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ON $\lambda(\phi, P)$ -NUCLEARITY

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Introduction

We consider sequence spaces $\lambda(\phi, P, \infty)$ and $\lambda(\phi, P, k)$ and define $\lambda(\phi, P, \infty)$ -nuclear resp. $\lambda(\phi, P, \mathbb{N})$ -nuclear spaces in order to unify the concepts of $\lambda(P)$ -nuclearity, $\Lambda_n(\alpha)$ -nuclearity, and ϕ -nuclearity considered in [7, 8, 9]; we are especially interested in obtaining universal generators for the varieties of $\lambda(\phi, P, \infty)$ -nuclear and $\lambda(\phi, P, \mathbb{N})$ -nuclear spaces. Section 1 of the paper contains various definitions and some remarks on the operator ideal of $\lambda(P, \infty)$ -nuclear maps, P a stable, nuclear G_∞ -set; in section 2 we consider $\lambda(\phi, P, \infty)$ -nuclear spaces and show that $\lambda(\phi, P, \infty)$ -nuclearity is the same as $\lambda(Q, \infty)$ -nuclearity, Q a suitably chosen G_∞ -set. In section 3 we introduce the concept of $\lambda(\phi, P, \mathbb{N})$ -nuclearity and extend a result in [8] by showing that $\lambda(Q, \infty) - Q$ suitably chosen G_∞ -set – is a universal generator for the variety of $\lambda(\phi, P, \mathbb{N})$ -nuclear spaces whenever P is a countable, monotone, stable, nuclear G_∞ -set.

1. Definitions, notations, and some remarks on $\lambda(P, \infty)$ -nuclear maps

For terminology and notations not explained here we refer to Köthe [3], Pietsch [4], Dubinsky and Ramanujan [1], and Terzioglu [12].

Let X and Y be Banach spaces, λ a normal sequence space, and λ^\times

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its Köthe-dual. A continuous linear map $T \in L(X, Y)$ is said to be
 (i) λ -nuclear (written $T \in N_\lambda(X, Y)$) if there exists a representation

$$Tx = \sum_{n=0}^{\infty} \gamma_n \langle x, a_n \rangle y_n \quad \text{for } x \in X$$

with $\{\gamma_n\}_n \in \lambda$, $a_n \in X'$, $\|a_n\| \leq 1$, and $y_n \in Y$, $\{\langle y_n, b \rangle\}_n \in \lambda^\times$ for each $b \in Y'$;

(ii) pseudo- λ -nuclear or $\tilde{\lambda}$ -nuclear (written $T \in \tilde{N}_\lambda(X, Y)$) if T has a representation

$$Tx = \sum_{n=0}^{\infty} \gamma_n \langle x, a_n \rangle y_n \quad \text{for } x \in X$$

with $\{\gamma_n\}_n \in \lambda$, $a_n \in X'$, $\|a_n\| \leq 1$, $y_n \in Y$, and $\|y_n\| \leq 1$;

(iii) of type λ if $\{s_n^{\text{app}}(T)\}_n \in \lambda$ where $s_n^{\text{app}}(T)$ denotes the n -th approximation number of T .

For a locally convex Hausdorff space (l.c.s.) E , $\mathcal{U}(E)$ will denote a neighbourhood base of 0 of absolutely convex, closed sets; E_U will denote the completion of the normed space $E/p_U^{-1}(0)$ $U \in \mathcal{U}(E)$; $\delta_n(V, U)$ denotes the n -th Kolmogorov-diameter of $V \in \mathcal{U}(E)$ with respect to $U \in \mathcal{U}(E)$; $\Delta(E)$ denotes the Δ -diametral dimension of E , viz., the sequences $\{\gamma_n\}_n$ such that given $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ with $\gamma_n \delta_n(V, U) \rightarrow 0$.

Let $\lambda(P)$ be a Köthe space with its generating Köthe set P . The Köthe set P is called a power set of infinite type if it satisfies the following additional conditions:

- (i) for each $a \in P$, $0 < a_n \leq a_{n+1}$, $n \in \mathbb{N}$;
- (ii) for each $a \in P$, there exists a $b \in P$ with $a_n^2 \leq b_n$, $n \in \mathbb{N}$.

The corresponding space $\lambda(P, \infty)$ is called a smooth sequence space of infinite type or a G_∞ -space. For an example of a G_∞ -space which is not a power series space of infinite type see [1; theorem 2.25].

Throughout, $\lambda(P, \infty)$ is assumed to be a G_∞ -space. The nuclearity and related concepts of such spaces are discussed in [1, 12, 13]; we only need the following result.

1.1 LEMMA: $\lambda(P, \infty)$ is nuclear if and only if there exists a sequence $\{p_n\}_n \in P$ such that $\{1/p_n\}_n \in l_1$.

We shall frequently say “ P is a nuclear G_∞ -set” to mean that the corresponding $\lambda(P, \infty)$ is a nuclear G_∞ -space; P is said to be a countable, monotone, nuclear G_∞ -set if $P = \{p_n^i\}_n$: $i = 1, 2, \dots$ with

$p_n^i \leq p_n^{i+1}$ for each $i, n \in \mathbb{N}$ and P is a nuclear G_∞ -set; note that P is a G_∞ -set already implies $0 < p_n^i \leq p_{n+1}^i$.

Let $\lambda(P, \infty)$ be nuclear; then a l.c.s. E is said to be $\lambda(P, \infty)$ -nuclear if for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ such that V is absorbed by U and the canonical map $K(V, U)$ on E_V to E_U is a $\lambda(P, \infty)$ -nuclear map. In [1] it is shown that a l.c.s. E is $\lambda(P, \infty)$ -nuclear if and only if for each $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ such that $\{\delta_n(V, U)\}_n \in \lambda(P, \infty)$. We shall denote the class of all $\lambda(P, \infty)$ -nuclear spaces by $\mathcal{N}_{\lambda(P, \infty)}$.

A l.c.s. E is said to be *stable* if $E \times E$ is isomorphic to E . It is known that a nuclear G_∞ -space is stable if and only if for each $p \in P$ there exists a $q \in P$ such that $\{p_{2n}/q_n\}_n \in l_\infty$ [14]. We say “ P is a stable G_∞ -set” to mean that $\lambda(P, \infty)$ is a stable G_∞ -space. Stability plays an important role in the study of various permanence properties of $\lambda(P, \infty)$ -nuclear spaces; the following result can be found in [7; Proposition 4.5].

1.2 PROPOSITION: *Let P be a countable, monotone, nuclear G_∞ -set, $P = \{p_n^i : i = 1, 2, \dots\}$. Then the following statements are equivalent:*

- (i) *For each $j \in \mathbb{N}$ there exists a $\gamma(j) \in \mathbb{N}$ such that $\{p_{2n}^j/p_n^{\gamma(j)}\}_n \in l_\infty$.*
- (ii) *If $\xi^k \in \lambda(P, \infty)$ for each $k \in \mathbb{N}$ and $\beta: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is a bijection defined by $\beta^{-1}(k, m) = 2^{k-1}(2m - 1)$ then there exist $t_k > 0, k \in \mathbb{N}$, such that the sequence $\{t_{\beta_1(n)} \xi_{\beta_2(n)}^{\beta_1(n)}\}_n \in \lambda(P, \infty)$ where $\beta(n) = (\beta_1(n), \beta_2(n))$.*
- (iii) *If $\xi, \eta \in \lambda(P, \infty)$ and $\zeta = \xi * \eta = (\xi_1, \eta_1, \xi_2, \eta_2, \dots)$ then there exists a bijection $\pi: \mathbb{N} \rightarrow \mathbb{N}$ such that $\{\xi_{\pi(n)}\}_n \in \lambda(P, \infty)$.*
- (iv) $\mathcal{N}_{\lambda(P, \infty)}$ *is closed under the operation of forming countable direct sums.*
- (v) $\mathcal{N}_{\lambda(P, \infty)}$ *is closed under the operation of forming finite Cartesian products.*
- (vi) $\mathcal{N}_{\lambda(P, \infty)}$ *is closed under the operation of forming arbitrary Cartesian products.*
- (vii) *The sum of two $\lambda(P, \infty)$ -nuclear maps is a $\lambda(P, \infty)$ -nuclear map.*
- (viii) $\lambda(P, \infty)$ *is stable.*

An operator ideal A is said to be

- (i) *surjective* if for each closed subspace N of a Banach space X and each $T \in L(X/N, Y)$, Y a Banach space, $TQ_N^X \in A(X, Y)$ implies $T \in A(X/N, Y)$, $Q_N^X: X \rightarrow X/N$ denotes the canonical map onto X/N ;

(ii) *injective* if for each closed subspace M of a Banach space Y and each $T \in L(X, M)$, X a Banach space, $J_M^X T \in A(X, Y)$ implies $T \in A(X, Y)$, $J_M^X: M \rightarrow Y$ denotes the injection.

The following result shows that the operator ideal of $\lambda(P, \infty)$ -nuclear maps – P a stable, countable, monotone, nuclear G_∞ -set – is injective and surjective.

1.3 PROPOSITION: *Let P be a stable, countable, monotone, nuclear G_∞ -set, X and Y be Banach spaces with closed unit balls U_X and U_Y . Then the following statements are equivalent.*

- (i) $T \in L(X, Y)$ is $\lambda(P, \infty)$ -nuclear.
- (ii) $T \in L(X, Y)$ is of type $\lambda(P, \infty)$.
- (iii) $\{s_n^{\text{gel}}(T)\}_n \in \lambda(P, \infty)$ where $s_n^{\text{gel}}(T) := \inf \{\|TJ_M^X\| : \text{codim } M < n, M \text{ closed subspace of } X\}$ denotes the n -th Gelfand-number.
- (iv) $\{s_n^{\text{kol}}(T)\}_n \in \lambda(P, \infty)$ where $s_n^{\text{kol}}(T) := \delta_n(TU_X, U_Y)$.

PROOF: (i) \Leftrightarrow (ii) is shown in [1] and [7]. (ii) \Rightarrow (iii) and (ii) \Rightarrow (iv) are consequences of the facts that $s_n^{\text{gel}}(T) \leq s_n^{\text{app}}(T)$ and $s_n^{\text{kol}}(T) \leq s_n^{\text{app}}(T)$ for each $n \in \mathbb{N}$ [5]. (iii) \Rightarrow (ii) and (iv) \Rightarrow (ii) are consequences of the facts that $s_n^{\text{app}}(T) \leq (n+1) s_n^{\text{gel}}(T)$ and $s_n^{\text{app}}(T) \leq (n+1) s_n^{\text{kol}}(T)$ for each $n \in \mathbb{N}$ [5] and the well known fact that $\{(n+1)\xi_n\}_n \in \lambda(P, \infty)$ for $\{\xi_n\}_n \in \lambda(P, \infty)$.

1.4 COROLLARY: *Let P be a stable, countable, monotone, nuclear G_∞ -set. Then the operator ideal of $\lambda(P, \infty)$ -nuclear maps is injective and surjective.*

PROOF: Let X and Y be Banach spaces, M a closed subspace of Y , and X/N a quotient space of X . Then $s_n^{\text{gel}}(T) = s_n^{\text{gel}}(J_M^Y T)$ for all $T \in L(X, M)$ and $s_n^{\text{kol}}(SQ_N^X) = s_n^{\text{kol}}(S)$ for all $S \in L(X/N, Y)$ [5]. Now the proof easily follows from Proposition 1.3.

1.5 COROLLARY: *Let P be a stable, countable, monotone, nuclear G_∞ -set, X and Y be Banach spaces such that each map $T \in L(X, Y)$ is $\lambda(P, \infty)$ -nuclear. Then X or Y must be finite-dimensional.*

PROOF: By Proposition 1.3 every operator $T \in L(X, Y)$ is of type $\lambda(P, \infty)$, hence of type l_1 . By a result of Pietsch [6], X or Y must be finite-dimensional.

REMARK: With methods used in [11], it can be also shown that X or Y is finite-dimensional if each nuclear map $T \in L(X, Y)$ is $\lambda(P, \infty)$ -nuclear, P as in Corollary 1.5.

2. On $\lambda(\phi, P, \infty)$ -Nuclearity

Throughout, let Φ denote the set of all functions $\phi: [0, \infty] \rightarrow [0, \infty]$ which are continuous, strictly increasing, subadditive with $\phi(0) = 0$ and $\phi(1) = 1$, and satisfy the additional condition (+) there exist constants $M \geq 1$ and $t_\phi \in [0, \infty]$ such that $\phi(\sqrt{t}) \leq \sqrt{\phi(Mt)}$ for $t \in [0, t_\phi]$.

Note that all examples given in [10;2.6] do have the additional property (+). So far we have been unable to find an example of a continuous, strictly increasing, subadditive function $\phi: [0, \infty] \rightarrow [0, \infty]$ with $\phi(0) = 0$ which does not fulfill condition (+).

2.1 DEFINITION: Let $\lambda(P, \infty)$ be nuclear. For $\phi \in \Phi$ define the sequence space $\lambda(\phi, P, \infty)$ by

$$\lambda(\phi, P, \infty) = \{ \{ \xi_n \}_n : \{ \phi(|\xi_n|) \}_n \in \lambda(P, \infty) \}.$$

A l.c.s. E is called $\lambda(\phi, P, \infty)$ -nuclear if for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ such that $\{ \delta_n(V, U) \}_n \in \lambda(\phi, P, \infty)$.

For $\phi = \text{id}$ this definition agrees with the definition of a $\lambda(P, \infty)$ -nuclear space, so we write $\lambda(P, \infty)$ -nuclear instead of $\lambda(\text{id}, P, \infty)$ -nuclear.

We now show that the concept of $\lambda(\phi, P, \infty)$ -nuclearity is exactly the same as the concept of $\lambda(Q, \infty)$ -nuclearity where Q is suitably chosen depending on ϕ .

2.2 LEMMA: Let P a countable, monotone, nuclear G_∞ -set, $P = \{ \{ p_n^i \}_n : i = 1, 2, \dots \}$. Take $\phi \in \Phi$; define q_n^i by $1/q_n^i := \phi^{-1}(1/p_n^i)$, $i \in \mathbb{N}$. Then $Q := \{ \{ q_n^i \}_n : i = 1, 2, \dots \}$ is a countable, monotone, nuclear G_∞ -set.

PROOF: (i) It is obvious that Q is countable and monotone and that for each n , $i \in \mathbb{N}$ $q_n^i > 0$ and $q_n^i \leq q_{n+1}^i$.

(ii) Given $q^k, q^l \in Q$, we have to show the existence of a $q^r \in Q$ such that $q^k q^l < q^r$, i.e. $q_n^k q_n^l \leq M q_n^r$ for a constant $M > 0$ and each $n \in \mathbb{N}$. Since P is a Köthe set we find $p^m \in P$ and $M_1 > 0$ such that $p_n^k \leq M_1 p_n^m$ and $p_n^l \leq M_1 p_n^m$ for each $n \in \mathbb{N}$. Since $\lambda(P, \infty)$ is nuclear,

there exists a $p^s \in P$ with $\{p_n^m/p_n^s\}_n \in l_1$. This can be seen as follows. By Lemma 1.1 there exists a $p^j \in P$ with $\{1/p_n^j\}_n \in l_1$; we choose $p^h \in P$ with $p_n^j \leq M_2 p_n^h$ and $p_n^m \leq M_2 p_n^h$ for $n \in \mathbb{N}$, we also choose $p^s \in P$ with $(p_n^h)^2 \leq p_n^s$ for all $n \in \mathbb{N}$. (This can be done by condition (ii) of a G_∞ -set.) From the inequalities

$$p_n^m/p_n^s \leq p_n^m/(p_n^h)^2 \leq M_2/p_n^h \leq M_2^2/p_n^j$$

we get $\{p_n^m/p_n^s\}_n \in l_1$. We also find an integer n_0 such that $p_n^k/p_n^s \leq 1$, $p_n^l/p_n^s \leq 1$, and $1/p_n^s < t_\phi$ for each $n \geq n_0$ (here t_ϕ is determined by condition (+)). Because of condition (+) for the function ϕ we then have

$$M_\phi^{-1}(1/p_n^k)\phi^{-1}(1/p_n^l) \geq M\phi^{-1}(1/p_n^s)\phi^{-1}(1/p_n^s) \geq \phi^{-1}(1/p_n^s p_n^s)$$

for each $n \geq n_0$. Again because of the nuclearity of $\lambda(P, \infty)$ and the second additional condition for a G_∞ -set we find $p^r \in P$, $n_1 \in \mathbb{N}$, $n_1 \geq n_0$, such that $p_n^s p_n^s \leq p_n^r$ for each $n \geq n_1$. Therefore there exists $K > 0$ such that $\phi^{-1}(1/p_n^r) \leq K\phi^{-1}(1/p_n^k)\phi^{-1}(1/p_n^l)$ for each $n \in \mathbb{N}$, i.e. $q^k q^l < q^r$.

(iii) To prove the nuclearity of $\lambda(Q, \infty)$ we use Lemma 1.1 and show the existence of a sequence $\{q_n^k\}_n \in Q$ such that $\{1/q_n^k\}_n \in l_1$. Since $\lambda(P, \infty)$ is nuclear we can find $p^k \in P$ such that $\{1/p_n^k\}_n \in l_1$. Since ϕ is subadditive we have $1/q_n^k = \phi^{-1}(1/p_n^k) \leq M/p_n^k$ for each $n \in \mathbb{N}$, and $\lambda(Q, \infty)$ is nuclear.

If $\lambda(P, \infty)$ is a power series space of infinite type then $\lambda(Q, \infty)$ is in general not a power series space as the following example shows.

EXAMPLE: Define $p^i \in P$ by $p_n^i := n^i$, $n, i \in \mathbb{N}$. Let $\phi \in \Phi$ be the function ϕ_{\log} as given in [10], i.e.

$$\phi_{\log}(t) := \begin{cases} 0 & \text{for } t = 0 \\ -\alpha/\log t & \text{for } t \in (0, t_0], t_0 \text{ sufficiently small} \\ \beta t + \gamma & \text{for } t \geq t_0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ are chosen in such a way that ϕ_{\log} is continuous with $\phi(1) = 1$. Then $\phi_{\log} \in \Phi$ and $\phi_{\log}^{-1}(t) = \exp(-\alpha/t)$ for $t \in (0, t_0]$. Therefore $q_n^i = \exp(\alpha n^i)$ for $n \geq n_0, i \in \mathbb{N}$, and $\lambda(Q, \infty)$ is not a power series space [1].

The following lemma is well known and can be found in [4].

2.3 LEMMA: Let P be a countable, monotone, nuclear G_∞ -set. Then $x = \{\xi_n\}_n \in \lambda(P, \infty)$ if and only if $\{\xi_n p_n^i\}_n \in l_\infty$ for each $i \in \mathbb{N}$. The

topologies determined by the semi-norms $\pi_i(x) := \sum_n |\xi_n| p_n^i$ resp. $\sigma_i(x) := \sup |\xi_n| p_n^i$, $i \in \mathbb{N}$, are equivalent.

2.4 PROPOSITION: *Let P be a countable, nuclear G_∞ -set, $\phi \in \Phi$. Then the following statements are true.*

- (i) $\lambda(\phi, P, \infty) = \lambda(Q, \infty)$
- (ii) *On $\lambda(\phi, P, \infty)$ the two topologies given by the F -norms $\tilde{\pi}_i(x) := \sum_n \phi(|\xi_n|) p_n^i$ resp. $\tilde{\sigma}_i(x) := \sup \phi(|\xi_n|) p_n^i$, $i \in \mathbb{N}$, are equivalent.*
- (iii) $\lambda(\phi, P, \infty)$ with the above topologies is topologically equivalent to $\lambda(Q, \infty)$.

PROOF: (ii) For $x = \{\xi_n\}_n \in \lambda(\phi, P, \infty)$ we always have $\tilde{\sigma}_i(x) \leq \tilde{\pi}_i(x)$, $i \in \mathbb{N}$. On the other hand for $i \in \mathbb{N}$ there exists $p^r \in P$ and $p^l \in P$ such that $\{1/p_n^r\} \in l_1$ and $p^i p^r < p^l$; therefore

$$\begin{aligned} \tilde{\pi}_i(x) &= \sum_n \phi(|\xi_n|) p_n^i = \sum_n \phi(|\xi_n|) p_n^i p_n^r / p_n^r \leq M \sum_n \phi(|\xi_n|) p_n^l / p_n^r \\ &\leq M_0 \tilde{\sigma}_i(x) \quad \text{for } x \in \lambda(\phi, P, \infty). \end{aligned}$$

To prove (i) and (iii) notice that because of Lemma 2.3 $x = \{\xi_n\}_n \in \lambda(\phi, P, \infty) \Leftrightarrow \phi(|\xi_n|) p_n^k \leq 1$ for $k \in \mathbb{N}$ and $n \geq n_k \Leftrightarrow |\xi_n| \leq \phi^{-1}(1/p_n^k) = 1/q_n^k$ for $k \in \mathbb{N}$ and $n \geq n_k \Leftrightarrow x \in \lambda(Q, \infty)$.

2.5 COROLLARY: *Let P be a countable, nuclear G_∞ -set. $\phi \in \Phi$. Then $[\lambda(Q, \infty)]'_b = [\lambda(\phi, P, \infty)]'_b$.*

As an immediate consequence of Proposition 4.6 in [7] we finally get

2.6 PROPOSITION: *Let P be a stable, countable, monotone, nuclear G_∞ -set; $\phi \in \Phi$. Let E be a l.c.s. Then the following statements are equivalent.*

- (i) E is $\lambda(\phi, P, \infty)$ -nuclear.
- (ii) E is $\lambda(Q, \infty)$ -nuclear.
- (iii) E is isomorphic to a subspace of a suitable I -fold product $[\lambda(Q, \infty)]'_b{}^I$.

3. On $\lambda(\phi, P, \mathbb{N})$ -nuclearity

In this section we study the concepts of $\lambda(\phi, P, \mathbb{N})$ -nuclearity, $\phi \in \Phi$, and $\lambda(P, \mathbb{N})$ -nuclearity, P a countable, monotone G_∞ -set. As in

the case of $\lambda(\phi, P, \infty)$ -nuclearity we obtain that both concepts are closely related. The main result in this section is that whenever P is a countable, monotone, stable, nuclear G_∞ -set $\lambda(P, \infty)$ is a universal generator for the variety of $\lambda(P, \mathbb{N})$ -nuclear spaces. For $\lambda(P, \infty) = \Lambda_\infty(\alpha)$, α a stable exponent sequence, this result has been established in [8].

Let $P = \{ \{ p_n^i \}_n : i = 1, 2, 3, \dots \}$ be a countable, monotone G_∞ -set. Fix $k \in \mathbb{N}$. We define the sequence space $\lambda(P, k)$ by

$$\lambda(P, k) := \{ \{ \xi_n \}_n : \sum_n |\xi_n| p_n^k < \infty \}.$$

It is obvious that $\lambda(P, k+1) \subset \lambda(P, k)$ and $\lambda(P, \infty) = \bigcap_k \lambda(P, k)$. For $\phi \in \Phi$ define $\lambda(\phi, P, k)$ by $\lambda(\phi, P, k) := \{ \{ \xi_n \}_n : \{ \phi(|\xi_n|) \}_n \in \lambda(P, k) \}$.

3.1 DEFINITION: Let P be a countable, monotone, nuclear G_∞ -set, $\phi \in \Phi$. A l.c.s. E is said to be

- (i) $\tilde{\lambda}(\phi, P, k)$ -nuclear if for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ such that $\{ \delta_n(V, U) \}_n \in \lambda(\phi, P, k)$;
- (ii) $\lambda(\phi, P, \mathbb{N})$ -nuclear if E is $\tilde{\lambda}(\phi, P, k)$ -nuclear for each $k \geq k_0$, where $\{ 1/p_n^{k_0} \}_n \in l_1$ because of the nuclearity of $\lambda(P, \infty)$.

For $\phi = \text{id}$ we write $\tilde{\lambda}(P, k)$ -nuclear resp. $\lambda(P, \mathbb{N})$ -nuclear instead of $\tilde{\lambda}(\text{id}, P, K)$ -nuclear resp. $\lambda(\text{id}, P, \mathbb{N})$ -nuclear.

If α is an exponent sequence and if $P = \{ \{ k^{\alpha_n} \}_n : k \in \mathbb{N} \}$, then a l.c.s. E is $\lambda(P, \mathbb{N})$ -nuclear iff E is $\Lambda_\infty(\alpha)$ -nuclear, i.e. $\tilde{\Lambda}_k(\alpha)$ -nuclear for each $k > 1$. So we indeed generalize the concept of $\Lambda_\infty(\alpha)$ -nuclearity considered in [8].

As in section 2 we now show that the concepts of $\lambda(\phi, P, \mathbb{N})$ -nuclearity and $\lambda(Q, \mathbb{N})$ -nuclearity are the same if Q is suitably chosen.

3.2 PROPOSITION: Let P be a countable, monotone, nuclear G_∞ -set, $\phi \in \Phi$. Define $\{ q_n^i \}_n$ by $1/q_n^i := \phi^{-1}(1/p_n^i)$ and $Q := \{ \{ q_n^i \}_n \}$. Let E be a l.c.s. Then the following statements are equivalent.

- (i) E is $\lambda(\phi, P, \mathbb{N})$ -nuclear.
- (ii) E is $\lambda(Q, \mathbb{N})$ -nuclear.

PROOF: Note that E is $\lambda(\phi, P, \mathbb{N})$ -nuclear (resp. $\lambda(Q, \mathbb{N})$ -nuclear) will follow if E is shown to be $\tilde{\lambda}(\phi, P, k)$ -nuclear (resp. $\tilde{\lambda}(Q, k)$ -nuclear) for each $k \geq k_0$, k_0 as in 3.1. Therefore we will show (1) given $k \geq k_0$ there exists $l \in \mathbb{N}$ such that $\lambda(\phi, P, l+1) \subset \lambda(Q, k)$; (2) given $r \geq r_0$ there exists $s \in \mathbb{N}$ such that $\lambda(Q, s+1) \subset \lambda(\phi, P, r)$. Given k , by nuclearity of $\lambda(Q, \infty)$ (Lemma 2.2) there exists $l \in \mathbb{N}$ such that

$\{q_n^k/q_n^l\}_n \in l_1$. If $x = \{\xi_n\}_n \in \lambda(\phi, P, l + 1)$ then $\phi(|\xi_n|)p_n^l \leq 1$ for $n \geq n_l$ and therefore $\sum_n |\xi_n|q_n^k = \sum_n |\xi_n|q_n^k q_n^l/q_n^l < \infty$. On the other hand given r , by nuclearity of $\lambda(P, \infty)$, find $s \in \mathbb{N}$ such that $\{p_n^r/p_n^s\}_n \in l_1$. So if $x = \{\xi_n\}_n \in \lambda(Q, s + 1)$ we have $|\xi_n| \leq 1/q_n^s$ for $n \geq n_s$ and therefore $\sum_n \phi(|\xi_n|)p_n^r = \sum_n \phi(|\xi_n|)p_n^s p_n^r/p_n^s < \infty$; this shows (1) and (2).

To prove (i) \Rightarrow (ii) fix $k \geq k_0$, find $l \in \mathbb{N}$ such that $\lambda(\phi, P, l + 1) \subset \lambda(Q, k)$. Since E is $\lambda(\phi, P, \mathbb{N})$ -nuclear, E is $\tilde{\lambda}(\phi, P, l + 1)$ -nuclear and therefore $\lambda(Q, \mathbb{N})$ -nuclear. The implication (ii) \Rightarrow (i) can be proved in the same way.

REMARK: It is easily seen that for $\phi \in \Phi$ $\lambda(\phi, P, \mathbb{N})$ -nuclearity implies ϕ -nuclearity [9]. But the reverse of this implication is not true in general as the following example shows.

Take $\phi = \phi_{\log}$; consider the power series space $\Lambda_\infty(\alpha)$ with $\alpha_n := (n + 1)^2$. We show that $\Lambda_\infty(\alpha)$ is ϕ_{\log} -nuclear but not $\lambda(\phi_{\log}, \mathbb{R}, \mathbb{N})$ -nuclear if $R := \{(n + 1)^k\}_n : k = 1, 2, \dots\}$. The ϕ_{\log} -nuclearity of $\Lambda_\infty(\alpha)$ immediately follows from Korollar 3.6 in [9]. But $\Lambda_\infty(\alpha)$ is $\lambda(\phi_{\log}, \mathbb{R}, \mathbb{N})$ -nuclear if and only if there exists $M > 1$ such that $\{\phi_{\log}(1/M^{\alpha_n})\}_n \in \lambda(R, k)$ for each k [8]. Therefore $\Lambda_\infty(\alpha)$ is $\lambda(\phi_{\log}, \mathbb{R}, \mathbb{N})$ -nuclear if and only if for each $k \geq k_0$ the sum $\sum_n (n + 1)^k/\alpha_n \log M = \sum_n (n + 1)^{k-2}/\log M$ is finite which obviously is not true.

In order to answer the question what the model of a universal $\lambda(\phi, P, \mathbb{N})$ -nuclear space is, it is enough to describe the model of a universal $\lambda(Q, \mathbb{N})$ -nuclear space; so from now on P is always supposed to be a countable, monotone, nuclear G -set.

Since $\lambda(P, \mathbb{N})$ -nuclearity implies nuclearity one easily obtains following permanence properties.

3.3 PROPOSITION: *Subspaces, quotient spaces by closed subspaces, and completions of $\lambda(P, \mathbb{N})$ -nuclear spaces are $\lambda(P, \mathbb{N})$ -nuclear.*

3.4 LEMMA: *For each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $\{(n + 1)\xi_n\}_n \in \lambda(P, k)$ whenever $\{\xi_n\}_n \in \lambda(P, r + 1)$.*

PROOF: Fix $k \in \mathbb{N}$. Since $\lambda(P, \infty)$ is nuclear there exists $l \in \mathbb{N}$ such that $\{p_n^k/p_n^l\}_n \in l_1$. Therefore for each $n \in \mathbb{N}$ $(p_n^k/p_n^l) + \dots + (p_n^k/p_n^l) \leq M$ and $n + 1 \leq p_n^l M/p_n^k =: M_1 p_n^l$. This implies

$$\sum_n (n + 1) |\xi_n| p_n^k \leq M_1 \sum_n |\xi_n| p_n^k p_n^l \leq M_2 \sum_n |\xi_n| p_n^r < \infty.$$

3.5 LEMMA: *Let H_1, H_2 be Hilbert spaces and $T \in L(H_1, H_2)$. Then (i) given k , T is $\tilde{\lambda}(P, k)$ -nuclear if T is of type $\lambda(P, k)$;*

(ii) given k , there exists r so that T is of type $\lambda(P, k)$ if T is $\tilde{\lambda}(P, r + 1)$ -nuclear.

PROOF: (i): We recall that for a compact operator $T \in L(H_1, H_2)$ T has a representation $Tx = \sum_n \lambda_n(x, e_n)f_n$ for suitable orthonormal sequences $\{e_n\}, \{f_n\}$ in H_1 resp. H_2 ; also $\lambda_n = \delta_n(TU_{H_1}, U_{H_2}) = s_n^{\text{app}}(T)$.

(ii) Fix k . Lemma 3.4 guarantees the existence of $r \in \mathbb{N}$ such that $\{(n + 1)\xi_n\}_n \in \lambda(P, k)$ if $\{\xi_n\}_n \in \lambda(P, r + 1)$. Now let T be $\tilde{\lambda}(P, r + 1)$ -nuclear, then T has a representation $Tx = \sum_n \gamma_n(x, a_n)y_n, \{\gamma_n\}_n \in \lambda(P, r + 1), \|a_n\| = \|y_n\| = 1, a_n \in H'_1, y_n \in H'_2$. Since $s_n^{\text{app}}(T) \leq \sum_{i=n}^\infty \gamma_i$, we have

$$\begin{aligned} \sum_{n=0}^\infty s_n^{\text{app}}(T)p_n^k &\leq \sum_{n=0}^\infty p_n^k \sum_{i=n}^\infty \gamma_i = \sum_{i=0}^\infty \sum_{n=0}^i \gamma_i p_n^k \\ &\leq \sum_{i=0}^\infty (i + 1)\gamma_i p_i^k \leq M \sum_{i=0}^\infty \gamma_i p_i^r < \infty. \end{aligned}$$

3.6 PROPOSITION: Let P be a countable, monotone, stable, nuclear G_∞ -set. Then countable direct sums of $\lambda(P, \mathbb{N})$ -nuclear spaces are $\lambda(P, \mathbb{N})$ -nuclear.

PROOF: We only indicate a partial proof and refer the reader to [1, Theorem 2.8] for the rest of the proof.

In $E = \bigoplus_{i=1}^\infty E_i$ a typical fundamental system of neighbourhoods of 0 is of the form $U = \Gamma((U_i)_i)$ where each U_k is an absolutely convex, closed, and absorbing neighbourhood of 0 in E_k and Γ represents the closed convex hull of the union. Stability of $\lambda(P, \infty)$ now gives a single valued, increasing map $\gamma: \mathbb{N} \rightarrow \mathbb{N}$ such that $\gamma(j) > j$ and $p_{2^n}^j = 0(p_n^{\gamma(n)})$. Then

- (1) $p_{2^{j-1}(2m-1)}^k \leq p_{2^j m}^k \leq M_{k,j} p_m^{\gamma^j(k)}$ for each j, k, m where $\gamma^j := \gamma \circ \gamma \circ \dots \circ \gamma$ (j -times)
- (2) $p_{2^{j-1}(2m-1)}^k \leq p_{2^{j-1}(2m-1)}^j \leq M_j p_m^{\gamma^j(j)}$ for each $k \leq j$.

Define $\beta: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ by $\beta^{-1}(j, m) := 2^{j-1}(2m - 1)$; β is a bijection and $\beta(n) = (\beta_1(n), \beta_2(n))$.

We have to show that $E = \bigoplus_i E_i$ is $\tilde{\lambda}(P, k)$ -nuclear for each $k \geq k_0$. Fix k , assume $i < k$. Using the fact that E_i is $\lambda(P, \mathbb{N})$ -nuclear and therefore nuclear for each neighbourhood $U_i \in \mathcal{U}(E_i)$ we find a neighbourhood $W_i \in \mathcal{U}(E_i)$ so that the canonical map $K_i: (E_i)_{W_i} \rightarrow (E_i)_{U_i}$ is represented by

$$K_i(x_i) = \sum_{m=0}^\infty \xi_m^i(x, a_m^i)y_m^i$$

where $\{\xi_m^i\}_m \in \lambda(P, \gamma^i(k) + 1), \|a_m^i\| \leq 1, \|y_m^i\| \leq 1$; then we have $\sum_{m=0}^\infty \xi_m^i p_m^{\gamma^i(k)} < \infty$. Now pick $t_i > 0$ so that $\sum_{m=0}^\infty t_i \xi_m^i p_m^{\gamma^i(k)} \leq 1/2^i M_{k,i}$. For

$i \geq k$ we get a representation

$$K_i(x_i) = \sum_{m=0}^{\infty} \xi_m^i \langle x, a_m^i \rangle y_m^i$$

where $\{\xi_m^i\}_m \in \lambda(P, \gamma^i(i) + 1)$, $\|a_m^i\| \leq 1$, $\|y_m^i\| \leq 1$; then we have $\sum_{m=0}^{\infty} \xi_m^i p_m^{\gamma^i(i)} < \infty$. Pick $t_i > 0$ so that $\sum_{m=0}^{\infty} t_i \xi_m^i p_m^{\gamma^i(i)} \leq 1/2^i M_i$. We then get

$$\begin{aligned} \sum_{n=1}^{\infty} t_{\beta_1(n)} |\xi_{\beta_2(n)}^{\beta_1(n)}| p_n^k &= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| p_{\beta^{-1}(j,m)}^k \\ &= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| p_{2^{j-1}(2m-1)}^k \\ &= \sum_{j=1}^{k-1} \sum_{m=1}^{\infty} t_j |\xi_m^j| p_{2^{j-1}(2m-1)}^k + \sum_{j=k}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| p_{2^{j-1}(2m-1)}^k \\ &\leq \sum_{j=1}^{k-1} \sum_{m=1}^{\infty} t_j |\xi_m^j| M_{k,j} p_m^{\gamma^j(k)} + \sum_{j=k}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| M_j p_m^{\gamma^j(j)} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} < \infty. \end{aligned}$$

Thus we have shown that $\{t_{\beta(n)} \xi_{\beta(n)}^{\beta(n)}\} \in \lambda(P, k)$. As in the proof of Theorem 2.8 in [1], we now can show that E is $\tilde{\lambda}(P, k)$ -nuclear and since this can be done for each $k \geq k_0$, E is $\lambda(P, \mathbb{N})$ -nuclear.

3.7 COROLLARY: *Arbitrary products of $\lambda(P, \mathbb{N})$ -nuclear spaces again are $\lambda(P, \mathbb{N})$ -nuclear whenever P is a countable, monotone, stable, nuclear G_{∞} -set.*

3.8 PROPOSITION: *Let P be a countable, monotone, nuclear G_{∞} -set. Then $\lambda(P, \infty)$ is $\lambda(P, \mathbb{N})$ -nuclear.*

PROOF: Since $\lambda(P, \infty)$ is a nuclear G_{∞} -space, there exists a $p^i \in P$ such that $\{1/p_n^i\} \in l_1$. Given $k, r \in \mathbb{N}$, there exists a $p^r \in P$ such that $p^r p^k < p^r$; we also find a $p^s \in P$ so that $p^r p^i < p^s$. We then have

$$\sum_n p_n^k p_n^r / p_n^s \leq M \sum_n p_n^i / p_n^s = M \sum_n (p_n^r p_n^i) / (p_n^s p_n^i) \leq M_0 \sum_n 1/p_n^i < \infty,$$

so $\lambda(P, \infty)$ is $\tilde{\lambda}(P, k)$ -nuclear for each k .

Our next result shows that $\lambda(P, \infty)$ is a universal generator for the variety of $\lambda(P, \mathbb{N})$ -nuclear spaces if P is stable.

3.9 PROPOSITION: *Let P be a countable, monotone, stable, nuclear G_{∞} -set; then each $\lambda(P, \mathbb{N})$ -nuclear space E is isomorphic to a subspace of $[\lambda(P, \infty)]^I$ for a suitable I .*

PROOF: Let $k \in \mathbb{N}$ be fixed. By stability let $\gamma: \mathbb{N} \rightarrow \mathbb{N}$ be a single valued, increasing function such that $\gamma(j) > j$ and $p_{2n}^j = 0 (p_n^{\gamma(j)})$. Write $\bar{k} := \gamma^k(k)$. Since E is $\lambda(P, \mathbb{N})$ -nuclear we have $p^{\bar{k}} \in \Delta(E)$, and there exists an absolutely convex, closed, and absorbing neighbourhood $U \in \mathcal{U}(E)$ so that E'_{U^0} (U^0 is the polar of U) is a Hilbert space; now by Proposition IV.1 of [12] there exists an orthonormal basis $\{e_n^k\}_n$ in E'_{U^0} so that the set

$$A_k := \left\{ \sum_{n=0}^{\infty} \xi_n p_n^{\bar{k}} e_n^k : \sum_n |\xi_n|^2 \leq 1 \right\}$$

is equicontinuous in E' . Rearrange the set $\{e_n^k: k, n = 1, 2, \dots\}$ into a single sequence by using the bijection map $\beta: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ with $\beta^{-1}(k, n) := 2^{k-1}(2n - 1)$; use the Gram-Schmidt process to obtain a new orthonormal basis $\{e_m\}$ for E'_{U^0} . We then have $e_m = \sum_{n \geq m/2^k} (e_m, e_n^k) e_n^k$ and therefore

$$\sum_{n \geq m/2^k} |(e_m, e_n^k)|^2 / (p_n^{\bar{k}})^2 \leq \sum_{n \geq m/2^k} |(e_m, e_n^k)|^2 / (p_{m/2^k}^{\bar{k}})^2.$$

Since $p_{2^k m}^k \leq M_k p_m^{\gamma^k(k)} = M_k p_m^{\bar{k}}$ and since without loss of generality we may assume $M_k = 1$, we get

$$\sum_{n \geq m/2^k} |(e_m, e_n^k)|^2 / (p_{m/2^k}^{\bar{k}})^2 \leq \sum_{n \geq m/2^k} |(e_m, e_n^k)|^2 / (p_m^k)^2 \leq 1 / (p_m^k)^2$$

and therefore $p_m^k e_k \in A_k$.

So we have shown that there exists an orthonormal basis $\{e_m\}$ of E'_{U^0} such that $\{p_m^k e_m: m = 1, 2, \dots\}$ is equicontinuous in E'_{U^0} , for each fixed k .

Let $\mathcal{U} = (U_i: i \in I)$ be a base of neighbourhoods of 0 in E so that each U_i is absolutely convex, closed, and absorbing and E'_{U_i} is a Hilbert space. For each $i \in I$ we can get an orthonormal basis $\{e_m^i: m = 1, 2, \dots\}$ of E'_{U_i} such that the sets $B_{i,k} := \{p_m^k e_m^i: m = 1, 2, \dots\}$ are equicontinuous for each fixed k ; for each $i \in I$, define the map $T_i: E \rightarrow \lambda(P, \infty)$ by $T_i x := \{(x, e_m^i)\}_m$; T_i goes into $\lambda(P, \infty)$ and is continuous. Define $T: E \rightarrow [\lambda(P, \infty)]^I$ by $Tx := \{T_i x\}_i$. Then T is continuous and one-to-one. With obvious changes, the rest of the proof is exactly the same as of Proposition 3.4 in [8].

In [2], Fenske and Schock consider the class Ω of all l.c.s. E such that the set ω of all strictly positive, non-decreasing sequences of reals is contained in $\Delta(E)$. They prove Ω to be a stability class of nuclear spaces and therefore closed under the operations of forming completions, subspaces, quotients by closed subspaces, arbitrary products, countable direct sums, tensor products, and isomorphic

images; more precisely they show $\Omega = \bigcap_{\phi \in \Phi} \mathcal{N}_\phi$ where \mathcal{N}_ϕ denotes the class of ϕ -nuclear spaces. The following proposition extends this result.

3.10 PROPOSITION: *Let P be a countable, monotone, stable, nuclear G_∞ -set. Then $\Omega = \bigcap_{\phi \in \Phi} \mathcal{N}_{\lambda(\phi, P, \mathbb{N})}$.*

PROOF: Let E be a l.c.s. in $\bigcap_{\phi \in \Phi} \mathcal{N}_{\lambda(\phi, P, \mathbb{N})}$, then $E \in \bigcap_{\phi \in \Phi} \mathcal{N}_\phi$. If $\tilde{\Phi}$ denotes the set of all continuous, strictly increasing, subadditive functions ϕ with $\phi(0) = 0$, then obviously $\bigcap_{\phi \in \Phi} \mathcal{N}_\phi = \bigcap_{\phi \in \tilde{\Phi}} \mathcal{N}_\phi$. Therefore $\bigcap_{\phi \in \Phi} \mathcal{N}_{\lambda(\phi, P, \mathbb{N})} \subset \Omega$. Now fix E in Ω , then $\omega \subset \Delta(E)$. Take $k \in \mathbb{N}$, $\phi \in \tilde{\Phi}$, then $\{1/\phi^{-1}(1/p_n^k)\} \in \omega$, so given $U \in \mathcal{U}(E)$ one may find $V_k \in \mathcal{U}(E)$ such that $\delta_n(V_k, U)/\phi^{-1}(1/p_n^k) \rightarrow 0$ for $n \rightarrow \infty$. For $n \geq n_0$ we then have $\phi(\delta_n(V_k, U))p_n^k \leq 1$, and E is $\lambda(\phi, P, \mathbb{N})$ -nuclear for each $\phi \in \tilde{\Phi}$.

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