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CATEGORIES OF WALLMAN EXTENDIBLE FUNCTIONS

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It is shown in [1] that the collection of all T_1 spaces and all *WI* functions is a category which contains all Wallman embeddings and in which each function has a unique Wallman extension. The question is then raised as to whether this category is maximal with respect to these properties. In [2] a larger category is constructed, which, from its construction, is clearly not maximal. It will be shown in this paper that in fact there is no unique category which is maximal with respect to satisfying these two properties.

All spaces will be assumed to be T_1 spaces and all functions will be assumed to be continuous. Recall that for a space X, the Wallman compactification WX is the collection $\{\mu : \mu$ is an ultrafilter in the lattice of all closed subsets of $X\}$ with topology generated by $\{C(A) = \{\mu \in WX : A \in \mu\} : A$ is a closed subset of $X\}$ as a base for the closed subsets of WX. With this topology WX is a compact T_1 space and the Wallman embedding $\varphi_X : X \to WX$ defined by $\varphi_X(x) = \{A : A$ is a closed subset of X and $x \in A\}$ is a dense embedding. When no ambiguity seems likely, the distinction between $\varphi_X[X]$ and X will be ignored and X will be spoken of as subset of WX. It is easily seen that for any closed subset A of X, if B is a compact subset of WX and $A \subseteq B$ then $C(A) \subseteq B$. Also for any two closed subsets A and B of X, $C(A \cap B) = C(A) \cap C(B)$. A function $f^* : WX \to WY$ is a Wallman extension of $f : X \to Y$ if $f^* \circ \varphi_X = \varphi_Y \circ f$. A filter $\mathfrak{F}$ in the lattice of closed subsets of X is said to be indicative in X if $\cap_{A \in \mathfrak{F}} C(A)$ is a singleton. A function $f : X \to Y$ is said to be a *WI* function provided that f has a Wallman extension $f^* : WX \to WY$ and that for any indicative filter $\mathfrak{F}$ in X, $\{B : B$ closed in Y and $f[A] \subseteq B$ for some $A \in \mathfrak{F}\}$ is indicative in Y.

In order to show that there is no maximal category of uniquely Wallman extendible functions another category of uniquely Wallman extendible functions will be defined and it will be shown that any category which contains both this and the *WI* functions must contain functions which are not uniquely Wallman extendible.

A function $f:X \to Y$ will be called a *WK* function if it has a Wallman extension $f^*: WX \to WY$ such that for any compact subset $A \subseteq WY$, $f^{*-1}[A]$ is compact. It is immediate that the *WK* functions form a category which contains all Wallman embeddings. It must now be shown that *WK* functions have unique Wallman extensions.

THEOREM 1: *If $f:X \to Y$ is a WK function, then f has precisely one Wallman extension.*

PROOF: From the definition of a *WK* function we know that f has a Wallman extension $f^*: WX \to WY$ such that the inverse image under f^* of any compact set is compact. Suppose that f has another Wallman extension $g: WX \to WY$. Since g and f^* are distinct there is some $\mu \in WX$ such that $g(\mu) \neq f^*(\mu)$. Then $g^{-1}(f^*(\mu))$ is a closed subset of WX which does not contain μ; so $\{C(A): A \in \mu\}$ is a collection of closed sets whose intersection with the compact set $g^{-1}(f^*(\mu))$ is empty. Hence there is a finite subcollection $\{C(A_i): i = 1_{n,..,}n\}$ such that $\bigcap_{i=1}^n C(A_i) = C(\bigcap_{i=1}^n A_i)$ has empty intersection with $g^{-1}(f^*(\mu))$; so $g[C(\bigcap_{i=1}^n A_i)]$ is a compact subset of WY which does not contain $f^*(\mu)$. Then $f^{*-1}[g[C(\bigcap_{i=1}^n A_i)]]$ is a compact subset of WX which contains $\bigcap_{i=1}^n A_i$ but does not contain μ. However every compact subset of WX which contains $\bigcap_{i=1}^n A_i$ must contain $C(\bigcap_{i=1}^n A_i)$ and $\mu \in C(\bigcap_{i=1}^n A_i)$. Therefore f^* is the only Wallman extension of f.

From [1] we know that the collection of *WI* functions forms a category of uniquely Wallman extendible functions which contains all Wallman embeddings, and it has just been established that the collection of *WK* functions also satisfies these properties.

THEOREM 2: *Any category which contains all WI functions and all WK functions must contain functions which are not uniquely Wallman extendible.*

PROOF: Let Q denote the rational numbers with the usual topology. Let Q' denote the space whose points are the elements of WQ and whose topology is generated by the open sets in Q together with all sets of the form $\{\mu\} \cup U$ where $\mu \in WQ \sim Q$ and U is an open subset of Q containing an element of μ. Let Q^+ denote the space whose points are $Q' \cup \{a\}$ and whose open sets are open subsets of Q' or sets of the form $\{a\} \cup U$ where $Q' \sim U$ is finite. Let N denote the positive integers and i the identity map from N into Q^+. Note that $WQ \sim Q$ is a closed discrete subspace of Q'. Thus if A is a compact subset of $Q^+ \sim \{a\}$ then $A \cap (WQ \sim Q)$ must be finite. The space Q' is Hausdorff; so any compact subset of $Q^+ \sim \{a\}$ is closed in Q'. The closure in Q', however, of any noncompact subset of Q contains infinitely many points of $WQ \sim Q$. Therefore if A is a compact subset of $Q^+ \sim \{a\}$ then $A \cap Q$ is compact. The function $i : N \to Q^+$ has an obvious Wallman extension $i^* : WN \to Q^+$ which carries each element of $WN \sim N$ to the cofinite point a. The inverse image under i^* of any compact subset of Q^+ is either the intersection of a closed subset and a compact subset of Q or contains all of $WN \sim N$, and any subset of WN which contains $WN \sim N$ must be compact. Thus i is a WK function. Let Q^* denote the space whose elements are the elements of WQ and whose topology is generated by the open sets in WQ together with the open sets in Q. Let Q^c denote the set $Q^* \cup \{c\}$ with topology the open sets in Q^* together with all sets of the form $\{c\} \cup U$ where $Q^* \sim U$ is finite. Since, as is easily shown, any indicative filter in Q^+ must contain a singleton or a set of the form $\{a\} \cup A$ where $A \subseteq WQ \sim Q$, the function $f : Q^+ \to Q^c$ defined by $f(x) = x$ if $x \in Q$ and $f(x) = c$ if $x \notin Q$ is a WI function. The composition $f \circ i$ has as a Wallman extension the identity from WN to $cl_{WQ}(N)$ and also the function which is the identity on N and which carries all elements of $WN \sim N$ to c.

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