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FOUNDATIONS OF THE THEORY OF FANO SCHEMES

Allen B. Altman and Steven L. Kleiman*

The family of lines on a cubic hypersurface with a finite number of singularities was studied by Fano for what it says about the hypersurface [F1; 1904] and for its own sake [F2; 1904]. The family is now named after him. Recently the study has been taken up again. Bombieri and Swinnerton-Dyer [B, S-D; 1967] were especially interested in verifying the Weil conjectures for a smooth cubic threefold. Clemens and Griffiths [C, G; 1972] established the irrationality of the cubic threefold over the complex numbers (which Fano claimed to have done); Murre [Mr 2; 1973] did this in characteristic different from 2. Clemens and Griffiths [C, G; 1972] also proved a Torelli theorem for the cubic threefold; Tjurin [Tj 2; 1971] did too, following an earlier suggestion of Griffiths [Gr; 1970]. Below some basic properties and invariants of Fano schemes are properly treated using schemes.

The chief results for the case of a base field are presented in section 1. Most of these are proved in greater generality than before, notably, in arbitrary characteristic (except for one minor failure (1.16, (iii)) in characteristic 3) and for hypersurfaces with singularities. The proofs are not *ad hoc*, but applications of general principles, and the methods might apply to other problems. Often the proofs in section 1 refer ahead to more general results in later sections.

Let F denote the Fano scheme of lines on a cubic hypersurface X over a field. The first main result (1.3) is that F is the scheme of zeroes of a regular section of the locally free sheaf $\mathrm{Sym}_3(Q)$, where Q is the universal quotient bundle on the ambient grassmannian. This result yields a formula for the dualizing sheaf of F and a formula for the class of F in the Chow ring of the grassmannian. Then Schubert calculus

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yields the degree of F, the genus of a 1-dimensional linear section, and, if X is a threefold, it yields the (weighted) number 6 of lines through a general point. The idea of expressing the family of lines on a quartic hypersurface as the zeroes of a regular section of $\mathrm{Sym}_4(Q)$ and the idea of using Schubert calculus appear (independently) in Tennison's article [Tn; 1974].

The second major result (1.10) is in two parts. Part (i) asserts that F is smooth at a point if and only if X is smooth along the corresponding line. The proof given below, using the universal family of Fano schemes, is an abstract version of one communicated privately by Lipman [Lp; 1974]. Part (ii), the so-called tangent bundle theorem, asserts that the sheaf of differentials on the smooth locus of F is the restriction of the universal quotient Q, if X is a threefold. This result is obtained by a simple direct computation made possible by a trick, (2.10) with $s = 3$ and $t = 2$, which turns certain short exact sequences into others. The result was earlier obtained using more roundabout means by Tjurin [Tj 1; 1970] and independently by Clemens and Griffiths [C, G; 1972]. Lieberman [Lb; 1973] inspired the work by pointing out that the tangent bundle theorem is one of the key ingredients, which, when combined with the general theory of the intermediate jacobian, yields the irrationality of the smooth cubic threefold.

Assume $\dim(X) \geq 3$. The third major result (1.16) asserts that F is geometrically connected, that F is linearly normal in the Plücker embedding (that is, it lies in no hyperplane and the linear system of hyperplanes is complete) and that, at least if the characteristic is not 3, every quadric hypersurface containing F contains the grassmannian. This result follows from a study of the cohomology groups $H^i(F, O_F(n))$ for $n = 0$, 1, 2. For example, $h^0(F, O_F) = 1$ is proved, whence F is geometrically connected. When X is a cubic threefold, and so F is a surface, these cohomology groups also yield the values of the various classical invariants of F (see (1.21) and (1.23)).

Section 2 contains miscellaneous general facts needed elsewhere. Some of them are more or less well-known, but there are no convenient references. Section 3 discusses the universal family W/H of degree d hypersurfaces, the universal family $\mathbb{P}(Q)/G$ of r-planes, and the universal family Z/H of Fano schemes of r-planes on degree d hypersurfaces. Some of the assertions are up-to-date, finer versions of familiar facts; others are new results. Section 4 discusses the smoothness of the families W/H and Z/H, and it computes the relative differentials of the smooth locus of Z/H for $r = 1$ and $d = 3$, obtaining a generalized version (4.4) of the tangent bundle theorem. The context of both sections 3 and 4 is set over an arbitrary locally noetherian base (the hypothesis "locally noetherian" can be eliminated in the usual way if desired). Section 5 discusses the elementary but lengthy computations of the cohomology groups of the sheaf $E(n) = (\mathrm{Sym}_3(Q)^{\vee})(n)$ for $n = 0, 1, 2$. The results are used in section 1 to study the cohomology groups $H^i(F, O_F(n)$ via the Koszul resolution $\Lambda^{\cdot} E \to O_F \to 0$ of the regular section whose scheme of zeroes is F.

Blanket notation. Fix a locally noetherian base scheme S and a locally free O_S-Module V with rank $(m+1)$ for fixed $m \geq 3$. Set

$$P = \mathbb{P}(V) \qquad \text{and} \qquad G = \mathrm{Grass}_{(r+1)}(V)$$

for fixed $r \geq 0$ (see [EGA I], 9.7.5) and let

$$0 \to M \to V_G \to Q \to 0$$

denote the *fundamental sequence* on G. Regard G as embedded in $\mathbb{P}(\Lambda^{(m+1)} V)$ by the *Plücker embedding* (see [EGA I], 9.8.1), and note the formula,

$$O_G(1) = \Lambda^{(r+1)} Q.$$

Fix $d \geq 1$ and set

$$E = \mathrm{Sym}_d(Q)^{\vee} \qquad \text{and} \qquad H = \mathbb{P}(\mathrm{Sym}_d(V)^{\vee}).$$

The scheme H parametrizes degree d hypersurfaces (see (3.1, (i))). Let

$$W \subset P \times_S G$$

denote the *universal family* of degree d hypersurfaces. Let

$$Z \subset H \times_S G$$

denote the scheme which parametrizes pairs (X, L) where X is a degree d hypersurface and L is an r-plane contained in X (see (3.3, (i))). The subscheme Z is an example of an incidence correspondence. Below, Z/H is usually thought of as the *universal family of Fano schemes.*

1. Some basic properties and invariants of Fano schemes

(1.0) NOTATION: In this section take

$$r = 1 \quad \text{and} \quad S = \mathrm{Spec}\,(k)$$

where k is a field. Let X be a cubic hypersurface with only finitely many singularities (nonsmooth points) in the projective m-space over S. Exclude the case that X is a surface (i.e., $m = 3$) with a triple point (except in (1.5)).

(1.1) DEFINITION: The *Fano scheme* F of X is the S-scheme parametrizing the lines of P which lie entirely in X.

(1.2) REMARK: Suppose X is a cubic threefold in 4-space (i.e., $m = 4$). Then the Fano scheme F of X is a surface by (1.3), and it is reduced if X has no triple points by (1.19a). This surface was studied by G. Fano ([F1], [F2]).

(1.3) THEOREM: *The Fano scheme F of X exists and is equal to the subscheme $Z_G(s)$ of G of zeroes of a regular section s of the locally free O_G-Module* $\mathrm{Sym}_3(Q)$. (*See* (2.2) *for a definition of a scheme of zeroes.*) *Moreover, each component has dimension* $2(m-3)$.

PROOF: Clearly F is equal to the fiber of Z/H over the point x of H corresponding to X. Note $k(x) = k$ holds because X is an S-scheme. Hence the assertions will all follow from the equivalence of (a), (b) and (c′) of (3.3, (iv)) once F is proved nonempty with dimension $\leq 2(m-3)$.

By (5.1) the dimension $h^0(F, O_F)$ is equal to 1, and so F is nonempty. Alternately, by (1.6, (ii)) the degree of F is positive, and so F is nonempty.

Let Σ denote the set of points ℓ of F such that the corresponding line of $X \otimes k(\ell)$ contains a nonsmooth point of $X \otimes k(\ell)$. Then $(F - \Sigma)$ is smooth with dimension $2(m-3)$ by (4.2), if it is nonempty. Moreover, the dimension of Σ is less than or equal to $2(m-3)$ by (1.5). Therefore F has dimension $\leq 2(m-3)$, and the proof is complete.

(1.4) COROLLARY: *The Fano scheme F is locally a complete intersection with pure dimension* $2(m-3)$. *Its normal sheaf is given by the* *formula,*

$$N(F, G) = \mathrm{Sym}_3(Q)|F.$$

PROOF: The assertions follow immediately from (1.3) and (2.5).

(1.5) LEMMA: *Let Σ denote the set of points ℓ of F such that the corresponding line of $X \otimes k(\ell)$ contains a nonsmooth point of $X \otimes k(\ell)$.*

(i) *Assume X has a triple point x. Then X is a cone over a smooth cubic hypersurface X_1 in $\mathbb{P}_k^{(m-1)}$ and Σ has dimension $(m-2)$.*

(ii) *Assume X has no triple point. Then either X is smooth or Σ has dimension $(m-3)$.*

PROOF: We may assume that the ground field k is algebraically closed.

(i) Take an affine space $\mathbb{A}^m \subset P$ with x as origin and take $C = 0$ as the equation defining X in $\mathbb{A}^m$. Then, since x is a triple point and X is a cubic, C is a homogeneous cubic polynomial. Let X_1 denote the hypersurface in the $\mathbb{P}^{m-1}$ at infinity defined by the equation $C = 0$. Then X is obviously equal to the cone over X_1, and X_1 is smooth because X has only finitely many singularities.

By the theory of cones (see [J], (C 11)) there is an injective map from X_1 to F sending a point x_1 to the generator through x_1. Clearly every line through x is a generator. Now, x is clearly the only singularity of X. Hence the family of generators consists of all lines through a nonsmooth point of X. Therefore the image of this map is equal to Σ. Since the dimension of X_1 is equal to $(m-2)$, the dimension of Σ is also equal to $(m-2)$.

(ii) Let x be a double point of X. Then the cone T_x in P of tangent lines to X at x is a quadric hypersurface. It will now be proved that $T_x \cap X$ is a cone with vertex x over a scheme Y_x with dimension $(m-3)$ (cf. [F1], §4, p. 602 and [C, G], p. 306). Clearly any line on X passing through x lies in T_x. On the other hand, let y be a point of $T_x \cap X$ and let ℓ be the line through x and y. Since the intersection multiplicity of ℓ and X at x is at least 3, clearly ℓ must lie in X by Bézout's theorem. So, each point of $T_x \cap X$ lies on a line in X through x. Hence $T_x \cap X$ is a cone. The dimension of the base Y_x is obviously equal to $(m-3)$.

By the theory of cones, there is an injective map,

$$\mu_x : Y_x \to F.$$

Hence there is a formula,

$$\dim(\mu_x(Y_x)) = (m-3).$$

Let $\{x_1, \ldots, x_n\}$ be the singular set of X. Since X has no triple point and since X is a cubic, each x_i is a double point of X. Now, Σ is equal to $\cup \mu_{x_i}(Y_{x_i})$. Hence the dimension of Σ is equal to $(m-3)$.

(1.6) PROPOSITION: (i) *The class of F in the Chow ring $A(G)$ is given by the formula,*

$$c\ell(F) = 9c_2(Q)\cdot(2c_1(Q)^2 + c_2(Q)).$$

(ii) *The degree of F is given by the formula,*

$$\deg(F) = \begin{cases} 27, & m=3; \\ 45, & m=4; \\ 9\left(6\dfrac{(2m-3)!}{m!(m-2)!} - 12\dfrac{(2m-4)!}{m!(m-3)!} + \dfrac{(2m-6)!}{(m-2)!(m-3)!}\right. \\ \qquad \left. + 3\dfrac{(2m-6)!}{(m-1)!(m-4)!} + 5\dfrac{(2m-6)!}{m!(m-5)!}\right), & m \geq 5. \end{cases}$$

PROOF: (i) Since F is equal to the subscheme of G of zeroes of a regular section of $\mathrm{Sym}_3(Q)$ by (1.3), the class of F in $A(G)$ is equal to $c_4(\mathrm{Sym}_3(Q))$ by ([Ch], 18 bis p. 153). So (2.14) yields (i).

(ii) The Chern classes $c_1(Q)$ and $c_2(Q)$ are equal to $\sigma_1 = (m-2, m)$ and $(m-2, m-1)$ (see the proof of Proposition 5.6, [Kℓ 1].) So using Pieri's formula ([Kℓ 2], p. 18) and Giambelli's (determinantal) formula ([Kℓ 2], p. 18) yields the formulas,

$$\begin{aligned} \tfrac{1}{9}\deg(F) &= \langle \sigma_1^{2(m-3)}\cdot(m-2,m)\cdot(2(m-2,m)^2 + (m-2,m-1))\rangle && \text{by (i)} \\ &= \langle \sigma_1^{2(m-3)}\cdot(\sigma_1^2-\sigma_2)\cdot(3\sigma_1^2-\sigma_2)\rangle && \text{(Giambelli)} \\ &= \langle \sigma_1^{2(m-3)}\cdot(3\sigma_1^4 - 4\sigma_1^2\sigma_2 + (m-3,m-2) \\ &\qquad + (m-4,m-1) + (m-5,m))\rangle, && \text{(Pieri)} \end{aligned}$$

where $\sigma_2 = (m-3, m)$ and where the last two terms (resp. the last term) in the third line are zero for $m=3$ (resp. $m=4$). The formula for the degree of a Schubert cycle ([Kℓ 2], p. 20) now yields the result.

(1.7) PROPOSITION: *Assume X is a cubic threefold and the ground* *field k is algebraically closed. Then the number of lines, counted with appropriate multiplicity, passing through a general point of X is equal to* 6. *The* 6 *lines are distinct if the characteristic is different from* 2, *from* 3, *and from* 5, *or if the characteristic is different from* 2 *and from* 3 *and F is irreducible* (*e.g., X smooth by* (1.12) *and* (1.16, (i))).

PROOF: The class in $A(G)$ of the Schubert variety of lines which meet a given line is $\sigma_2 = (1, 4)$. Thus, by (1.6, (i)), the number of lines in X which meet a general line in 4-space is equal to

$$\langle c\ell(F)\cdot\sigma_2\rangle = \langle 9\sigma_2\cdot c_2(Q)\cdot(2c_1(Q)^2 + c_2(Q))\rangle.$$

By Pieri's formula and Giambelli's formula, this number is equal to 18.

Let p, g denote the two projections from the product $P\times G$, let L be a general line in P, and consider the following subvarieties of $P\times G$:

$$Y = \mathbb{P}(Q); \quad C = \mathbb{P}(Q|F).$$

Let $C_1, \ldots, C_n$ denote the irreducible components of C. Then σ_2 is given by the formula,

$$\sigma_2 = g_*[c\ell(Y)\cdot p^*(c\ell(L))].$$

This formula yields

$$\begin{aligned}
\langle c\ell(F)\cdot\sigma_2\rangle_G &= \langle g^*c\ell(F)\cdot c\ell(Y)\cdot p^*c\ell(L)\rangle_{P\times G} && \text{(proj. form.)}\\
&= \langle c\ell(C)\cdot p^*c\ell(L)\rangle_{P\times G}\\
&= \langle p_*c\ell(C)\cdot c\ell(L)\rangle_P && \text{(proj. form)}\\
&= \sum [k(C_j):k(X)]\langle c\ell(X)\cdot c\ell(L)\rangle_P\\
&= 3\sum [k(C_j):k(X)].
\end{aligned}$$

Equating the two values of $\langle c\ell(F)\cdot\sigma_2\rangle$ yields

$$\sum [k(C_j):k(X)] = 6.$$

In particular, since one of the degrees $[k(C_i):k(X)]$ is therefore nonzero, the restriction $p': C\to X$ is surjective.

There is an open set U of X such that each fiber $(p')^{-1}(x)$, for $x\in U$, is finite by ([EGA IV_3], 13.1.4), and U is nonempty because it contains the generic point ξ of X since both C and X have dimension 3 and p' is surjective. Now, by ([EGA IV_2], 4.5.10), the fiber $(p')^{-1}(\xi)$ has

$$t = \sum [k(C_j):k(X)]_s$$

geometric components; hence, by ([EGA IV_3], 9.7.8), there is a nonempty open set U' of X such that the fiber $(p')^{-1}(x)$ over each point x of U' has t geometric points. However, the number of geometric points of $(p')^{-1}(x)$ is clearly equal to the number of distinct lines passing through x. Therefore the number of lines, counted with multiplicity, passing through a point x of U' is equal to 6.

The multiplicity of a line represented by a point of $C_j \cap (p')^{-1}(x)$ is equal to the degree of inseparability $[k(C_j):k(X)]_i$. Hence the multiplicity is equal to 1 if the characteristic is different from 2, 3, and 5 because 2, 3 and 5 are the only primes dividing a summand of 6. The multiplicity is equal to 1 also if the characteristic is different from 2 and 3 and F is irreducible because then $C = \mathbb{P}(Q|F)$ is irreducible too and 2 and 3 are the only primes dividing 6.

(1.8) PROPOSITION: *The dualizing sheaf of F is given by the formula,*

$$\omega_F = O_F(5-m).$$

PROOF: There are formulas,

$$\omega_F = (\Lambda^4 N(F,G)) \otimes (\Lambda^{2(m-1)}\Omega^1_G)|F \qquad \text{(Fund. Loc. Is. ([GD], I, 4.5, p. 13))}$$

$$= (\Lambda^4 \mathrm{Sym}_3(Q)) \otimes O_G(-m-1)|F \qquad (1.4) \text{ and } (2.7)$$

$$= O_F(5-m). \qquad (2.12)$$

(1.9) COROLLARY: *The dualizing sheaf of a linear space section F' of F of codimension p is given by the formula,*

$$\omega_{F'} = O_{F'}(5-m+p).$$

If F' is a curve, its genus is given by the formula,

$$p_a(F') = \tfrac{1}{2}(m-2)\deg(F) + 1,$$

(see (1.6, (ii)) for the value of $\deg(F)$).

PROOF: There are formulas,

$$\omega_{F'} = \overset{p}{\Lambda}\, N(F', F) \otimes \omega_F \qquad \text{(Fund. Loc. Is. ([GD], I, 4.5, p. 13))}$$

$$= \overset{p}{\Lambda}\, (O_F(1)^{\oplus p}) \otimes O_F(5-m) \qquad \text{(2.5) and (1.8)}$$

$$= O_{F'}(5-m+p).$$

Consequently there are formulas,

$$\begin{aligned} 2p_a(F') - 2 &= \deg(\omega_{F'}) \\ &= (5-m+2(m-3)-1)\deg(F) \\ &= (m-2)\deg(F). \end{aligned}$$

(1.10) THEOREM: (i) *Let ℓ be a point of F, and let L denote the corresponding line on $X \otimes k(\ell)$. Then ℓ is a smooth point of F if and only if L lies entirely in the smooth locus of $X \otimes k(\ell)$.*

(ii) (*The Tangent Bundle Theorem*). *Assume X is a threefold in 4-space (i.e., $m = 4$), and let F^0 denote the smooth locus of F. Then there is a canonical isomorphism,*

$$\Omega^1_{F^0/k} = Q \,|\, F^0.$$

PROOF: (i) By (1.3), F has dimension $2(m-3)$ at ℓ. Hence the assertion results from (4.2).

(ii) The assertion results immediately from (4.4) by restriction to the fiber.

(1.11) COROLLARY: *The nonsmooth locus of F is equal to the set Σ of points $\ell \in F$ such that the corresponding line on $X \otimes k(\ell)$ passes through a nonsmooth point of $X \otimes k(\ell)$.*

(1.12) COROLLARY: *Assume X is smooth. Then its Fano scheme F is smooth.*

(1.13) COROLLARY: *Assume X is a smooth cubic surface (in 3-space) and the ground field k is algebraically closed. Then X contains precisely 27 distinct lines.*

PROOF: The Fano scheme F is smooth by (1.2), it has dimension 0 by (1.3), and it has degree 27 by (1.6). Hence it consists of 27 distinct points.

(1.14) LEMMA: *Recall* $E = \mathrm{Sym}_d\,(Q)^\vee$. *For each integer n, there is a spectral sequence,*

$$E_1^{p,q}(n) = H^q(G, \Lambda^{-p}E(n)) \Rightarrow H^{p+q}(F, O_F(n)).$$

PROOF: Since F is equal to the subscheme of G of zeroes of a regular section of $\mathrm{Sym}_3(Q)$ by (1.3), the Koszul complex $K_p = (\Lambda^{-p}E)(n)$ of the section is a resolution of $O_F(n)$. Taking an injective Cartan-Eilenberg resolution and proceeding in the usual way yields the asserted spectral sequence.

(1.15) PROPOSITION: (i) *For* $\dim(X) = 3$, *there are formulas (valid in any characteristic),*

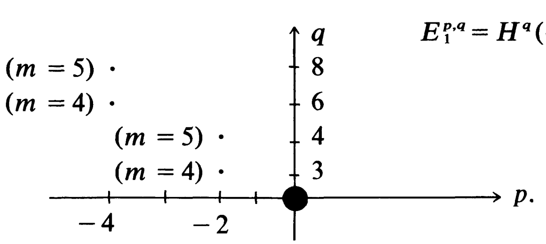
$$\begin{array}{lll} h^0(F, O_F) = 1, & h^1(F, O_F) = 5, & h^2(F, O_F) = 10, \\ h^0(F, O_F(1)) = 10, & h^1(F, O_F(1)) = 5, & h^2(F, O_F(1)) = 1, \\ h^0(F, O_F(2)) = 51, & h^1(F, O_F(2)) = 0, & h^2(F, O_F(2)) = 0. \end{array}$$

(ii) *For* $\dim(X) \geq 3$, *the canonical map,*

$$H^0(G, O_G(n)) \to H^0(F, O_F(n)),$$

is bijective for $n = 0, 1$, *and if* $\mathrm{char}(k) \neq 3$, *it is injective for* $n = 2$.

PROOF: By (1.14), there is a spectral sequence,



$$E_1^{p,q} = H^q(G, \Lambda^{-p}E) \Rightarrow H^{p+q}(F, O_F)$$

By (5.1) all the $E_1^{p,q}$-terms are zero except for those represented by the black spots in the above figure, and the latter have the appropriate ranks. Clearly the spectral sequence degenerates and yields assertions (i) and (ii) for $n = 0$. The proofs for $n = 1$ and $n = 2$ are similar.

(1.16) THEOREM: *Assume* $\dim(X) \geq 3$ *holds.*

(i) *F is geometrically connected.*

(ii) *F is linearly normal in the Plücker embedding; that is, no hyperplane contains F and the linear system of hyperplane sections of F is complete.*
(iii) *Every quadric hypersurface containing F contains G, at least in characteristic different from* 3.

PROOF: Assertion (iii) follows directly from (1.15, (ii)) for $n = 2$. The map,

$$H^0(\mathbb{P}(\overset{r+1}{\Lambda} V), O(n)) \to H^0(G, O_G(n)),$$

is bijective for $n = 0, 1$ (cf. proof of Prop. 13, p. 8, [Lk]). Hence in view of (1.15, (ii)), the map,

$$H^0(\mathbb{P}(\overset{r+1}{\Lambda} V), O(n)) \to H^0(F, O_F(n)),$$

is bijective for $n = 0, 1$. The bijectivity for $n = 0$ implies $h^0(F, O_F)$ is equal to 1, and so F is geometrically connected. The bijectivity for $n = 1$ is equivalent to (ii).

(1.17) REMARK (Fano [F2], p. 79): There is a cubic hypersurface containing F but not containing G. Recall that the set of lines that meet a subvariety X' of P with $\operatorname{codim}(X', P) = 2$ and with $\deg(X') = d$ is a section G' of G by a hypersurface with degree d. In our case, take X' to be a hyperplane section of X. Clearly X' has codimension 2 and degree 3. Moreover, every line in X meets X' because every line in P meets every hyperplane. Thus G' is a cubic hypersurface section of G that contains F.

(1.18) REMARK: For a cubic threefold X, the dualizing sheaf ω_F of F is equal to $O_F(1)$ by (1.8). So, by duality ([GD], I, 1.3, p. 5), there are formulas,

$$h^i(F, O_F) = h^{2-i}(F, O_F(1)) \qquad i = 0, 1, 2.$$

These are clearly consistent with (1.15, (i)).

(1.19) PROPOSITION: *Assume either*:
(a) $m = 4$ (*resp.* $m = 5$) *holds and X has no triple point*; *or*
(b) $m \geq 5$ (*resp.* $m \geq 6$) *holds.*
Then F is geometrically reduced (*resp. geometrically normal and geometrically irreducible*).

PROOF: We may assume k is algebraically closed. Both (a) and (b) imply that the codimension of the singular locus of F is at least one (resp. two) by (1.3), by (1.11), and by (1.5). Thus F satisfies condition R_0 (resp. R_1). Since F is locally a complete intersection by (1.4), it also satisfies condition S_1 (resp. S_2). Thus F is reduced by ([EGA IV_2], 5.8.5) (resp. normal by ([EGA IV_2], 5.8.6)). Finally a normal connected scheme is obviously irreducible.

(1.20) REMARK: (i) Suppose $\dim(X) = 3$. If X has a triple point, then the Fano surface F is not reduced because the nonsmooth locus $\operatorname{Sing}(F)$ is 2-dimensional by (1.11) and (1.5, (i)). If F is normal, then it is smooth because the dimension of $\operatorname{Sing}(F)$ cannot be zero by (1.5) and (1.11).

(ii) If $m \geq 5$ holds, then the smooth locus F^0 of F is connected by ([EGA IV_2], 5.10.7) because F is connected by (1.16, (i)) and locally a complete intersection by (1.4). Hence F^0 is irreducible. Moreover, F^0 is dense in F because F is geometrically reduced by (1.19). Hence F is irreducible. (If $m \geq 6$ holds, this provides an alternate proof to that in (1.19).) If $m = 4$ holds and X has 5 or fewer double points, then Fano shows F is irreducible ([F1], p. 603), and he shows F is not necessarily irreducible if it has 6 double points ([F1], §6, p. 605). Clemens and Griffiths show that if X has one ordinary double point, then F is irreducible ([C, G], p. 315, [It is an algebro-geometric proof]).

(1.21) PROPOSITION: *Assume X is a cubic threefold. Then F is a surface with geometric genus $p_g = 10$, with irregularity $q = 5$, and with arithmetic genus $p_a = 5$. Moreover, the Hilbert polynomial is given by the formula,*

$$\chi(F, O_F(n)) = 45\binom{n+1}{2} - 45\binom{n}{1} + 6. \tag{1.21.1}$$

PROOF: The first assertion follows immediately from (1.3) and (1.15, (i)); for p_g is equal to $h^2(F, O_F)$, and q is equal to $h^1(F, O_F)$, and p_a is equal to $p_g - q$. Now, $\chi(F, O_F(n))$ has the form,

$$\chi(F, O_F(n)) = a\binom{n+1}{2} + b\binom{n}{1} + c.$$

So, formula (1.21.1) results straightforwardly from (1.15, (i)).

(1.22) REMARK: Since the degree of F is equal to the coefficient of $\binom{n+1}{2}$ in the Hilbert polynomial, Proposition (1.21) yields an alternate derivation of the formula $\deg(F) = 45$.

(1.23) PROPOSITION: *Assume X is a smooth threefold. Then F is a smooth, geometrically irreducible surface, and its topological Euler characteristic $E(F)$ is equal to 27.*

PROOF: By (1.12), F is smooth; by (1.16, (i)), geometrically irreducible; by (1.3), a surface. Now, there are formulas,

$$\begin{aligned} E(F) &= (\Delta \cdot \Delta) && \text{(Lefschetz Fixed-Point Formula)} \\ &= \deg(c_2(\Omega^1_F)) && \text{(Riemann-Roch Theorem)} \\ &= \deg(c_2(Q)) && (1.10, (ii)) \\ &= 9c_2(Q)^2(2c_1(Q)^2 + c_2(Q)) && (1.6, (i)) \\ &= 9c_2(Q)^2(2\sigma_2 + 3c_2(Q)) && \text{with } \sigma_2 = (1, 4) \\ &= 27. \end{aligned}$$

(1.24) REMARK: Assume X is a smooth cubic threefold. An alternate derivation of $\chi(O_F) = 6$ comes from Noether's formula,

$$12\chi(O_F) = c_1(T_F)^2 + c_2(T_F)$$

a special case of the Riemann-Roch Theorem. Indeed, by (1.10, (ii)), there is a formula $T_F = Q^\vee$. Now, $c_1(Q)^2$ is equal to the degree of F, so to 45 by (1.6) and $c_2(T_F)$ is equal to 27 by the proof of (1.23).

The Riemann-Roch formula now yields

$$\begin{aligned} \chi(O_F(n)) &= \tfrac{1}{2}\langle nc_1(O_F(1)) \cdot (nc_1(O_F(1)) + c_1(T_F))\rangle + \chi(O_F) \\ &= 45\binom{n+1}{2} - 45\binom{n}{1} + 6. \end{aligned}$$

This result agrees with (1.21). This derivation requires X to be smooth (for Noether's formula but not for the Riemann-Roch formula); however, it does not require the lengthy computations of §5.

Another derivation of the Hilbert polynomial of F was found by Libgober [Lr; 1973].

2. Miscellany

(2.1) LEMMA: *Let $p : Y \to S$ be a morphism of schemes and let $u : p^*E \to F$ be an O_Y-homomorphism.*

(i) *For each base change $g : T \to S$, the triangle,*

$$\begin{array}{ccc} g^*E & \xrightarrow{\quad (u_T)^\flat \quad} & \\ {\scriptstyle (u^\flat)_T}\downarrow & \searrow & \\ g^*p_*F & \xrightarrow{\quad b \quad} & (p_T)_*(g_Y)^*F, \end{array}$$

is commutative where b is the base change map. In short, adjunction commutes with base change up to the base change map.

(ii) *Set $\sigma(F) = (id_{(p_*F)})^\#$. For each base change $g : T \to S$, the square,*

$$\begin{array}{ccc} (p_T)^*g^*p_*F & \xrightarrow{(p_T)^*b} & (p_T)^*(p_T)_*(g_Y)^*F \\ \| & & \downarrow {\scriptstyle \sigma((g_Y)^*F)} \\ (g_Y)^*p^*p_*F & \xrightarrow{(g_P)^*(\sigma(F))} & (g_Y)^*F, \end{array}$$

is commutative where b is the base change map. In short, the formation of σ commutes with base change up to the base change map.

PROOF: The assertions are easy consequences of the definition of b ([EGA I], 9.3.1) and of the three formulas (see [EGA O_I], 3.5.3, 3.5.4, and 3.5.5) for the adjoint of a composition. In fact, (ii) follows from (i) with $u = \sigma(F)$ and from ([EGA O_I], 3.5.4.2). (Note that in ([EGA O_I], 3.5.3) the maps v and w refer to maps,

$$v : \psi^{-1}(H) \to \psi^{-1}(G) \quad \text{and} \quad w : \psi^{-1}(G) \to F,$$

where G and H are sheaves on X and that the map v in (3.5.3.4) should be replaced by w.)

(2.2) DEFINITION: Let $p : Y \to S$ be a morphism of schemes, and let $u : A \to B$ be an O_Y-homomorphism of O_Y-Modules. A closed subscheme of S is called the *scheme of zeroes* of u if it has the universal property that a map $g : T \to S$ factors through it if and only if $u_T = g_Y^*(u)$ is equal to zero. If it exists, the scheme of zeroes is denoted $Z_S(u)$.

(2.3) PROPOSITION: *Let $p : Y \to S$ be a morphism of schemes, and let $u : p^*E \to F$ be an O_Y-homomorphism. Assume that E is quasi-coherent and that p_*F is locally free and its formation commutes with base change. Then the scheme $Z_S(u)$ of zeroes of u exists and it is equal to the scheme $Z_S(u^\flat)$ of zeroes of the adjoint $u^\flat : E \to p_*F$.*

PROOF: By ([EGA I], 9.7.9.1), there exists a scheme of zeroes $Z_S(u^\flat)$. By (2.1), it is equal to $Z_S(u)$.

(2.4) DEFINITION (see [EGA IV$_4$], 16.1.2): Let Y be a closed subscheme of a scheme S and denote by

$$I = I(Y, S)$$

its Ideal. The *conormal sheaf* of Y in S is the O_Y-Module defined by the formula,

$$N(Y, S) = (I/I^2).$$

The *normal sheaf* of Y in S is the dual O_Y-Module and is defined by the formula,

$$\check{N}(Y, S) = \underline{\mathrm{Hom}}_{O_Y}(I/I^2, O_Y).$$

(2.5) LEMMA: *Let E be a locally free O_S-Module, let u be a section of E, and consider the scheme $Y = Z_S(u)$ of zeroes of u. Assume u is regular. Then there is a canonical isomorphism*

$$E^\vee|Y = N(Y, S).$$

PROOF: By construction (see [EGA I], 9.7.9.1), the Ideal I of Y is equal to the image of the map,

$$s^\vee : E^\vee \to O_S.$$

Since s is regular, (I/I^2) is a locally free O_Y-Module whose rank is equal to the rank of E (see [EGA IV$_4$], 16.9.2). Hence $s^\vee$ induces an isomorphism from $E^\vee|Y$ to (I/I^2), and the assertion holds.

(2.6) LEMMA: *Let*

$$A \xrightarrow{u} B \to C \to 0$$

be an exact sequence of quasi-coherent O_S-Modules. Set

$$Y = \mathbb{P}(B), \qquad Y' = \mathbb{P}(C).$$

(i) *The image of the composition*

(2.6.1) $$A_Y(-1) \xrightarrow{u_Y(-1)} B_Y(-1) \xrightarrow{\alpha(-1)} O_Y,$$

is equal to the Ideal of Y' in Y, where α denotes the fundamental surjection.

(ii) *If u is injective and if B and C are locally free, then Y' is equal to the scheme of zeroes $Z_Y(s)$ of a regular section s of $A^{\vee}_Y(1)$ and there is a formula for the conormal sheaf,*

$$N(Y', Y) = A_{Y'}(-1).$$

PROOF: (i) Applying tilde to the natural exact sequence,

$$A[-1]\otimes \mathrm{Sym}(B) \xrightarrow{v} \mathrm{Sym}(B) \to \mathrm{Sym}(C) \to 0,$$

yields an exact sequence,

$$A_Y(-1) \xrightarrow{\tilde{v}} O_Y \to O_{Y'} \to 0,$$

by ([J], A1.2) and ([EGA II], 3.5.2, (i)). Hence $\mathrm{Im}(\tilde{v})$ is equal to the Ideal of Y' in Y. Since v factors through $B[-1]\otimes \mathrm{Sym}(B)$, clearly $\tilde{v}$ is equal to the composition (2.6.1).

(ii) It follows from (i) that Y' is equal to the scheme of zeroes $Z_Y(s)$ where s is equal to the image of 1 under the dual of the composition (2.6.1).

Since Y is flat over S, we may replace S by the spectrum of a field to verify that s is regular by ([EGA IV_3], 11.3.8). Now Y is Cohen-Macaulay because it is smooth over a field. Since $\mathrm{codim}(Y', Y)$ is equal to the rank of A, the section s is regular by ([EGA O_{IV}], 16.5.6). An alternate proof that s is regular may be based on the fact that the sections of B defining Y' come from indeterminates in the homogeneous coordinate (polynomial) ring of Y.

Finally, $N(Y', Y)$ is equal to $A_{Y'}(-1)$ by (2.5).

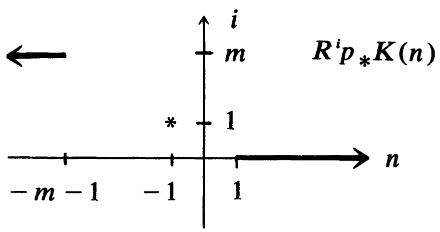
(2.7) PROPOSITION: *There are formulas,*

$$\Omega^1_{G/S} = \underline{\mathrm{Hom}}_{O_G}(Q, M) = Q^{\vee}\otimes M.$$

PROOF: One proof, due to Porteous ([P], Prop. 0.2, p. 292), runs briefly as follows. The tangent bundle of G/S is equal to the normal bundle $\check{N}(\Delta, G\times_S G)$ of the diagonal by ([EGA IV_4], 16.3.1). Let p_1, $p_2: G\times_S G \to G$ denote the projections. Then the diagonal is clearly equal to the subscheme of zeroes of the composition, $p_1^*M \to V_{G\times G} \to p_2^*Q$, so to the zeroes of a global section $\underline{\mathrm{Hom}}(p_1^*M, p_2^*Q)$. This section is clearly regular. Hence $\check{N}(\Delta, G\times G)$ is equal to $\underline{\mathrm{Hom}}(M, Q)$ by (2.5).

A second proof can be based on the deformation theory of quotient Modules ([SGA 3], p. 127), and a third can use the second fundamental form ([Gr 1], (2.19), p. 199 or [GD], I, 3.1, p. 11). A fourth is virtually given in the course of proving (4.2) (see especially the middle isomorphism in (4.2.1) and (4.2.5)).

(2.8) LEMMA: *Let $p: P \to S$ denote the structure map, and let K denote the kernel of the fundamental surjection $\alpha_1^{\#}: V_P \to O_P(1)$. For $m \geq 1$, the only nonzero sheaves of the form $R^i p_* K(n)$ are indicated by the darkened portions of the following figure*:



Moreover, each sheaf $R^i p_* K(n)$ is locally free, and the Euler characteristic is given by

$$\chi(K(n)) \left(= \sum (-1)^i \operatorname{rank} (R^i p_* K(n))\right)$$
$$= (m+1) \binom{m+n}{m} - \binom{m+n+1}{m}.$$

PROOF: For each integer n, there is an exact sequence,

$$0 \to K(n) \to V_P(n) \to O_P(n+1) \to 0.$$

The associated long exact sequence of derived functors $R^i p_*$ and Serre's explicit computation yield the assertions. (Note that the maps $p_* V(n) \to p_* O_P(n)$ are surjective because the compositions,

$$V^{\otimes(n+1)} \to p_* V_P(n) \to p_* O_P(n+1),$$

are surjective for $n \geq 0$.)

(2.9) LEMMA: *Let $0 \to A \to B \to C \to 0$ be an exact sequence of locally free O_S-Modules, with A invertible. Then the following natural sequence* *is exact for* $n \geq 0$:

$$0 \to \mathrm{Sym}_n(B) \otimes A \to \mathrm{Sym}_{n+1}(B) \to \mathrm{Sym}_{n+1}(C) \to 0.$$

PROOF: The exactness is local, so we may assume $B = A \oplus C$ holds. The assertion follows from the following formula (see [EGA I], 9.4.4, p. 371):

$$\mathrm{Sym}(A \oplus C) = \mathrm{Sym}(A) \otimes \mathrm{Sym}(C).$$

(2.10) LEMMA: *Let* $0 \to N \to B \to L \to 0$ *be an exact sequence of locally free* O_S*-Modules, with* L *invertible. For each pair of positive integers* (s, t)*, there is an exact sequence,*

$$0 \to \Lambda^s N \otimes L^{-t} \to \cdots \to \Lambda^{s-j} B \otimes L^{-t+j} \to \cdots \to \Lambda^{s-t} N \to 0, \quad 0 \leq j \leq t-1.$$

PROOF: For each positive integer p, there is an exact sequence (see [SGA 6], V, Lemma 2.2.1, p. 315),

$$0 \to \Lambda^p N \to \Lambda^p B \to \Lambda^{p-1} N \otimes L \to 0.$$

Tensoring these sequences with appropriate powers of L and piecing the results together yield the required exact sequence.

(2.11) PROPOSITION: *For each integer* $n > 0$*, there is an exact sequence,*

(2.11.1) $$0 \to \Lambda^n V \to \cdots \to (\Lambda^{n-j} V) \otimes \mathrm{Sym}_j(V) \to \cdots \to \mathrm{Sym}_n(V) \to 0.$$

PROOF (Kempf [K], p. 8): Applying (2.10) with $s = t = (m+1)$ to the fundamental exact sequence on P and tensoring the result with $O_P(n)$ yield the following exact sequence (the Koszul complex of $\alpha_1^{\#} \otimes O_P(n-1)$),

(2.11.2) $$0 \to \Lambda^{m+1} V_P(n-m-1) \to \cdots \to \Lambda^2 V_P(n-2) \to V_P(n-1) \to O_P(n) \to 0.$$

By Serre's explicit computation there is a formula,

$$R^i p_*((\Lambda^{n-j} V_P)(j)) = \begin{cases} 0 & \text{for all } i \text{ and } j < 0, \text{for } i > 0 \text{ and } j \geq 0 \\ \Lambda^{(n-j)} V \otimes \mathrm{Sym}_j(V) & \text{for } i = 0 \text{ and } j \geq 0, \end{cases}$$

where $p : P \to S$ denotes the structure map. So, the spectral sequence of hyperdirect images of the complex (2.11.2) degenerates and yields the complex (2.11.1). So, (2.11.1) is exact because (2.11.2) is so.

(2.12) LEMMA: *Let R be a locally free O_S-Module with rank* 2. *Then there is a formula,*

$$\Lambda^{n+1}(\mathrm{Sym}_n\,(R)) = (\Lambda^2 R)^{n(n+1)/2} \qquad n \geq 0.$$

PROOF: By ([SGA 6], X, 6.2.3, p. 554), there are an integer t (independent of R) and a formula,

$$\Lambda^{n+1}(\mathrm{Sym}_n\,(R)) = (\Lambda^2 R)^t.$$

To determine t, we may take $R = A \oplus B$ with A and B invertible. Then, since the symmetric algebra of a sum is equal to the tensor product of the symmetric algebras, there is a formula,

$$\Lambda^{n+1}\,\mathrm{Sym}_n\,(A \oplus B) = A^{n(n+1)/2} \oplus B^{n(n+1)/2}.$$

Hence t is equal to $n(n+1)/2$.

(2.13) LEMMA (Svanes [S]): *For any locally free O_S-Module R with rank s, there is a functorial map,*

$$\mathrm{Sym}_t\,(R^\vee) \to \mathrm{Sym}_t\,(R)^\vee, \tag{2.13.1}$$

and it is an isomorphism for $t = 3$ and $s = 2$ if 3 is invertible in O_S.

PROOF: The natural pairing,

$$R^{\otimes t} \otimes (R^\vee)^{\otimes t} \to O_S$$

yields an isomorphism,

$$(R^{\otimes t})^\vee \xrightarrow{\sim} (R^\vee)^{\otimes t}, \tag{2.13.2}$$

because R is locally free with a finite rank. Define the map (2.13.1) as the composition,

$$\mathrm{Sym}_t\,(R)^\vee \to (R^{\otimes t})^\vee \xrightarrow{\sim} (R^\vee)^{\otimes t} \to \mathrm{Sym}_t\,(R^\vee),$$

where the middle map is (2.13.2) and the two end maps come from the natural surjection from the tensor product to the symmetric product.

To show that (2.13.1) is an isomorphism, we may assume S is affine and L is free. Let (u, v) be a basis of L and $(\bar{u}, \bar{v})$ the dual basis of $L^{\vee}$. Then (u^3, u^2v, uv^2, v^3) is a basis of $\mathrm{Sym}_3(L)$. It is easy to see that (2.13.1) carries the dual basis $(\bar{u}^3, \bar{u}^2\bar{v}, \bar{u}\bar{v}^2, \bar{v}^3)$ to the set $(\bar{u}^3, 3\bar{u}^2\bar{v}, 3\bar{u}\bar{v}^2, \bar{v}^3)$. So, if 3 is invertible, then (2.13.1) is an isomorphism.

(2.14) LEMMA: *Let R be a locally free O_S-Module with rank 2 and assume S admits a theory of Chern classes. Then there are formulas,*

$$c_{n+1}(\mathrm{Sym}_n(R)) = \prod_{j=0}^{(n-1)/2} (j(n-j)c_1(R)^2 + (n-2j)^2c_2(R)) \text{ for } n \text{ odd},$$

$$c_{n+1}(\mathrm{Sym}_n(R)) = \frac{n}{2}c_1(R)\prod_{j=0}^{n/2-1} (j(n-j)c_1(R)^2 + (n-2j)^2c_2(R)) \quad \text{for } n \text{ even}.$$

PROOF: By the splitting principle, we may assume R is equal to $A \oplus B$ with A and B invertible. Then the formula for the symmetric product of a sum yields the relation,

$$c_{n+1}(\mathrm{Sym}_n(A \oplus B)) = c_1(A^{\otimes n}) \ldots c_1(A^{\otimes(n-j)} \otimes B^{\otimes j}) \ldots c_1(B^{\otimes n}),$$

by additivity. So the linearity of $c_1(-)$ yields this formula,

$$c_{n+1}(\mathrm{Sym}_n(R)) = [nc_1(A) + nc_1(B)][((n-1)c_1(A) + c_1(B)) \times ((n-1)c_1(B) + c_1(A))] \ldots.$$

It is easy to verify the formula,

$$((n-j)X + jY)((n-j)Y + jX) = j(n-j)(X+Y)^2 + (n-2j)^2XY$$

for indeterminates X, Y. So combining the formulas,

$$c_1(R) = c_1(A) + c_1(B); \qquad c_2(R) = c_1(A)c_1(B),$$

with those above yields the assertion.

(2.15) LEMMA: *Let $p: A \to B$ be a projective morphism of locally noetherian schemes.*

(i) *Let F be a coherent O_A-Module that is flat over B. Then there exist a coherent O_B-Module $Q(F)$ and an element $q(F) \in H^0(A, F \otimes Q(F))$ such that the Yoneda map and the induced map of sheaves,*

$$y(q(F)): \mathrm{Hom}(Q(F), M) \to H^0(A, F \otimes M) \quad \textit{and}$$
$$\underline{y}(q(F)): \underline{\mathrm{Hom}}(Q(F), M) \to p_*(F \otimes M),$$

are isomorphisms for each quasi-coherent O_B-Module M. The pair $(Q(F), q(F))$ is functorial in F and determined up to a unique isomorphism, and its formation commutes with base change. Moreover, if $H^1(A(b), F \otimes k(b))$ is equal to zero for each point b of B, then $Q(F)$ is locally free and there is a canonical functorial isomorphism,

$$Q(F) = (p_* F)^{\vee}.$$

(ii) *Let I, J be coherent O_A-Modules and assume J is flat over B. Then there exists a coherent O_B-Module $H(I, J)$ and an element $h(I, J) \in$* Hom $(I, J \otimes H(I, J))$ *such that the Yoneda map and the induced map of sheaves,*

$$y(h(I,J)) : \mathrm{Hom}\,(H(I,J), M) \to \mathrm{Hom}\,(I, J \otimes M) \quad \textit{and}$$
$$\underline{y}(h(I,J)) : \underline{\mathrm{Hom}}\,(H(I,J), M) \to p_* \,\underline{\mathrm{Hom}}\,(I, J \otimes M),$$

are isomorphisms for each quasi-coherent O_B-Module M. The pair $(H,(I, J), h(I, J))$ is functorial in I and J and determined up to a unique isomorphism, and its formation commutes with base change. Moreover, if I is locally free, then there is a canonical functorial isomorphism,

$$H(I, J) = Q(\underline{\mathrm{Hom}}\,(I, J)).$$

PROOF: Almost all the assertions are discussed in ([ASDS], (12), (13), (14)) and are virtually proved in ([EGA III_2], 7.5.5, 7.7.6, 7.7.8, and 7.7.9).

(2.16) PROPOSITION: *Let A be a projective S-scheme, set D =* $\mathrm{Hilb}_{(A/S)}$, *and let*

$$Y \subset A \times D$$

denote the universal subscheme of A/S. Then there is a canonical isomorphism,

$$H(N(Y, A \times D), O_Y) = \Omega^1_{D/S},$$

and $h = h(N(Y, A \times D), O_Y)$ is the unique map fitting into the following natural commutative diagram:

(2.16.1)
$$\begin{array}{ccc} I(Y, A \times D) \to O_{A \times D} & \xrightarrow{d_D \otimes O_A} & \Omega^1_{D/S} \otimes O_A \\ \big\downarrow & & \big\downarrow \\ N(Y, A \times D) & \xrightarrow{h} & \Omega^1_{D/S} \otimes O_Y. \end{array}$$

Proof: This is a rephrasing of the usual infinitesimal theory of the Hilbert scheme (see [FGA], 221–22, 23, or [SGA 3], p. 130, or [A], p. 62).

(2.17) Theorem: *Let A be a projective S-scheme, set $D = \mathrm{Hilb}_{(A/S)}$ and let $Y \subset A \times D$ denote the universal subscheme of A/S. Let B be a closed subscheme of A, and set $F = \mathrm{Hilb}_{(B/S)}$.*

(i) *F is a closed subscheme of D, and $C = Y \times_D F$ is the universal subscheme of B/S. Moreover, F is equal to the scheme of zeroes $Z_D(v)$, where*

$$v : I(B \times D, A \times D) \to O_Y$$

denotes the natural map.

(ii) *There is a natural commutative diagram with exact rows and canonical isomorphisms in the middle and on the right,*

$$\begin{array}{ccccccc}
H(N(B \times D, A \times D)|C, O_C) & \to & H(N(Y, A \times D)|F, O_C) & \to & H(N(C, B \times D), O_C) & \to & 0 \\
\downarrow & & \downarrow \cong & & \downarrow \cong & & \\
N(F, D) & \longrightarrow & \Omega^1_{D/S}|F & \longrightarrow & \Omega^1_{F/S} & \longrightarrow & 0.
\end{array} \qquad (2.17.1)$$

Proof: The bottom row is exact by ([EGA IV$_4$], 16.4.21) and the two canonical isomorphisms exist by (2.16) (note that the top middle term is equal to $H(N(Y, A \times D), O_Y)|F$ because $H(-,-)$ commutes with base change and because of the defining formula $C = Y \times_D F$). The existence of the top row and its exactness follow from the usual exact sequence of normal sheaves ([EGA IV$_4$], 16.2.7) and abstract nonsense. Alternately, the exactness of the top row will follow from the exactness of the bottom once the commutativity is established. Set

$$I = I(B \times D, A \times D), \qquad \text{and}$$
$$N = N(B \times D, A \times D).$$

Let $w : H(I, O_Y) \to O_D$ denote the map corresponding under the Yoneda map to the natural map, $v : I \to O_Y$. Then $\mathrm{Im}(w)$ defines a closed subscheme F' of D, and there is a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
 & & H(I, O_Y) & \xrightarrow{w} & O_D & \to & O_{F'} & \to & 0 \\
 & & \downarrow w' & & \downarrow id & & \downarrow id & & \\
0 & \to & I(F', D) & \to & O_D & \to & O_{F'} & \to & 0.
\end{array} \qquad (2.17.2)$$

Let $g: T \to D$ be a morphism of schemes. The following four statements are equivalent (the parenthetical comments indicate why a statement and the one before it are equivalent):

(1) g factors through F';

(2) The map, $w_T: H(I, O_Y)_T \to O_T$, is equal to zero (in view of the exactness of the top row in (2.17.2));

(3) The map $v_T: I_T \to O_{Y_T}$ is equal to zero (because the formation of $(H(I, O_Y), h(I, O_Y))$ commutes with base change);

(4) The family Y_T/T is contained in $B \times T/T$. Therefore, F' is equal to $F = \mathrm{Hilb}_{(B/S)}$ and to $Z_D(v)$, and C is equal to the universal subscheme of B/S; that is, (i) holds.

There are formulas,

$$\begin{aligned} H(I, O_Y)|F &= H(I|F, O_Y|F) \\ &= H(N|C, O_C). \end{aligned}$$

The first results from the commutativity of $(H(I, O_Y), h(I, O_Y))$ with base change; the second results from the relation $C = Y \times_D F$ and from the canonical isomorphism,

$$\mathrm{Hom}_{O_{A \times F}}(I|F, O_C \otimes M) = \mathrm{Hom}_{O_C}(N|C, O_C \otimes M)$$

for each O_F-Module M. So w', which is the left hand vertical map in (2.17.2), induces, by restriction to F, a surjection,

$$w'': H(N|C, O_C) \to N(F, D).$$

This is the map that fits into the left hand square of diagram (2.17.1). We now prove that square is commutative.

To show that the two maps in

$$\mathrm{Hom}_{O_F}(H(N|C, O_C), \Omega^1_{D/S}|F)$$

are equal is equivalent to showing that their images in

$$\mathrm{Hom}_{O_C}(N|C, O_C \otimes \Omega^1_{D/S}|F)$$

under the Yoneda map are equal. Now, one image is defined as the composition at the bottom of the natural diagram,

$$\begin{array}{ccccccc} I|F & \to & I(Y, A \times D)|F & \to & O_{A \times F} & \to & O_A \otimes \Omega^1_{D/S}|F \\ \downarrow & & \downarrow & & & & \downarrow \\ N|C & \to & N(Y, A \times D)|C & & \longrightarrow & & O_C \otimes \Omega^1_{D/S}|F. \end{array}$$

The right hand square is commutative because it is the restriction over F of (2.16.1). The left hand square is obviously commutative.

The other image is defined as the composition at the bottom of the natural diagram,

$$\begin{array}{ccccc} I|F & \xrightarrow{v'|F} & I(F,D)\otimes O_C & \to & O_C\otimes(O_D|F) \\ \downarrow & & \downarrow = & & \downarrow \\ N|C & \xrightarrow{v''} & N(F,D)\otimes O_C & \to & O_C\otimes(\Omega^1_{D/S}|F) \end{array}$$

where $v':I\to I(F,D)\otimes O_Y$ is the image of w' under the Yoneda map and v'' is the image of w'' under the Yoneda map. Since the formation of $(H(I,O_Y), h(I,O_Y))$ commutes with base change, $v'|F$ is equal to the image of $w'|F$ under the Yoneda map, and so the left hand square is commutative in view of the way w' induces w''. The right hand square is obviously commutative.

In each diagram the composition at the top followed by the right right hand vertical map is equal to $[O_C\otimes(d_D|F)]\circ(v|F)$. Hence since the two diagrams are commutative and since the left hand vertical map in each is surjective, the two images are equal.

A similar but easier argument shows the right hand square of (2.17.1) is commutative.

3. Schemes of r-planes

(3.1) PROPOSITION: (i) *There is a universal flat family W/H of hypersurfaces of P with degree d.*

(ii) *Let $f:P\to S$ denote the structure map, and for each base change $g:T\to S$, let*

$$b^T:\mathrm{Sym}_d\,(V)_T\to f_{T*}O_{P\times T}(d)$$

denote the base change map. Let

$$\alpha:(\mathrm{Sym}_d\,(V))^\vee_H\to O_H(1)$$

denote the fundamental surjection and set

$$s=(b^H\circ\alpha^\vee)^{\#}.$$

There is a canonical exact sequence on $P_H=P\times_S H$,

$$(3.1.1)\qquad 0\to(O_H(-1))_P\xrightarrow{s}O_{P_H}(d)\to O_W(d)\to 0.$$

(iii) *W is a divisor on P_H, its associated invertible sheaf is equal to $O_H(1)\otimes O_P(d)$, and there is a formula for the conormal sheaf,*

$$N(W, P\times H) = (O_H(-1)\otimes O_P(-d))|W.$$

PROOF: The formation of s from α commutes with an arbitrary base change $T\to H$; that is, there is a formula,

$$s_T = (b^T \circ \alpha_T^\vee)^{\#}. \tag{3.1.2}$$

The formula results from (2.1, (i)) with s for u because of the way the base change maps are compatible with composition ([BC], (6.5), p. 35). Moreover, b^T is an isomorphism for each T by Serre's explicit computation.

Consequently, the restriction of s to each fiber of P_H/H is nonzero as the restriction of α is nonzero. So the restriction of s is injective because each fiber is integral. Hence, since P is flat over S, the map s is injective and coker(s) is flat over H by ([EGA IV_3], 11.3.8).

Define W so that the sequence (3.1.1) is right exact. Then, in view of the above, W is flat over H and (3.1.1) is exact. Obviously W is a family of hypersurfaces with degree d.

Let W'/T be any flat family of hypersurfaces of P with degree d. Then its ideal $I(W', P_T)$ is isomorphic to $O_P(-d_T)$ on the fibers of P_T/T. Hence there are an invertible O_T-Module L and an isomorphism ([M 2], p. 54),

$$I(W', P_T) \cong O_P(-d)\otimes L_P.$$

The inclusion of $I(W', P_T)$ in O_{P_T} gives rise to an injection,

$$s' : L_P \to O_{P_T}(d),$$

and thence to the composite map,

$$(b^T)^{-1}\circ(s')^\flat : L \to f_{T*}O_{P_T}(d) \to \mathrm{Sym}_d\,(V)_T.$$

Clearly $(s')^\flat$ is nonzero on each fiber of P_T/T. Hence $(b^T)^{-1}\circ(s')^\flat$ is also. Therefore, by ([EGA O_I], 5.5.4), $(b^T)^{-1}\circ(s')^\flat$ is left invertible and so its dual is surjective. Therefore, by ([EGA II], 4.2.3), there is a unique S-morphism $T\to H$ with $\alpha_T^\vee = (b^T)^{-1}\circ(s')^\flat$. So s' is equal to s_T by formula (3.1.2). Hence W' is equal to the pullback of W to T. Thus W is the universal flat family of hypersurfaces with degree d.

(3.2) PROPOSITION: (i) *The grassmannian G parametrizes the r-planes of P, and $\mathbb{P}(Q) \subset P \times G$ is the universal flat family of r-planes.*

(ii) *There is a canonical isomorphism,*

$$G = \mathrm{Hilb}^{\varphi}_{(P/S)} \qquad \text{with} \qquad \varphi(n) = \binom{r+n}{r}.$$

PROOF: (i) Obviously $\mathbb{P}(Q)$ is flat over G and is a family of r-planes of P.

Let Y/T be a flat family of r-planes of P. Let R denote the direct image of $O_Y(1)$ on T. By Serre's explicit computation, $H^1(Y(t), O_{Y(t)}(1))$ is equal to zero for each point t of T. Hence R is locally free with rank $(r+1)$ and its formation commutes with base change ([M 1], 3° Lecture 7, pp. 50–53).

Consider the following canonical map and its adjoint:

$$u : V_Y \to O_Y(1)$$
$$u^\flat : V_T \to R.$$

Fix $t \in T$. By (2.1, (i)), the map $u(t)^\flat$ factors as follows:

$$u(t)^\flat : V \otimes k(t) \xrightarrow{(u^\flat)(t)} R(t) \xrightarrow{\sim} H^0(Y(t), O_{Y(t)}(1)).$$

Now, $u(t)^\flat$ is surjective because $Y(t)$ is an r-plane of $P \otimes k(t)$ by hypothesis. Hence $(u^\flat)(t)$ is surjective. Therefore, by Nakayama's lemma, $u^\flat$ is surjective.

With $\sigma = (id_R)^{\#}$, there is a natural commutative diagram (see [EGA $\mathrm{O_I}$], 3.5.4.2),

$$\begin{array}{ccc} V_Y & \xrightarrow{u} & O_Y(1) \\ {\scriptstyle (u^\flat)_Y}\downarrow & \nearrow{\scriptstyle \sigma} & \\ R_Y & & \end{array}$$

There is a corresponding commutative diagram of schemes (see [EGA II], 3.7.1 and 2.8.4),

$$\begin{array}{ccc} P_T & \xleftarrow{\theta(u)} & Y \\ {\scriptstyle \mathbb{P}(u^\flat)}\uparrow & \swarrow{\scriptstyle \theta(\sigma)} & \\ \mathbb{P}(R) & & \end{array}$$

where $\theta(u)$ denotes the composition of Proj(u) with the projection $P_T \times Y \to P_T$.

Let t be a point of T. Set $\sigma' = (id_{R'})^{\#}$ with R' the direct image of $O_{Y(t)}(1)$. Then by (2.1, (ii)) there is a formula,

$$\sigma(t) = \sigma' \circ b_{Y(t)},$$

where b is the base change map for $O_Y(1)$. So, there is a commutative diagram (see [EGA II], 2.8.4),

$$\begin{array}{ccc} & Y(t) & \\ {\scriptstyle \theta(\sigma(t))} \swarrow & & \searrow {\scriptstyle \theta(\sigma')} \\ \mathbb{P}(R(t)) & \xleftarrow[\mathbb{P}(b)]{\sim} & \mathbb{P}(R'). \end{array}$$

By ([EGA II], 4.2.3), the map $\theta(\sigma')$ is an isomorphism. So since $\theta(\sigma(t))$ is equal to $\theta(\sigma)(t)$ (see [EGA II], 3.5.3), the map $\theta(\sigma)(t)$ is an isomorphism.

By ([EGA II], 3.7.1), the map $\theta(u)$ is equal to the given closed embedding of Y in P_T. Hence $\theta(\sigma)$ is an embedding (see [EGA I], 5.1.8). The formation of the Ideal $I(Y, \mathbb{P}(R))$ commutes with base change because Y is flat over T. Each of its fibers is equal to zero because each fiber of $\theta(\sigma)$ is an isomorphism. Hence the ideal itself is equal to zero by Nakayama's lemma. Thus Y is equal to $\mathbb{P}(R)$.

Finally, let $T \to G$ be the map defined by $u^{\flat}$. Then Q_T is equal to R and so $\mathbb{P}(Q)_T$ is equal to Y. Thus $\mathbb{P}(Q)$ is the universal flat family of r-planes.

(ii) A subscheme of a projective space is an r-plane if and only if its Hilbert polynomial is equal to φ. Hence, by (i), G also parametrizes the subschemes with Hilbert polynomial φ.

(3.3) THEOREM: (i) *The pairs consisting of a flat family of hypersurfaces with degree d and a flat family of r-planes contained in it, both parametrized by the same S-scheme T, are the T-points of a closed subscheme Z of* $H \times G$.

(ii) *There is a canonical isomorphism of G-schemes,*

$$Z = \mathbb{P}(K^{\vee}),$$

where K is defined by the following exact sequence on G:

(3.3.1) $$0 \to K \to \mathrm{Sym}_d\,(V)_G \to \mathrm{Sym}_d\,(Q) \to 0.$$

(iii) *Z is the scheme of zeroes of a regular section of the locally free sheaf* $\mathrm{Sym}_d\,(Q)\otimes O_H(1)$ *on* $H\times G$, *and there is a formula for the conormal sheaf,*

$$N(Z, H\times G) = (\mathrm{Sym}_d\,(Q)\otimes O_H(1))^{\vee}|Z.$$

(iv) *Let* ℓ *be a point of* Z, *and* F *the fiber of* Z/H *through* ℓ. *The following conditions are equivalent*:

(a) *Z/H is flat at ℓ.*

(b) *F is the scheme of zeroes of a section of* $\mathrm{Sym}_d\,(Q(x))$, *which is regular at* ℓ, *where x is the image of ℓ in H.*

(c) $\dim_\ell\,(F) = (r+1)(m-1) - \binom{d+r}{r}$.

(c′) $\dim_\ell\,(F) \leq (r+1)(m-1) - \binom{d+r}{r}$.

PROOF: Let Y be a flat family of hypersurfaces with degree d, and ℓ a flat family of r-planes, both parametrized by the same S-scheme T. Then ℓ is contained in Y if and only if the composition,

$$I(Y, P_T) \to O_{P_T} \to O_\ell, \tag{3.3.2}$$

is equal to zero. By (3.1, (i)) and (3.2), there is a map $T \to H\times G$ such that $Y = W_T$ and $\ell = \mathbb{P}(Q)_T$ hold. Hence, by (3.1, (ii)), the twist of (3.3.2) by $O_{P_T}(d)$ is equal to the pullback of the composition,

$$u : O_H(-1)_{P_G} \xrightarrow{s_G} O_{P_H}(d)_G \to O_{\mathbb{P}(Q)_H}(d).$$

Therefore the sought scheme Z is equal to the scheme of zeroes $Z_{H\times G}(u)$. The latter exists by (2.3) because, by Serre's explicit computation, the direct image $\mathrm{Sym}_d\,(Q)$ of $O_{\mathbb{P}(Q)}(d)$ is locally free and its formation commutes with base change. Thus, (i) holds.

By (2.3) as well, Z is equal to $Z_{H\times G}(u^\flat)$. By the formula for the adjoint of a composition ([EGA $\mathrm{O_I}$], 3.5.3.2) and by (3.1.2), $u^\flat$ is canonically isomorphic to the following composition (the two base change maps involved are isomorphisms by Serre's explicit computation):

$$O_H(-1)_G \xrightarrow{\alpha_g^\vee} \mathrm{Sym}_d\,(V)_{H\times G} \to \mathrm{Sym}_d\,(Q)_H.$$

Hence, by construction (see [EGA I], 9.7.9.1), the Ideal of Z is equal to the image of the composition,

$$\mathrm{Sym}_d\,(Q)^\vee_H(-1) \to \mathrm{Sym}_d\,(V)^\vee_{H\times G}(-1) \xrightarrow{\alpha(-1)_G} O_{H\times G}.$$

Apply (2.6) to the sequence dual to (3.3.1). First (2.6, (i)) yields that $\mathbb{P}(K^\vee)$ has the same Ideal as Z, and so these schemes are equal, as asserted in (ii). Next, (2.6, (ii)) yields (iii).

Statements (a) and (b) of (iv) are equivalent by ([EGA IV$_3$], 11.3.8). Statements (b), (c) and (c′) are equivalent by ([EGA O$_{\text{IV}}$], 16.5.6) because of the formulas,

$$\dim(G(z')) = (r+1)(m-1) \text{ and } \operatorname{rank}(\operatorname{Sym}_d(Q)) = \binom{d+r}{r}.$$

(3.4) Remark: It is easy to generalize (3.3) to the case of r-planes contained in several hypersurfaces of various degrees.

4. Infinitesimal theory

(4.1) Lemma: *Set $Y = \mathbb{P}^r_S$, let $p: Y \to S$ denote the structure map, and let s be a point of S. Let*

$$u: O_Y(1)^{\oplus n} \to O_Y(j)$$

be an O_P-homomorphism. If $p_*(u)$ is surjective at s, then u is surjective along $p^{-1}(s)$ for $j \geq 0$, and the converse holds for $r = 1$ and $j \leq 3$.

Proof: Consider the commutative diagram,

$$\begin{array}{ccc} p^*p_*(O_Y(1)^{\oplus n}) & \xrightarrow{p^*p_*(u)} & p^*p_*(O_Y(j)) \\ \downarrow & & \downarrow \\ O_Y(1)^{\oplus n} & \xrightarrow{u} & O_Y(j). \end{array}$$

The right hand vertical map is surjective (see [EGA II], 4.1.6 for the case $j = 1$, the general case is similar). Therefore, if $p_*(u)$ is surjective at s, then u is surjective at each point of $p^{-1}(s)$.

To prove the converse, we may replace S by an infinite extension field k of $k(s)$. Indeed, $p_*(u \otimes k)$ is equal to $p_*(u) \otimes k$ because the formation of $p_* O_Y(i)$ commutes with base change by Serre's explicit computation. Hence, if $p_*(u \otimes k)$ is surjective, $p_*(u)$ is also surjective by Nakayama's lemma.

The map $u(-1): O_Y^{\oplus n} \to O_Y(j-1)$ is defined by n global sections of $O_Y(j-1)$. Since u is surjective, they have no common zero. Therefore, since the set of zeroes of each section is finite, because $r = 1$ holds, and since k is infinite, taking a general linear combination yields a pair of sections with no common zero. So, we may assume $n = 2$ holds.

Since $K = \operatorname{Ker}(u)$ has rank 1, the exact sequence,

$$0 \to K \to O_Y(1)^{\oplus 2} \xrightarrow{u} O_Y(j) \to 0,$$

yields the formula,

$$K = \Lambda^2(O_Y(1)^{\oplus 2}) \otimes O_Y(-j) = O_Y(2-j).$$

Assume $j \leq 3$ holds. Then by Serre's explicit computation, $R^1 p_* K$ is equal to zero. Hence $p_*(u)$ is surjective.

(4.2) THEOREM: *Let F be a fiber of Z/H, and let X be the fiber of W/H over the same point of H. Let ℓ be a point of F, and let L be the corresponding r-plane on $X \otimes k(\ell)$. For $d = 1$, 2, or 3 and $r = 1$, if $X \otimes k(\ell)$ is smooth along L, then W/H is smooth along the image of L and Z/H is smooth at ℓ with relative dimension $(r+1)(m-1) - \binom{d+r}{r}$. Conversely, for any $d \geq 1$ and $r \geq 1$, if F is smooth at ℓ with dimension $(r+1)(m-1) - \binom{d+r}{r}$, then Z/H is smooth at ℓ and $X \otimes k(\ell)$ is smooth along L.*

PROOF: Recall (3.2) that G is equal to $\operatorname{Hilb}^{\varphi}_{(P/S)}$ for $\varphi(n) = \binom{r+n}{n}$ and $\mathbb{P}(Q)$ is the universal flat family of r-planes of P/S. So, since Hilb commutes with base change, $G \times H$ is equal to $\operatorname{Hilb}^{\varphi}_{(P \times H/H)}$ and $Y = \mathbb{P}(Q_H)$ is the universal flat family of r-planes of $P \times H/H$. Let $f: Y \to G \times H$ denote the structure map.

Apply (2.17) with H for S, with $P \times H$ for A, and with W for B. By (2.17, (i)) the scheme Z is equal to $\operatorname{Hilb}^{\varphi}_{(W/H)}$ and $C = \mathbb{P}(Q_Z)$ is the universal flat family of r-planes. By (2.17, (ii)) there is a natural commutative diagram with exact rows,

$$\begin{array}{ccccccc}
H(N(W \times_H Z, P \times Z)|C, O_C) & \to & H(N(Y, P \times_S G)|Z, O_C) & \to & H(N(C, W \times_H Z), O_C) & \to & 0 \\
\downarrow & & \downarrow \cong & & \downarrow \cong & & \\
N(Z, G \times H) & \longrightarrow & \Omega^1_{G \times H/H} & \longrightarrow & \Omega^1_{Z/H} & \longrightarrow & 0.
\end{array} \tag{4.2.1}$$

There are formulas for the restriction of the conormal sheaf,

(4.2.2) $N(W \times_H Z, P \times Z)|C = N(W, P \times H)_Z|C$ by the flatness of W/H

$= (O_H(-1) \otimes O_C(-d))$ by (3.1, (iii)).

Hence there are canonical isomorphisms,

$$(4.2.3)\quad \begin{aligned} H(N(W\times_H Z, P\times Z)|C, O_C) &= Q(\check{N}(W\times_H Z, P\times Z)|C) && \text{by (2.15, (ii))}\\ &= f_*(\check{N}(W\times_H Z, P\times Z)|C)^{\vee} && \text{by (2.15, (i)), (4.2.2), and Serre's computation}\\ &= f_*(O_H(1)\otimes O_C(d))^{\vee} && \text{by (4.2.2)}\\ &= O_H(-1)\otimes \mathrm{Sym}_d\,(Q_Z)^{\vee} && \text{by Serre's computation.} \end{aligned}$$

Similarly, there are canonical isomorphisms,

$$(4.2.4)\quad \begin{aligned} N(Y, P\times G)|Z &= N(C, P\times Z) && \text{by the flatness of } Y/G\times H\\ &= M\otimes O_C(-1) && \text{by (2.6, (ii)) applied to } 0\to M_Z\to V_Z\to Q_Z\to 0. \end{aligned}$$

Hence, as above, there are canonical isomorphisms,

$$(4.2.5)\quad H(N(Y, P\times G)|Z, O_C) = f_*(\check{N}(C, P\times Z))^{\vee} = M\otimes \check{Q}_Z.$$

By (4.2.3) and by (3.3, (iii)), the source and target of the left hand vertical map in (4.2.1) are isomorphic locally free sheaves. Since the map is surjective, it is therefore an isomorphism.

The following eleven statements are equivalent for all $r\geq 1$ and $d\geq 1$, except that the implication (5)$\Rightarrow$(4) holds only for $r=1$ and $d=1$, 2 or 3. The equivalence of (0), (1), (9) and (10) yields the assertions. The parenthetical comments indicate why a statement and the one before it are equivalent.

(0) F is smooth at ℓ with dimension $(r+1)(m-1)-\binom{d+r}{r}$.

(1) Z/H is smooth at ℓ (3.3, (iv) and [EGA IV$_4$], 17.8.2).

(2) The bottom left hand map of (4.2.1) is left invertible at ℓ [EGA IV$_4$], 17.12.1).

(3) The natural map, $f_*(\check{N}(W\times_H Z, P\times Z)|C)^{\vee}\to f_*(\check{N}(C, P\times Z))^{\vee}$, is left invertible at ℓ ((4.2.1), where the left hand vertical map is an isomorphism, (4.2.3) and (4.2.5)).

(4) The natural map, $f_*(\check{N}(C, P\times Z))\to f_*(\check{N}(W\times_H Z, P\times Z)|C)$, is surjective at ℓ (both terms are locally free by (4.2.3) and (4.2.5)).

(5) The natural map, $\check{N}(C, P\times Z)\to \check{N}(W_Z, P\times Z)|C$, is surjective along $f^{-1}(\ell)$ (by (4.2.4), (4.2.2), and (4.1), applied locally).

(6) The natural map, $N(W\times_H Z, P\times Z)|C\to N(C, P\times Z)$, is left invertible along $f^{-1}(\ell)$ (both terms are locally free by (4.2.2) and (4.2.4)).

(7) The composition, $N(W \times_H Z, P \times Z)|C \to N(C, P \times Z) \to \Omega^1_{P\times Z/Z}|C$, is left invertible along $f^{-1}(\ell)$ (the second map is left invertible by ([EGA IV$_4$], 17.12.1) because $P \times Z/Z$ is smooth).

(8) $W \times_H Z/Z$ is smooth along $f^{-1}(\ell)$ ([EGA IV$_4$], 17.12.1).

(9) W/H is smooth along the image of L (because $f^{-1}(\ell) = L$ obviously holds).

(10) $X \otimes k(\ell)$ is smooth along L ([EGA IV$_4$], 17.8.2 because W/H is flat).

(4.3) COROLLARY (of the proof): *Let U denote the smooth locus of Z/H. Set $C = \mathbb{P}(Q_Z)$ and let $f: C \to Z$ denote the structure map. There is a canonical left-exact sequence,*

$$(4.3.1) \quad 0 \to \check{N}(C, W \times_H Z) \to M^\vee \otimes O_C(1) \to O_C(d) \otimes O_H(1) \to 0,$$

and it is exact on $f^{-1}(U)$. Moreover there is a formula,

$$(4.3.2) \qquad f_* \check{N}(C, W \times_H Z) = (\Omega^1_{Z/H})^\vee.$$

PROOF: Sequence (4.3.1) is simply the usual left-exact sequence of conormal sheaves ([EGA IV$_4$], 16.2.7) with the second and third terms identified by (4.2.4) and (4.2.2). Sequence (4.3.1) is exact on $f^{-1}(U)$ by the implication (1)$\Rightarrow$(5) of the proof of (4.2). Finally, formula (4.3.2) follows formally from the right hand vertical isomorphism in diagram (4.2.1) and from (2.15, (ii)) with $M = O_Z$.

(4.4) THEOREM (generalized Tangent Bundle Theorem): *Assume W is the universal family of cubic threefolds (i.e., $d = 3$ and $m = 4$). Let U denote the smooth locus of Z/H. Then there is a canonical isomorphism,*

$$\Omega^1_{Z/H}|U = Q \otimes \Lambda^5 V \otimes O_H(2)|U.$$

PROOF: By (4.3), the sequence (4.3.1) is exact on $f^{-1}(U)$. Applying (2.10) with $s = 3$ and $t = 2$ yields an exact sequence on $f^{-1}(U)$,

$$(4.4.1) \quad 0 \to \Lambda^3(M^\vee) \otimes O_C(-3) \otimes O_H(-2) \to \Lambda^2(M^\vee) \otimes O_C(-1) \to \check{N}(C, W \times_H Z) \to 0,$$

because $\Lambda^j(M^\vee \otimes O_C(-1))$ is equal to $\Lambda^j(M^\vee) \otimes O_C(-j)$ for all j and because the rank of $\check{N}(C, W \times_H Z)$ is equal to 2. By Serre's explicit computation, applying f_* and $R^1 f_*$ to the middle term yields 0. Hence, the long exact sequence of derived images associated to (4.4.1) yields a canonical isomorphism,

(4.4.2) $$f_*\check{N}(C, W\times_H Z)|U \xrightarrow{\sim} \Lambda^3(M^\vee)\otimes O_H(-2)\otimes R^1f_*O_C(-3)|U.$$

It follows from Serre duality (see [EGA III_1], 2.1.16) that $R^1f_*O_C(-3)$ is isomorphic to $\Lambda^2Q_Z^\vee\otimes Q_Z^\vee$. So, the fundamental exact sequence,

$$0\to M\to V_G\to Q\to 0,$$

yields a canonical isomorphism,

(4.4.3) $$\Lambda^3M_Z^\vee\otimes R^1f_*O_C(-3)=\Lambda^5V_Z^\vee\otimes Q_Z^\vee.$$

Combining (4.4.3), (4.4.2) and (4.3.2) yields the assertion.

5. Cohomological study

In this section, take S to be the spectrum of a field k, and take $r=1$ and $d=3$. Recall,

$$E=\mathrm{Sym}_d\,(Q)^\vee.$$

The object is to prove the following theorem, which summarizes the results in Propositions (5.8)–(5.13).

(5.1) THEOREM: *For $n=0$, 1, 2 and $m\geq 4$, the dimensions $h^i(\Lambda^jE(n))$ of the only nonzero vector spaces of the form $H^i(G,(\Lambda^jE)(n))$ are the following: for $m=4$,*

$n=0$, $\quad h^6(\Lambda^4E)=10,\quad h^3(\Lambda^2E)=5,$

$n=1$, $\quad h^6(\Lambda^4E(1))=1,\quad h^3(\Lambda^2E(1))=5,$

$n=2$, $\quad h^3(\Lambda^3E(2))=1;$

for $m=5$,

$n=0$, $\quad h^8(\Lambda^4E)=1,\quad h^4(\Lambda^2E)=1;$

for all $m\geq 4$,

$$h^0(O_G)=1,\quad h^0(O_G(1))=\frac{m(m+1)}{2},\quad h^0(O_G(2))=\frac{m(m+1)^2(m+2)}{12};$$

except that in characteristic 3 for all $m\geq 4$ both $h^1(E(2))$ and $h^0(E(2))$ may be nonzero.

(5.2) Setup: Let Γ denote the incidence correspondence subscheme of $P \times G$ that parametrizes the pairs consisting of a point and a line through it. The subscheme Γ exists because it is a flag scheme ([EGA I], 9.9.3).

$$\begin{array}{ccccl} R & & N & & \text{fundamental sheaves for } q \text{ and } p \\ \mathbb{P}(Q) & = \Gamma = & \mathbb{P}(K) & & \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} & & \\ G & & P & & \end{array}$$

$$0 \to M \to V_G \to Q \to 0;$$

$$0 \to K \to V_P \to O_P(1) \to 0 \quad \text{fundamental sequences}$$

fig. 1.

It is easy to establish the two canonical isomorphisms involving Γ in figure 1. It is easy to establish a canonical isomorphism,

(5.2.1) $$R = p^*O_P(1),$$

and the existence of the following commutative diagram with exact rows and columns,

(5.2.2) $$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ 0 \to & q^*M & \xrightarrow{id} & q^*M & \longrightarrow 0 \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & p^*K & \longrightarrow & V_\Gamma & \longrightarrow R \to 0 \\ & \downarrow{\scriptstyle \alpha_1^{\#}(K)} & & \downarrow & \downarrow{\scriptstyle id} \\ 0 \to & N & \longrightarrow & q^*Q & \xrightarrow{\alpha_1^{\#}(Q)} R \to 0 \\ & \downarrow & & \downarrow & \downarrow \\ & 0 & & 0 & 0 \end{array}$$

(One proves $\operatorname{Ker}(\alpha_1^{\#}(Q)) = N$ and $V_\Gamma/\operatorname{Ker}(\alpha_1^{\#}(K)) = q^*Q$.) Note for future reference the formula,

(5.2.3) $$\Lambda^2 q^*Q = N \otimes R.$$

(5.3) Lemma: *The dualizing sheaves are given by the following formulas*:

$$\omega_{\Gamma/G} = N \otimes R^{-1}$$
$$\omega_{\Gamma/P} = N^{-m} \otimes R^{-1}$$
$$\omega_{\Gamma/k} = N^{-m} \otimes R^{-m-2}.$$

PROOF: By (2.7), there are formulas,

(5.3.1)
$$\Omega^1_{\Gamma/G} = N \otimes R^{-1}$$
$$\Omega^1_{\Gamma/P} = N^{-1} \otimes q^*M$$
$$\Omega^1_{P/k} = O_P(-1) \otimes K.$$

In view of (5.2.1), the third formula yields the following one:

$$p^*\Omega^1_{P/k} = R^{-1} \otimes p^*K.$$

The formula for the determinant of a tensor product and the additivity of the determinant yield formulas,

$$\Omega^{m-1}_{\Gamma/P} = N^{1-m} \otimes \Lambda^{(m-1)} q^*M$$
$$= N^{1-m} \otimes (\overset{2}{\Lambda} q^*Q)^{-1} \qquad \text{(see (5.2.2))}$$

(5.3.2)
$$\Omega^{m-1}_{\Gamma/P} = N^{-m} \otimes R^{-1}. \qquad \text{(see (5.2.3))}$$

Since Γ/P is smooth, the sequence

$$0 \to p^*\Omega^1_{P/k} \to \Omega^1_{\Gamma/k} \to \Omega^1_{\Gamma/P} \to 0$$

is exact (EGA IV_4, 17.2.3). So there are formulas,

$$\Omega^{(2m-1)}_{\Gamma/k} = (\Lambda^m p^*K \otimes R^{-m}) \otimes (N^{-m} \otimes R^{-1})$$

(5.3.3)
$$\Omega^{(2m-1)}_{\Gamma/k} = N^{-m} \otimes R^{-m-2}. \qquad \text{(see (5.2.2))}$$

Finally, since $\Omega^{\text{top}}_{A/B}$ is equal to $\omega_{A/B}$ when A/B is smooth ([RD], p. 140), the assertion holds.

(5.4) LEMMA: *There are formulas,*

(5.4.1)
$$\Lambda^4 E = (\Lambda^2 Q)^{-6},$$

(5.4.2)
$$\Lambda^j E(n) = \Lambda^{(4-j)}(\mathrm{Sym}_3(Q)) \otimes (\Lambda^2 Q)^{\otimes(n-6)}.$$

PROOF: The first formula follows immediately from (2.12). Now, by the duality of the exterior algebra, there is a formula,

$$\Lambda^j E = \underline{\mathrm{Hom}}\,(\Lambda^{4-j}E, \Lambda^4 E) \qquad 0 \le j \le 4.$$

Since $\Lambda^2 Q$ is equal to $O_G(1)$, the second formula now follows from the first.

(5.5) LEMMA: *For each integer j, there are exact sequences,*

(5.5.1) $$0 \to (\Lambda^5 V_P)(j-3) \to (\Lambda^4 V_P)(j-2) \to (\Lambda^3 V_P)(j-1) \to (\Lambda^2 K)(j) \to 0,$$

(5.5.2) $$0 \to (\Lambda^2 K)(j) \to (\Lambda^2 V_P)(j) \to V_P(j+1) \to O_P(j+2) \to 0.$$

PROOF: These two sequences result from (2.10) applied to the fundamental short exact sequence on P (see fig. 1).

(5.6) LEMMA: *For $m \geq 3$, there are formulas,*

(5.6.1) $$h^i(P, \mathrm{Sym}_2(K)) = \begin{cases} \dfrac{m(m+1)}{2} & i = 1 \\ 0 & i \neq 1 \end{cases}$$

(5.6.2) $$h^i(P, \mathrm{Sym}_2(K)(-3)) = 0 \qquad \text{all } i.$$

PROOF: The exact sequence,

$$0 \to K \otimes K(j) \to V_P \otimes K(j) \to K(j+1) \to 0,$$

with $j = 0$ and with $j = -3$ yields the formulas,

$$h^i(P, K \otimes K) = \begin{cases} \dfrac{m(m+1)}{2} & i = 1 \\ 0 & i \neq 1 \end{cases}$$

$$h^i(P, K \otimes K(-3)) = 0 \qquad \text{all } i,$$

in view of (2.8).

The exact sequences, (5.5.1) with $j = 0$ and (5.5.2) with $j = -3$ yield the formulas,

$$h^i(P, \Lambda^2 K) = 0 \qquad \text{all } i$$

$$h^i(P, \Lambda^2 K(-3)) = 0, \qquad \text{all } i,$$

in view of Serre's explicit computation.

Finally consider the exact sequence,

$$0 \to \Lambda^2 K(j) \to K \otimes K(j) \to \mathrm{Sym}_2(K)(j) \to 0,$$

obtained from (2.11). With $j = 0$, it yields (5.6.1), and with $j = -3$ it yields (5.6.2).

(5.7) LEMMA: *There are formulas,*

$$(5.7.1) \qquad H^i(\Gamma, N^s \otimes R^t) = \begin{cases} H^i(P, \mathrm{Sym}_s(K)(t)) & s \geq 0 \\ 0 & -1 \geq s \geq -(m-1) \\ H^{i-(m-1)}(P, R^{(m-1)}p_*(M^s)(t)) & -m \geq s \end{cases}$$

$$(5.7.2) \qquad H^i(\Gamma, N^s \otimes R^t) = \begin{cases} H^i(G, \mathrm{Sym}_{(t-s)}(Q)(s)) & t \geq s \\ 0 & t = s-1 \\ H^{i-1}(G, R^1q_*(R^{t-s})(s)). & t < s-1 \end{cases}$$

PROOF: By (5.2.1) there is a canonical isomorphism,

$$(R^jp_*N^s)(t) = R^jp_*(N^s \otimes R^t).$$

So Serre's explicit computation for $R^jp_*N^s$ implies that the Leray spectral sequence,

$$H^i(P, R^jp_*(N^s \otimes R^t)) \Rightarrow H^{i+j}(\Gamma, N^s \otimes R^t),$$

degenerates and yields the formula (5.7.1). The formula (5.7.2) follows similarly because of the formulas,

$$\begin{aligned} q^*O_G(1) &= \Lambda^2 Q_\Gamma \\ &= N \otimes R. \end{aligned} \qquad \text{(see (5.2.2))}.$$

(5.8) PROPOSITION: *There are formulas,*

$h^i(G, O_G) = 0$ *for* $i \geq 1$ *and* $h^0(G, O_G) = 1$

$h^i(G, O_G(1)) = 0$ *for* $i \geq 1$ *and* $h^0(G, O_G(1) = \dfrac{m(m+1)}{2},$

$h^i(G, O_G(2)) = 0$ *for* $i \geq 1$ *and* $h^0(G, O_G(2)) = \dfrac{m(m+1)^2(m+2)}{12}.$

PROOF: For arbitrary $n \geq 0$, Laksov [Lk] has verified the formulas,

$$h^i(G, O_G(n)) = \begin{cases} \binom{m+n}{n}\binom{m+n-1}{n} - \binom{m+n-1}{n-1}\binom{m+n}{n+1} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

ALTERNATE PROOF: Formulas (5.7) yield the formulas,

$$h^i(G, O_G) = h^i(P, O_P), \qquad h^i(G, O_G(1)) = h^i(P, K(1)).$$

So the cases $n = 0$ and $n = 1$ follow from Serre's explicit computation and from (2.8). The case $n = 2$ is similar but more involved.

(5.9) PROPOSITION: *For $n = 0, 1, 2$ and for $m \geq 4$, the dimensions $h^i(\Lambda^4 E(n))$ of the only nonzero vector spaces of the form $H^i(G, (\Lambda^4 E)(n))$ are the following: $h^8(\Lambda^4 E) = 1$ for $m = 5$; and $h^6(\Lambda^4 E) = 10$ and $h^6(\Lambda^4 E(1)) = 1$ for $m = 4$.*

PROOF: There are canonical isomorphisms,

$$\begin{aligned} H^i(G, \Lambda^4 E(n)) &= H^i(G, (\Lambda^2 Q)^{(n-6)}) && \text{by (2.12)} \\ &= H^i(\Gamma, N^{(n-6)} \otimes R^{(n-6)}). && \text{by (5.7.2)} \\ &= H^{(2m-1)-i}(\Gamma, N^{6-m-n} \otimes R^{4-m-n}). && \text{by duality and (5.3)} \end{aligned}$$

The last term is equal to zero for $5 \geq n \geq 7 - m$ by (5.7.1). The assertions now result from (5.7.1), from Serre's explicit calculation, from (2.8) and from (5.6).

(5.10) PROPOSITION: *For $n = 0, 1, 2$ and for $m \geq 4$, the dimension $h^i(\Lambda^3 E(n))$ of the only nonzero vector space of the form $H^i(G, (\Lambda^3 E)(n))$ is given by the formula,*

$$h^3(\Lambda^3 E(2)) = 1 \qquad \text{for } m = 4.$$

PROOF: By (5.4.2), there is a canonical isomorphism,

$$\Lambda^3 E(n) = \mathrm{Sym}_3(Q) \otimes (\Lambda^2 Q)^{(n-6)}.$$

Therefore there are formulas,

$$\begin{aligned} H^i(G, \Lambda^3 E(n)) &= H^i(\Gamma, N^{(n-6)} \otimes R^{(n-3)}) && \text{by (5.7.2)} \\ &= H^{(2m-1)-i}(\Gamma, N^{6-m-n} \otimes R^{1-m-n}) && \text{by duality and (5.3).} \end{aligned}$$

The last term is equal to zero for $5 \geq n \geq 7-m$ by (5.7.1). The assertion now results from (5.7.1), from Serre's explicit computation, from (2.8), and from (5.6).

(5.11) Proposition: *For $n = 0, 1, 2$ and for $m \geq 4$, the dimensions $h^i(\Lambda^2 E(n))$ of the only nonzero vector spaces of the form $H^i(G, (\Lambda^2 E)(n))$ are the following*: $h^2(\Lambda^2 E) = 1$ *for* $m = 5$; *and* $h^3(\Lambda^2 E) = 5$ *and* $h^3(\Lambda^2 E(1)) = 5$ *for* $m = 4$.

Proof: By Serre's explicit computation, the spectral sequence,

$$H^i(S, R^j p_* O_P \otimes L) \Rightarrow H^{i+j}(P, p^* L),$$

degenerates and yields canonical isomorphisms,

$$H^i(S, L) = H^i(P, p^* L), \tag{5.11.1}$$

for any locally free sheaf L with a finite rank.

There are canonical isomorphisms,

$$\begin{aligned} H^i(G, \Lambda^2 E(n)) &= H^i(G, \Lambda^2(\mathrm{Sym}_3(Q)) \otimes (\Lambda^2 Q)^{\otimes n-6}) && \text{by } (5.4.2) \text{ and } O_G(n) = (\Lambda^2 Q)^{\otimes n} \\ &= H^i(\Gamma, \Lambda^2(\mathrm{Sym}_3(Q_\Gamma)) \otimes N^{(n-6)} \otimes R^{(n-6)}) && \text{by } (5.11.1) \text{ and } (5.2.3). \end{aligned} \tag{5.11.2}$$

For each n, the bottom sequence in (5.2.2) yields the exact sequences,

$$0 \to N^{(n-2)} \otimes R^{(n-4)} \to Q_\Gamma \otimes N^{(n-3)} \otimes R^{(n-4)} \to N^{(n-3)} \otimes R^{(n-3)} \to 0$$
$$0 \to N^{(n-3)} \otimes R^{(n-3)} \to Q_\Gamma \otimes N^{(n-4)} \otimes R^{(n-3)} \to N^{(n-4)} \otimes R^{(n-2)} \to 0.$$

By (5.7.1), all the end terms have zero cohomology for $n = 0, 1, 2$ except R^{-4} and $N^{-4} \otimes R^{-2}$. However, it is easy to see that these terms have zero cohomology as well using duality on Γ, (5.7.1), and Serre's explicit computation. These formulas follow:

$$h^i(\Gamma, Q_\Gamma \otimes N^{(n-3)} \otimes R^{(n-4)}) = 0 \quad \text{for } n = 0, 1, 2, \quad \text{all } i \tag{5.11.3}$$

$$h^i(\Gamma, Q_\Gamma \otimes N^{(n-4)} \otimes R^{(n-3)}) = 0 \quad \text{for } n = 0, 1, 2, \quad \text{all } i. \tag{5.11.4}$$

By (2.9) applied to the bottom sequence in (5.2.2.), there is an exact sequence,

(5.11.5) $$0 \to Q_\Gamma \otimes N \to \mathrm{Sym}_2(Q_\Gamma) \to R^2 \to 0.$$

Applying (2.10) and tensoring with $N^{(n-4)} \otimes R^{(n-4)}$ yield the exact sequence,

$$0 \to N^{(n-1)} \otimes R^{(n-5)} \to \Lambda^2\,\mathrm{Sym}_2(Q_\Gamma) \otimes N^{(n-4)} \otimes R^{(n-6)} \to Q_\Gamma \otimes N^{(n-3)} \otimes R^{(n-4)} \to 0.$$

For $n = 0, 1, 2$, the end terms have zero cohomology by (5.11.3) and by (5.7.1), Serre's explicit computation, and (2.8). This formula follows:

(5.11.6) $$h^i(\Gamma, \Lambda^2\,\mathrm{Sym}_2(Q)_\Gamma \otimes N^{(n-4)} \otimes R^{(n-6)}) = 0 \quad \text{for } 0,\ 1,\ 2, \quad \text{all } i.$$

Tensoring (5.11.5) with $N^{(n-5)} \otimes R^{(n-3)}$ yields the exact sequence,

$$0 \to Q_\Gamma \otimes N^{(n-4)} \otimes R^{(n-3)} \to \mathrm{Sym}_2(Q_\Gamma) \otimes N^{(n-5)} \otimes R^{(n-3)} \to N^{(n-5)} \otimes R^{(n-1)} \to 0.$$

The left end terms have zero cohomology by (5.11.4). So there is a formula,

(5.11.7) $$H^i(\Gamma, \mathrm{Sym}_2(Q_\Gamma) \otimes N^{(n-5)} \otimes R^{(n-3)}) = H^i(\Gamma, N^{(n-5)} \otimes R^{(n-1)}) \quad \text{for } n = 0, 1, 2, \quad \text{all } i.$$

By (2.9) applied to the bottom sequence of (5.2.2), there is an exact sequence,

$$0 \to \mathrm{Sym}_2(Q_\Gamma) \otimes N \to \mathrm{Sym}_3(Q_\Gamma) \to R^3 \to 0 \quad \text{for } n = 0, 1, 2, \quad \text{all } i.$$

Applying (2.10) and tensoring with $N^{(n-6)} \otimes R^{(n-6)}$ yields the exact sequence,

$$0 \to \Lambda^2\,\mathrm{Sym}_2(Q_\Gamma) \otimes N^{(n-4)} \otimes R^{(n-6)} \to \Lambda^2\,\mathrm{Sym}_3(Q_\Gamma) \otimes N^{(n-6)} \otimes R^{(n-6)} \to \mathrm{Sym}_2(Q_\Gamma) \otimes N^{(n-5)} \otimes R^{(n-5)} \to 0.$$

Therefore, (5.11.2), (5.11.6), and (5.11.7) yield the formula,

(5.11.8) $$h^i(\Lambda^2 E(n)) = H^i(\Gamma, N^{(n-5)} \otimes R^{(n-1)}) \quad \text{for } n = 0,\ 1,\ 2, \quad \text{all } i.$$

By (5.7.1) the formula $h^i(\Lambda^2 E(n)) = 0$ for all i now holds if the inequalities $4 \geq n \geq 6 - m$ hold. For $n = 2$ the assertion thus holds. For

$n = 0, 1$, there is a formula,

$$h^i(\Lambda^2 E(n)) = h^{(2m-1)-i}(\Gamma, N^{5-m-n} \otimes R^{-1-m-n}) \quad \text{for all } i$$

by duality and (5.3). The remaining assertions now follow from (5.7.1) and Serre's explicit computation.

(5.12) LEMMA: *For $n = 0, 1, 2$ and for $m \geq 4$, all vector spaces of the form $H^i(G, E(n))$ are equal to zero with the possible exceptions of $H^i(G, E(2))$ for $i = 0, 1$.*

PROOF: Since R is the fundamental sheaf for Γ/G, there is a canonical isomorphism,

$$\mathrm{Sym}_3(Q) = q_*(R^3),$$

by Serre's explicit computation. So there are formulas,

$$\begin{aligned} E &= R^1 q_*(\omega_{\Gamma/G} \otimes R^{-3}) && \text{by duality} \\ &= R^1 q_*(N \otimes R^{-4}) && \text{by (5.3)} \\ &= R^1 q_*(R^{-5}) \otimes \Lambda^2 Q. && \text{by (5.2.3)} \end{aligned}$$

Hence there are formulas,

$$\begin{aligned} H^i(G, E(n)) &= H^i(G, R^1 q_*(R^{-5}) \otimes (\Lambda^2 Q)^{(n+1)}) \\ &= H^{i+1}(\Gamma, N^{(n+1)} \otimes R^{(n-4)}) && \text{by (5.7.2)} \\ &= H^{i+1}(P, \mathrm{Sym}_{(n+1)}(K)(n-4)) && \text{by (5.7.1)} \end{aligned} \tag{5.12.1}$$

for $n = 0, 1, 2$. The cases $n = 0$ and $n = 1$ now result from (2.8) and from (5.6.2).

The proof of the case $n = 2$ proceeds in several steps.

(a) Applying (2.10) to the fundamental sequence on P yields these exact sequences,

$$0 \to \Lambda^{m+1} V_P(-m) \to \cdots \to \Lambda^4 V_P(-3) \to \Lambda^3 K(-2) \to 0,$$

$$0 \to \Lambda^{m+1} V_P(-m) \to \cdots \to \Lambda^3 V_P(-2) \to \Lambda^2 K(-1) \to 0.$$

The first yields the formula,

$$h^i(P, \Lambda^3 K(-2)) = 0 \quad \text{for all } i \tag{5.12.2}$$

in view of Serre's explicit computation. Tensoring the second with $K(-1)$ yields the exact sequence,

$$0 \to \Lambda^{m+1} V_P \otimes K(-m-1) \to \cdots \to \Lambda^3 V_P \otimes K(-3) \to \Lambda^2 K \otimes K(-2) \to 0.$$

The spectral sequence of cohomology and (2.8) yield the formula,

$$(5.12.3) \quad h^i(P, \Lambda^2 K \otimes K(-2)) = \begin{cases} m+1 & i=2 \\ 0 & i \neq 2. \end{cases}$$

(b) Tensoring the fundamental exact sequence with $K(n)$ yields the exact sequence,

$$0 \to K \otimes K(n) \to V_P \otimes K(n) \to K(n+1) \to 0.$$

By (2.8), it yields the formulas,

$$h^i(P, K \otimes K(-2)) = \begin{cases} 1 & i=2 \\ 0 & i \neq 2 \end{cases}$$

$$h^i(P, K \otimes K(-1)) = \begin{cases} m+1 & i=1 \\ 0 & i \neq 1. \end{cases}$$

The exact sequence,

$$0 \to K \otimes K \otimes K(-2) \to V_P \otimes K \otimes K(-2) \to K \otimes K(-1) \to 0,$$

obtained by tensoring the fundamental exact sequence with $K \otimes K(-2)$, now yields the formula,

$$h^i(P, K \otimes K \otimes K(-2)) = \begin{cases} 2(m+1) & i=2 \\ 0 & i \neq 2. \end{cases}$$

The exact sequence,

$$0 \to \Lambda^2 K \otimes K(-2) \to K \otimes K \otimes K(-2) \to \mathrm{Sym}_2(K) \otimes K(-2) \to 0$$

obtained from (2.11), now yields the formula,

$$(5.12.4) \quad h^i(P, \mathrm{Sym}_2(K) \otimes K(-2)) = 0 \qquad i \neq 1, 2.$$

(c) Consider the exact sequence,

$$0 \to \Lambda^3 K(-2) \to \Lambda^2 K \otimes K(-2) \to K \otimes \mathrm{Sym}_2(K)(-2) \to \mathrm{Sym}_3(K)(-2) \to 0$$

obtained from (2.11). It yields a spectral sequence of cohomology whose E_1-terms are all zero except for $H^2(P, \Lambda^2 K \otimes K(-2))$ and possibly $H^i(P, K \otimes \mathrm{Sym}_2(K)(-2))$ for $i = 1, 2$ and whose end terms are zero. Since $h^i(E(2))$ is equal to $h^{i+1}(P, \mathrm{Sym}_3(K)(-2))$ by (5.12.1), the case $n = 2$ follows.

(5.13) PROPOSITION: *Assume* $\mathrm{char}(k) \neq 3$ *holds. Then there is a formula,*

$$h^i(E(2)) = 0 \qquad \textit{for all } i.$$

PROOF: Since Q has rank 2, there is a canonical isomorphism,

$$\Gamma = \mathbb{P}(Q^\vee),$$

because of the formula,

$$Q^\vee = Q \otimes (\Lambda^2 Q)^{-1}.$$

Moreover, N^{-1} is equal to the fundamental sheaf of $\mathbb{P}(Q^\vee)$ because of the formula (see (5.2.2)),

$$R \otimes (\Lambda^2 Q)^{-1} = N^{-1}.$$

Therefore, there are formulas,

$$\begin{aligned} q_*(N^{-1} \otimes R^2) &= q_*(N^{-3}) \otimes (\Lambda^2 Q)^2 \qquad (5.2.3)\\ &= \mathrm{Sym}_3(Q^\vee)(2). \end{aligned}$$

Since $\mathrm{char}(k) \neq 3$ holds, $\mathrm{Sym}_3(Q^\vee)$ is isomorphic to E. Therefore there are formulas,

$$\begin{aligned} h^i(E(2)) &= h^i(\Gamma, N^{-1} \otimes R^2) && \text{by Leray s.s.}\\ &= 0 && \text{by (5.7.1).} \end{aligned}$$

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