x \in \mathbb{R}$. This extends to a proper action of G on $\mathbb{R}$. The extended action of G on the end point compactification Y of $\mathbb{R}$ is almost proper and the following G-spaces are G-homeomorphic: $\mathfrak{E}_{\text{space}}(\mathbb{R}) = R_0 \cong \mathfrak{E}_{\text{top gr}}(G)$ and have the properties (C.2.a) and (C.2.b) resp. For $n = 3$ let X be the geometrical realization of the analogue of the Cayley diagram of G (cf. [2, Beispiel 2.7]) with the natural G-action. Let Y be the end point compactification of X. The following G-spaces are G-homeomorphic: $\mathfrak{E}_{\text{space}}(X) = R_0 \cong \mathfrak{E}_{\text{top gr}}(G)$. The orbit of any point $y \in R_0$ is y or dense in R_0 [1, 4.11.6]. So by theorems 2 and 4 it remains to show that $\mathfrak{E}_{\text{top gr}}(G)$ contains a fixed point in the exceptional case (C.3.b): $G = K *_\alpha = \langle K, x; x^{-1} \cdot k \cdot x = \alpha(k)$ for $k \in K\rangle$. The infinite cyclic subgroup H of G generated by x has two ends. One of them is $y_+ = \lim_{n\to\infty} x^n \in H \cup \mathfrak{E}_{\text{top gr}}(H) = \hat{H}$. The inclusion mapping $H \to G$ extends to a continuous mapping of the universal Specker compactifications $\hat{H} \to \hat{G} = G \cup \mathfrak{E}_{\mathrm{top\,gr}}(G)$. Let y be the image of y_+. Then $y = \lim_{n\to\infty} x^n$. For $k \in K$ we have $k \cdot x^n \in x^n \cdot K$ and $\lim_{n\to\infty} x^n \cdot K = y$ (Axiom (R) for a Specker compactification [2]). Since $x \cdot \lim_{n\to\infty} x^n = \lim_{n\to\infty} x^n$, y is G-fixed. 3.5. implies

3.6. The types of groups (C.n.x), $n \neq 1$, are mutually disjoint.

Because a non compactly generated group belongs to (C.3.a) Theorem 4 implies

3.7. For a group of type (C.n.x), $n > 1$, the set R_0 of limit points is as described in (C.n.x) or consists of one point.

3.8. If G is locally connected, e.g. Lie, G is not of type (C.3.b).

PROOF: Suppose $G = K *_\alpha$ as in (C.3.b) is locally connected. The connected component G_1 of G is an open normal subgroup of K, hence of finite index in K. The image $\alpha(G_1)$ is an open connected subgroup of K, so $\alpha(G_1) = G_1$. The induced homomorphism of the finite group $K/G_1 \to K/G_1$ is injective, thus surjective, which implies that α is surjective, a contradiction.

3.9. Groups as in (C.3.b) occur in nature, e.g. the group $G(\mathbb{Q}_p)$ of affine mappings of the line over the p-adic field $\mathbb{Q}_p$. $G(\mathbb{Q}_p) = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right);\ a \in \mathbb{Q}_p \setminus \{0\},\ b \in \mathbb{Q}_p\}$. For $K = G(\mathbb{Z}_p) = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right);\ a,\ a^{-1} \in \mathbb{Z}_p,\ b \in \mathbb{Z}_p\}$, $\alpha\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & p\cdot b \\ 0 & 1 \end{smallmatrix}\right)$, the mapping $K *_\alpha \to G(\mathbb{Q}_p)$ defined by $x \to \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\alpha\colon K \to G(\mathbb{Q}_p)$, is an isomorphism.

This example leads to conjectures on the structure of groups $G = K *_\alpha$ as in (C.3.b), which are then easily proved: Let G_0 be the kernel of the homomorphism $K *_\alpha \to \mathbb{Z}$, $K \to 0$, $x \to 1$. Then $G_0 = \bigcup_{n\in\mathbb{N}} x^n K x^{-n}$. Any compact subset of G_0 is contained in a compact subgroup. Any compact subgroup of G is contained in a conjugate of K.

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