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SOME LINEAR TOPOLOGICAL PROPERTIES OF THE HARDY SPACES H^p

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Abstract

The classical Hardy classes H^p $(1 \leq p < \infty)$ regarded as Banach spaces are investigated. It is proved: (1) Every reflexive subspace of L^1 is isomorphic to a subspace of H^1. (2) A complemented reflexive subspace of H^1 is isomorphic to a Hilbert space. (3) Every infinite dimensional subspace of H^1 which is isomorphic to a Hilbert space contains an infinite dimensional subspace which is complemented in H^1. The last result is a quantitative generalization of a result of Paley that a sequence of characters satisfying the Hadamard lacunary condition spans in H^1 a complemented subspace which is isomorphic to a Hilbert space.

Introduction

The purpose of the present paper is to investigate some linear topological and metric properties of the Banach spaces H^p, $1 \leq p < \infty$ consisting of analytic functions whose boundary values are p-absolutely integrable. The study of H^p spaces seems to be interesting for a couple of instances: (1) it requires a new technique which combines classical facts on analytic functions with recent deep results on L^p-spaces; several classical results on the Hardy classes seem to have natural Banach-space interpretation. (2) The spaces H^p and the Sobolev spaces are the most natural examples of "$\mathscr{L}_p$-scales" essentially different from the scale L^p.

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Boas [4] has observed that, for $1 < p < \infty$, the Banach space H^p is isomorphic to L^p. The situation in the "limit case" of H^1 is quite different. For instance H^1 is not isomorphic to any complemented subspace of L^1, more generally—to any $\mathscr{L}_1$-space (cf. [16], Proposition 6.1); H^1 is a dual of a separable Banach space (cf. [14]) while L^1 is not embeddable in any separable, dual cf. [23]; in contrast with L^1, by a result of Paley (cf. [21], [31], [7] p. 104), H^1 has complemented hilbertian subspaces hence it fails to have the Dunford-Pettis property.

On the other hand in Section 2 of the present paper we show that every reflexive subspace of L^1 is isomorphic to a subspace of H^1. Furthermore an analogue of the profound result of H. P. Rosenthal [27] on the nature of an embedding of a reflexive space in L^1 is also true for H^1. This implies that a complemented reflexive subspace of H^1 is necessarily isomorphic to a Hilbert space. In Section 3 we study hilbertian (= isomorphic to a Hilbert space) subspaces of H^1. We show that H^1 contains "very many" complemented hilbertian subspaces. Precisely: every subspace of H^1 which is isomorphic to ℓ^2 contains an infinite dimensional subspace which is complemented in H^1. This fact is a quantitative generalization of a result of Paley, mentioned above, on the boundedness in H^1 of the orthogonal projection from H^1 onto the closed linear subspace generated by a lacunary sequence of characters.

Section 4 contains some open problems and some results on the behaviour of the Banach-Mazur distance $d(H^p, L^p)$ as $p \to 1$ and as $p \to \infty$.

1. Preliminaries

Let $0 < p \leq \infty$. By L^p (resp. L^p_R) we denote the space of 2π-periodic complex-valued (resp. real-valued) measurable functions on the real line which are p-absolutely integrable with respect to the Lebesgue measure on $[0, 2\pi]$ for $0 < p < \infty$, and essentially bounded for $p = \infty$. $C_{2\pi}$ stands for the space of all continuous 2π-periodic complex-valued functions. We admit

$$\|f\|_p = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p dt \quad \text{for } 0 < p < 1,$$

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p dt\right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{0 \leq t \leq 2\pi} |f(t)|.$$

The n-th character χ_n is defined by

$$\chi_n(t) = e^{int} \quad (-\infty < t < +\infty;\ n = 0, \pm 1, \pm 2, \dots).$$

Given $f \in L^1$ we put

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt \quad (n = 0, \pm 1, \pm 2, \dots)$$

$$f_0 = f - \hat{f}(0) \cdot \chi_0.$$

If $0 < p < \infty$, then H^p is the closed linear subspace of L^p which is generated by the non-negative characters, $\{\chi_n : n \geq 0\}$. We define

$$H^\infty = \{f \in L^\infty : \hat{f}(n) = 0 \quad \text{for all } n < 0\}.$$

By A we denote the closed linear subspace of H^∞ generated by the non-negative characters. We put $H_0^p = \{f \in H^p : \hat{f}(0) = 0\}$ and $A_0 = \{f \in A : \hat{f}(0) = 0\}$.

Let $f \in H^p$. We denote by $\tilde{f}$ a unique analytic function on the unit disc $\{z : |z| < 1\}$ such that

$$\lim_{r=1} \tilde{f}(re^{it}) = f(t) \quad \text{for almost all } t. \tag{1.1}$$

For $u \in L_R^1$ we define $\mathcal{H}(u) = v$ to be the unique real 2π-periodic function such that for $f = u + iv$ there exists an $\tilde{f}$ analytic on the unit disc satisfying (1.1) and such that $\tilde{f}(0) = 2\pi^{-1} \int_0^{2\pi} u(t) dt$. Recall (cf. [33], Chap. VII and Chap. XII).

PROPOSITION 1.1: (i) *$\mathcal{H}$ is a linear operator of weak type* (1, 1).

(ii) *For every $p \in (0, 1)$ there exists a constant ρ_p such that*

$$\|\mathcal{H}(u)\|_p \leq \rho_p \|u\|_1^p \quad \text{for } u \in L_R^1$$

(iii) *For every $p \in (1, \infty)$ there exists a constant $\rho_p \leq C \max(p, p/(p-1))$, where C is an absolute constant, such that*

$$\|\mathcal{H}(u)\|_p \leq \rho_p \|u\|_p \quad \text{for } u \in L_R^p.$$

Next, for $f \in L^1$, we define $B(f)$ to be the unique function in $\bigcap_{0<p<1} H^p$ such that

$$B(f) = \sum_{n \geq 0} \hat{f}(n) \tilde{\chi}_{2n} + \sum_{n<0} \hat{f}(n) \tilde{\chi}_{-2n-1}.$$

Let $\mathcal{H}(f) = \mathcal{H}(\operatorname{Re} f) + i\mathcal{H}(\operatorname{Im} f)$ for $f \in L^1$. Then

$$B(f)(t) = \tfrac{1}{2}\{f_0(2t) + i\mathcal{H}(f_0)(2t) + [f_0(-2t) - i\mathcal{H}(f_0)(-2t)]e^{-it}\} + \hat{f}(0) \quad (-\infty < t < +\infty)$$

Clearly B is a one to one operator and if $g = B(f)$, then

$$f(t) = \frac{1}{2}\left[g\left(\frac{t}{2}\right) + g\left(\frac{t}{2} + \pi\right) + (\chi_1 g)\left(-\frac{t}{2}\right) + (\chi_1 g)\left(-\frac{t}{2} + \pi\right)\right]$$

$$(-\infty < t < +\infty).$$

Combining Proposition 1.1 with the above formulae we get (cf. Boas [4]).

PROPOSITION 1.2: (i) *B is a linear operator of weak type* $(1, 1)$ *from* L^1 *into* $\bigcap_{0<p<1} H^p$

(ii) *For every* $p \in (0, 1)$ *there exists a constant* β_p *such that*

$$\|B(f)\|_p \le \beta_p \|f\|_1^p$$

(iii) *For every* $p \subset (1, \infty)$ *B maps isomorphically* L^p *onto* H^p; *there exists a constant* $\beta_p \le 2\rho_p + 3$ *such that*

$$2^{-1}\|f\|_p \le \|B(f)\|_p \le \beta_p \|f\|_p. \tag{1.2}$$

A relative of B is the orthogonal projection $\mathcal{Q}$ defined by

$$\mathcal{Q}(f)(t) = 2^{-1}[B(f) + (B(f))^\pi]\left(\frac{t}{2}\right) \quad \text{for } f \in L^1, \tag{1.3}$$

$$-\infty < t < +\infty$$

where $g^\pi(t) = g(t + \pi)$. Clearly, by Proposition 1.2, $\mathcal{Q}(L^1) \subset \bigcap_{0<p<1} H^p$ and, for $1 < p < \infty$, $\mathcal{Q}$ regarded as an operator from L^p is a projection onto H^p with $\|\mathcal{Q}\|_p \le \|B\|_p$. In fact we have

$$\mathcal{Q}(f) = \sum_{n \ge 0} \hat{f}(n)\chi_n \quad \text{for } f \in L^p,\ 1 < p < \infty.$$

2. Reflexive subspaces of H^1

PROPOSITION 2.1: *A reflexive Banach space is isomorphic to a subspace of* H^1 *if (and only if) it is isomorphic to a subspace of* L^1.

PROOF: By a result of Rosenthal (cf. [27]) every reflexive subspace of L^1 is isomorphic to a reflexive subspace of L^r for some r with $1 < r \le 2$. Therefore it is enough to prove that, for every r with $1 < r \le 2$, the space L^r is isomorphic to a subspace of H^1. It is well known (cf. e.g. [27], p. 354) that, for $r \in [1, 2]$, there exists in $\bigcap_{0<p<r} L^p$ a subspace E_r which, for every fixed $p \in (0, r)$, regarded as a subspace of L^p is isometrically isomorphic to L^r. Moreover (if $r > 1$), for every p_1 and p_2 with $1 \le p_1 < p_2 < r$, there exists a constant

γ_{p_1,p_2} such that

$$\|f\|_{p_1} = \gamma_{p_1,p_2}\|f\|_{p_2} \quad \text{for } f \in E_r. \tag{2.1}$$

Now fix p_1 and p_2 with $1 < p_1 < p_2 < r$. By Proposition 1.2(iii), the operator B embeds isomorphically E_r regarded as a subspace of L^{p_1} into H^{p_1}. Clearly we have the set theoretical inclusion $H^{p_1} \subset H^1$. Thus it suffices to prove that the norm $\|\cdot\|_1$ and $\|\cdot\|_{p_1}$ are equivalent on $B(E_r)$. By (1.2) and (2.1), for every $g \in B(E_r)$ we have $\|g\|_{p_2} \leq k\|g\|_{p_1}$ where $k = \gamma_{p_1,p_2} \cdot 2\beta_{p_1}$. Letting $s = (p_1 - 1)(p_2 - 1)^{-1}$, in view of the logarithmic convexity of the function $p \to \|g\|_p^p$, we have

$$\|g\|_{p_1}^{p_1} \leq \|g\|_{p_2}^{sp_2}\|g\|_1^{1-s} \leq k^{sp_2}\|g\|_{p_1}^{sp_2}\|g\|_1^{1-s}$$

whence

$$\|g\|_1 \leq \|g\|_{p_1} \leq k^{p_2 s/1-s}\|g\|_1.$$

This completes the proof.

REMARK: Using the technique of [15] (cf. also [19]) instead of the logarithmic convexity of the function $p \to \|\cdot\|_p^p$ one can show that on $B(E_r)$ all the norms $\|\cdot\|_p$ are equivalent for $0 < p < r$ (in fact equivalent to the topology of convergence in measure). Hence if $0 < p \leq 1$, then H^p contains isomorphically every reflexive subspace of L^1. We do not know any satisfactory description of all Banach subspaces of H^p for $0 < p < 1$.

Our next result provides more information on isomorphic embeddings of reflexive spaces into H^1. It is a complete analogue of Rosenthal's Theorem on reflexive subspaces of L^1 (cf. [27]).

PROPOSITION 2.2: *Let X be a reflexive subspace of H^1. Then there exists a $p > 1$ such that for every r with $p > r > 1$ the natural embedding $j: X \to H^1$ factors through H^r, i.e. there are bounded linear operators $U: X \to H^r$ and $V: H^r \to H^1$ with $VU = j$. Moreover U and V can be chosen to be operators of multiplication by analytic functions.*

PROOF: By a result of Rosenthal ([27], Theorem 5 and Theorem 9), there exists a $p > 1$ such that for every r with $p > r > 1$ there exist a $K > 0$ and a non-negative function φ with $1/2\pi \int_0^{2\pi} \varphi(t)dt = 1$ such that

$$\left(\frac{1}{2\pi}\int_0^{\pi} |x(t)|^r[\varphi(t)]^{1-r}dt\right)^{1/r} \leq K\|x\|_1 \quad \text{for } x \in X$$

(In this formula we admit $0/0 = 0$). Let us set $\psi = \max(\varphi, 1)$. Let g be the outer function defined by

$$\tilde{g}(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+2}{e^{it}-2} \log\left[\psi(t)\frac{r-1}{r}\right] dt \quad \text{for } |z|<1$$

and let

$$g(t) = \lim_{\rho\to 1} \tilde{g}(\rho e^{it}) \quad \text{for } t \in [0, 2\pi]$$

Then (cf. [7], Chap. 2) $g \in H^{r/(r-1)}$, $|g(t)| = \psi(t)^{(r-1)/r}$ for t a.e., $|\tilde{g}(z)| \geq 1$ for $|z| < 1$ and $g^{-1} \in H^{\infty}$.

Let us set $U(x) = x/g$ for $x \in X$ and $V(f) = g \cdot f$ for $f \in H^r$. Since $\|g\|_{r/(r-1)} \leq 2^{(r-1)/r}$, V maps H^r into H^1 and $\|V\| \leq 2^{(r-1)/r}$. Finally, for every $x \in X$, we have

$$\|U(x)\|_r^r = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{x(t)}{g(t)}\right|^r dt = \frac{1}{2\pi}\int_0^{2\pi} |x(t)|^r [\psi(t)]^{1-r} dt$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} |x(t)|^r [\varphi(t)]^{1-r} dt \leq K^r \|x\|_1^r.$$

Thus $U(x) \in L^r$. Therefore $U(x) \in H^r$ because $U(x) \in H^1$ being a product of an $x \in H^1$ by $g^{-1} \in H^{\infty}$.

Corollary 2.1: *A complemented reflexive subspace of H^1 is isomorphic to a Hilbert space.*

Proof: Let X be a complemented reflexive subspace of H^1. Then, by Proposition 2.2, there exists a $p > 1$ such that for every r with $p > r > 1$ there are bounded linear operators U and V such that the following diagram is commutative

$$\begin{array}{ccccc} & & H^r & & \\ & {\scriptstyle U}\nearrow & & \searrow{\scriptstyle V} & \\ X & \xrightarrow[j]{} & H^1 & \xrightarrow[P]{} & X \end{array}$$

where $j: X \to H^1$ is the natural inclusion and $P: H^1 \xrightarrow[\text{onto}]{} X$ is a projection. Thus, for every $r \in (1, p)$, $Pj =$ the identity operator on X admits a factorization through H^r. Therefore X is isomorphic to a complemented subspace of L^r because, by Proposition 1.2(iii), H^r is isomorphic to L^r. Since this holds for at least two different $r \in (1, p)$, we infer that X is isomorphic to a Hilbert space (cf. [16] and [18]).

Remarks: (1) The following result has been kindly communicated to us by Joel Shapiro.

If $0 < p < 1$ and if a Banach space X is isomorphic to a complemented subspace of H^p, then either X is isomorphic to ℓ^1 or X is finite dimensional.

The proof (due to J. Shapiro) uses the result of Duren, Romberg and Shields [8], sections 2 and 3:

(D.R.S) the adjoint of the natural embedding $g \to \tilde{g}$ of H^p into the space B^p is an isomorphism between conjugate spaces. Here B^p denotes the Banach space of holomorphic functions on the open unit disc with the norm

$$\|f\|_{B_p} = \iint_{x^2+y^2\le 1} |f(x+iy)|(1-(x^2+y^2)^{1/2})^{(1/p)-2}dxdy.$$

It follows from (D.R.S) that a complemented Banach subspace of H^p $(0 < p < 1)$ is isomorphic to a complemented subspace of B^p. Next using technique similar to that of [17], Theorem 6.2 (cf. also [31]) one can show that B^p is isomorphic to ℓ^1. Now the desired conclusion follows from [22], Theorem 1.

Problem (J. Shapiro). Does H^p $(0 < p < 1)$ actually contain a complemented subspace isomorphic to ℓ^1?

(2) Slightly modifying the proof of Proposition 2.2 one can show the following

PROPOSITION 2.2a: *Let $1 \le p_0 < 2$. Let X be a subspace of H^{p_0} which does not contain any subspace isomorphic to ℓ^{p_0}. Then there exists a $p \in (p_0, 2)$ such that, for every r with $p_0 < r < p$ there exists an outer $g \in H^{p_0 r (r-p_0)^{-1}}$ with $g \neq 0$ such that $j = VU$ where $U: X \to H^r$ and $V: H^r \to H^{p_0}$ are operators of multiplication by $1/g$ and g respectively and $j: X \to H^{p_0}$ denotes the natural inclusion.*

The proof imitates the proof of Proposition 2.2; instead of Rosenthal's result we use its generalization due to Maurey (cf. [19], Théorème 8 and Proposition 97).

Our next result is in fact a quantitative version of Proposition 2.2a for hilbertian subspaces.

PROPOSITION 2.3: *Let $K \ge 1$ and let $1 \le p \le 2$. Let X be a subspace of H^p and let $T: \ell^2 \xrightarrow[\text{onto}]{} X$ be an isomorphism with $\|T\|\|T^{-1}\| \le K$. Then there exists an outer $\varphi \in H^1$ such that*

$$|\tilde{\varphi}(z)| \ge 1 \textit{ for every } z \textit{ with } |z| < 1 \tag{2.2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi(t)|dt = 1 \tag{2.3}$$

$$\left(\int_0^{2\pi} |f(t)|^2 |\varphi(t)|^{-(2/p)+1} dt\right)^{1/2} \le \gamma K \|f\|_p \quad \text{for every } f \in X \tag{2.4}$$

where γ is an absolute constant, in fact $\gamma \le 4/\sqrt{\pi}$.

PROOF: A result of Maurey ([19] Théorème 8, 50a, cf. also [20]), applied for the identity inclusion $X \to L^p$, yields the existence of a $g \in L^r$ where $1/r = 1/p - 1/2$ such that $\|g\|_r = 1$ and

$$\left(\frac{1}{\pi}\int_0^{2\pi}\left|\frac{f(t)}{g(t)}\right|^2 dt\right)^{1/2} \le C\|f\|_p \quad \text{for every } f \in X \tag{2.5}$$

where C is the smallest constant such that

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_j |f_j(t)|^2\right)^{p/2} dt\right)^{1/p} \le C\left(\sum_j \|f_j\|_p^2\right)^{1/2} \tag{2.6}$$

for every finite sequence (f_j) in X. A standard application of the integration against the independent standard complex Gaussian variables ξ_j gives

$$\begin{aligned}
\sum_j \|f_j\|_p^2 &\ge \|T^{-1}\|^{-2}\sum_j \|T^{-1}(f_j)\|^2 \\
&= \|T^{-1}\|^{-2}\int_\Omega \left\|\sum_j T^{-1}(f_j)\xi_j(s)\right\|^2 ds \\
&\ge (\|T^{-1}\|\|T\|)^{-2}\int_\Omega \left\|\sum_j f_j\xi_j(s)\right\|_p^2 ds \\
&\ge K^{-2}\left(\int_\Omega \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_j f_j(t)\xi_j(s)\right|^p dt ds\right)^{2/p} \\
&= K^{-2}k_p^2\left[\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_j |f_j(t)|^2\right)^{p/2} dt\right]^{2/p}
\end{aligned}$$

where $k_p = (1/\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x^2 + y2)^{p/2}e^{-(x^2+y^2)}dxdy)^{1/p}$. Since $k_p \ge k_1 = \sqrt{\pi}/2$, one can replace C in (2.5) and in (2.6) by $K/k_1 = 2K/\sqrt{\pi}$.

Now, by [14], p. 53, there exists an outer function $\varphi \in H^1$ satisfying (2.2), (2.3) and such that

$$|\varphi(t)| = \frac{\max(|g(t)|^r, 1)}{\left(\frac{1}{2\pi}\int_0^{2\pi}\max(|g(t)|^r, 1)dt\right)^{1/r}} \quad \text{for almost all } t \tag{2.7}$$

It can be easily checked that (2.7) and (2.5) imply (2.4) with $\gamma = 2/k_1$.

Our last result in this section gives some information on reflexive subspaces of the quotient L^1/H_0^1.

PROPOSITION 2.4: *Let X be a reflexive subspace of L^1 such that $\hat{f}(k) = 0$ for $k > 0$, $f \in X$. Then the sum $X + H_0^1$ is closed, equivalently the restriction of the quotient map $L^1 \to L^1/H_0^1$ to X is an isomorphic embedding.*

PROOF: Let $\mathcal{P}(f) = f - \mathcal{Q}(f)$ for $f \in L^1$ where $\mathcal{Q}$ is the projection defined, by (1.3). It follows from Proposition 1.2(ii) that there exists a constant $a > 0$ such that

$$\|\mathcal{P}(f)\|_{1/2} \leq a\|f\|_1^{1/2} \quad \text{for } f \in L^1.$$

On the other hand if X is a reflexive subspace of L^1, then X contains no subspace isomorphic to ℓ^1. Hence (cf. [15], [19]) the norm topology in X coincides with the topology of convergence in measure, in particular

$$\|f_n\|_1 \to 0 \text{ iff } \|f_n\|_{1/2} \to 0 \quad \text{for every sequence } (f_n) \subset X.$$

Thus there exists a constant $b_X = b > 0$ such that

$$\|f\|_1 \leq b\|f\|_{1/2}^2 \quad \text{for } f \in X.$$

Now fix $f \in X$ and $g \in H_0^1$. Then $\mathcal{P}(g) = 0$, and $\mathcal{P}(f) = f$ because $\hat{f}(k) = 0$ for $k > 0$. Hence

$$\|f+g\|_1 \geq a^2\|\mathcal{P}(f+g)\|_{1/2}^2 = a^2\|\mathcal{P}(f)\|_{1/2}^2 = a^2\|f\|_{1/2}^2 \geq \frac{a^2}{b}\|f\|_1.$$

Thus the sum $X + H_0^1$ is closed.

REMARK: Proposition 2.4 yields, in particular, the following "classical" result.

If (n_k) is a sequence of negative integers such that the space

$$\mathcal{X} = \{f \in L^1 : \hat{f}(n) = 0 \quad \text{for } n \neq n_k \ (k = 1, 2, \ldots)\}$$

is isomorphic to ℓ^2 (in particular if $\lim_k (n_{k+1}/n_k) > 1$) then the space $\mathcal{X} + H^1$ is closed or equivalently in the "dual language" the operator $A \to \ell^2$ defined by $f \to (\hat{f}(-n_k))$ is a surjection.

3. Hilbertian subspaces of H^1

The existence of infinite-dimensional complemented hilbertian subspaces of H^1 follows from the classical result of R.E.A.C. Paley (cf. [21], [29], [7] p. 104, [33], Chap. XII, Theorem 7.8) which yields (P). If $\lim_k (n_{k+1}/n_k) > 1$, then the closed linear subspace of H^1 spanned by the sequence of characters $(\chi_{n_k})_{1 \leq k < \infty}$ is isomorphic to ℓ^2 and complemented in H^1.

On the other hand there are subspaces of H^1 spanned by sequences of characters which are isomorphic to ℓ^2 but uncomplemented in H^1 (cf. Rudin [30] and Rosenthal [26]).

In this section we shall show that, in fact, H^1 contains "very many" complemented and "very many" uncomplemented hilbertian subspaces not necessarily translation invariant. The situation is similar to that in L^p (and therefore H^p, by Proposition 1.2(iii)) for $1 < p < 2$ (cf. [25], Theorem 3.1) but not in L^1 which contains no complemented infinite-dimensional hilbertian subspaces ([13], [22]).

If (x_n) is a sequence of elements of a Banach space X then $[x_n]$ denotes the closed linear subspace of X generated by the x_n's.

Let $1 \leq K < \infty$. Recall that a sequence (x_n) of elements of a Banach space is said to be K-equivalent to the unit vector basis of ℓ^2 provided there exist positive constants a and b with $ab = K$ such that

$$a^{-1}\left(\sum_n |t_n|^2\right)^{1/2} \leq \left\|\sum_n t_n x_n\right\| \leq b\left(\sum_n |t_n|^2\right)^{1/2}$$

for every finite sequence of scalars (t_n).

Now we are ready to state the main result of the present section

THEOREM 3.1: *Let $1 \leq K < \infty$. Let $(f_n)_{1 \leq n < \infty}$ be a sequence in H^1 which is K-equivalent to the unit vector basis of ℓ^2. Then, for every $\epsilon > 0$, there exists an infinite subsequence (n_k) such that the closed linear subspace $[f_{n_k}]$ spanned by the sequence (f_{n_k}) is complemented in H^1. Moreover, there exists a projection P from H^1 onto $[f_{n_k}]$ with $\|P\| < 4K + \epsilon$.*

The proof of Theorem 3.1 follows immediately from Propositions 3.1, 3.2 and 3.3 given below. We begin with the following general criterion

PROPOSITION 3.1: *Let X be a Banach space with separable conjugate X^*. Assume that there exists a constant $c = c_X$ such that every weakly convergent to zero sequence (y_m) in X contains an infinite subsequence (y_{m_k}) such that*

$$\left\|\sum t_k y_{m_k}\right\| \leq c \sup_m \|y_m\| \left(\sum |t_k|^2\right)^{1/2} \tag{3.1}$$

for every finite sequence of scalars (t_k). Then, for every $K \geq 1$ and for every $\epsilon > 0$, every sequence (x_n^*) in X^* which is K-equivalent to the unit vector basis of ℓ^2 contains an infinite subsequence $(x_{n_k}^*)$ such that the closed linear subspace $[x_{n_k}^*]$ admits a projection $P: X^* \xrightarrow[\text{onto}]{} [x_{n_k}^*]$ with $\|P\| < 2Kc + \epsilon$.

PROOF: Define $V: \ell^2 \to X^*$ *by* $V((t_n)) = \sum_n t_n x_n^*$ for $(t_n) \in \ell^2$. Clearly V is an isomorphic embedding with $\|V\|\|V^{-1}\| \leq K$ (V^{-1} acts from $V(\ell^2)$ onto ℓ^2). Since ℓ^2 is reflexive, V is weak-star continuous.

Hence there exists an operator $U : X \to \ell^2$ whose adjoint is V. It is easy to check that the operator U is defined by $U(x) = (x_n^*(x))_{1 \le n < \infty}$ for $x \in X$. Since $\|U^*((t_n))\| = \|V((t_n))\| \ge \|V^{-1}\|^{-1}(\Sigma_n |t_n|^2)^{1/2}$ for every $(t_n) \in \ell^2$, the operator U is a surjection such that, for every $r > \|V^{-1}\|$, the set $U(\{x \in X : \|x\| \le r\})$ contains the unit ball of ℓ^2 (cf. [32] Chap. VII, §5). Hence there exists a sequence (x_s) in X such that $\sup_s \|x_s\| \le r$ and $(U(x_s))$ is the unit vector basis of ℓ^2, equivalently $x_n^*(x_s) = \delta_n^s$ for $n, s = 1, 2, \ldots$. Since X^* is separable and $\sup_s \|x_s\| \le r$, there exists an infinite subsequence (x_{s_q}) which is a weak Cauchy sequence. Let us set $y_m = x_{s_{2m}} - x_{s_{2m-1}}$ for $m = 1, 2, \ldots$. Clearly the sequence (y_m) tends weakly to zero. Thus the condition imposed on X yields the existence of an infinite subsequence (y_{m_k}) satisfying (3.1). Let us set $n_k = s_{2m_k}$ for $k = 1, 2, \ldots$ and put

$$P(x^*) = \sum_{k=1}^{\infty} x^*(y_{m_k}) x_{n_k}^* \quad \text{for } x^* \in X^*.$$

Clearly we have

$$\|P(x^*)\| \le \|V\| \left(\sum_{k=1}^{\infty} |x^*(y_{m_k})|^2 \right)^{1/2}.$$

Thus, by (3.1),

$$\begin{aligned} \|P(x^*)\| &\le \|V\| \sup_{\Sigma |t_k|^2 = 1} \left| \sum_{k=1} x^*(y_{m_k}) t_k \right| \\ &\le \|V\| \|x^*\| \sup_{\Sigma |t_k|^2 = 1} \left\| \sum_{k=1} t_k y_{m_k} \right\| \\ & c \sup_k \|y_m\| \|V\| \|x^*\|. \end{aligned}$$

Thus P is a linear operator with $\|P\| \le 2cr\|V\|$ (because $\sup_k \|y_{m_k}\| \le 2 \sup_s \|x_s\| \le 2r$). Letting $r < \|V^{-1}\| + \epsilon(2c\|V\|)^{-1}$, we get $\|P\| < 2K + \epsilon$. Since $P(x^*) \in [x_{n_k}^*]$ for every $x^* \in X^*$ and since $P(x_{n_k}^*) = x_{n_k}^*$ for $k = 1, 2, \ldots$, we infer that P is the desired projection.

REMARK: The assertion of Proposition 3.1 remains valid if we replace the assumption of separability of X^* by the weaker assumption that X does not contain subspace isomorphic to ℓ^1. To extract a weak Cauchy subsequence from the sequence (x_s) we apply the result of Rosenthal [28].

To apply Proposition 3.1 we need a description of a predual of H^1. Our next proposition is known. Its part (ii) is a particular case of the Caratheodory-Fejer Theorem, cf. [1].

PROPOSITION 3.2: (i) *The conjugate space of the quotient* $C_{2\pi}/A_0$ *is isometrically isomorphic to* H^1.

(ii) *The space* $C_{2\pi}/A_0$ *is isometrically isomorphic to a subspace of the space of compact operators on a Hilbert space.*

PROOF: (i) The desired isometric isomorphism assigns to each $f \in H^1$ the linear functional x_f^* defined by

$$x_f^*(\{g + A_0\}) = \frac{1}{2\pi}\int_0^{\pi} f(t)g(t)dt \quad \text{for the coset } \{g + A_0\} \in C_{2\pi}/A_0.$$

The fact that this map is onto $(C_{2\pi}/A_0)^*$ follows from the F. and M. Riesz Theorem. For details cf. [14], p. 137, the second Theorem.

(ii) To each coset $\{f + A_0\}$ we assign the linear operator $T_f: H^2 \to H^2$ defined by

$$\langle T_f(g), h\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)g(t)\overline{h(-t)}dt \quad (g,\ h \in H^2).$$

Clearly the definition of T_f is independent of the choice of a representative in the coset $\{f + A_0\}$. Moreover, for every $f_1 \in \{f + A_0\}$, we have

$$|\langle T_f(g), h\rangle| \le \|f_1\|_\infty \|g\|_2 \|h\|_2 \quad (g,\ h \in H^2).$$

Thus $\|T_f\| \le \inf\{\|f_1\|_\infty : f_1 \in \{f + A_0\}\} = \|\{f + A_0\}\|$.

Conversely, it follows from part (i) and the Hahn Banach Theorem that there exists a $\varphi \in H^1$ with $\|\varphi\|_1 = 1$ such that $1/2\pi \int_0^{2\pi} f(t)\varphi(t)dt = \|\{f + A_0\}\|$. By the factorization theorem (cf. [14], p. 67), we pick functions g and h_1 in H^2 with $gh_1 = \varphi$ and $\|g\|_2 = \|h_1\|_2 = 1$ (cf. [14], p. 71), and we define $h \in H^2$ by $h(t) = \overline{h_1(-t)}$. Then $\langle T_f(g), h\rangle = \|\{f + A_0\}\| = \|\{f + A_0\}\| \|g\|_2 \|h\|_2$. Hence $\|T_f\| = \|\{f + A_0\}\|$. This shows that the map $\{f + A_0\} \to T_f$ is an isometrically isomorphic embedding of $C_{2\pi}/A_0$ into the space of bounded operators on H^2. Finally observe that each operator T_f is compact because the cosets $\{\{\chi_{-n} + A_0\}: n = 0, 1, 2, \ldots\}$ are linearly dense in $C_{2\pi}/A_0$ (by the Fejer Theorem) and $T_{\chi_{-n}} = \sum_{j=0}^n \langle \cdot, \chi_j\rangle \chi_{n-j}$ is an $(n + 1)$-dimensional operator ($n = 0, 1, \ldots$). This completes the proof.

To complete the proof of Theorem 3.1 it is enough to show that the space $K(\hbar)$ of the compact operators on an infinite-dimensional Hilbert space $\hbar$ (and therefore every subspace of $K(\hbar)$) satisfies the assumption of Proposition 3.1. Precisely we have

PROPOSITION 3.3: *Let* $\hbar$ *be an infinite-dimensional Hilbert space. Let* (T_m) *be a weakly convergent to zero sequence in* $K(\hbar)$. *Then, for* *every $\epsilon > 0$, there exists an infinite subsequence (m_k) such that*

$$\left\| \sum_k t_k T_{m_k} \right\| \le (2+\epsilon) \sup_m \|T_m\| \left(\sum |t_k|^2 \right)^{1/2}$$

for every finite sequence of scalars (t_k).

PROOF: The assumption that the sequence (T_m) converges weakly to zero in $K(\hbar)$ means

$$\lim_m \langle T_m(x), y \rangle = 0 \quad \text{for every } x, y \in \hbar. \tag{3.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\hbar$. Let $(e_\alpha)_{\alpha \in \mathfrak{A}}$ be an orthonormal basis for $\hbar$. Since each T_m is compact, the ranges of T_m and its adjoint T_m^* are separable. Hence there exists a countable set $\mathfrak{A}_0$ such that $\langle T_m(x), e_\alpha \rangle = \langle T_m^*(x), e_\alpha \rangle = 0$ for every $m = 1, 2, \ldots$ for every $x \in \hbar$ and for every $\alpha \in \mathfrak{A} \setminus \mathfrak{A}_0$. Let $j \to \alpha(j)$ be an enumeration of the elements of $\mathfrak{A}_0$. Let furthermore P_n denote the orthogonal projection onto the n-dimensional subspace generated by the elements $e_{\alpha(1)}$, $e_{\alpha(2)}, \ldots, e_{\alpha(n)}$. Since $\dim P_n(\hbar) = n$, it follows from (3.2) that

$$\lim_m \|P_n T_m P_n\| = 0 \quad \text{for } n = 1, 2, \ldots \tag{3.3}$$

Next the compactness of each T_m and the definition of the set $\mathfrak{A}_0$ yield

$$\lim_n \|T_m - P_n T_m P_n\| = 0 \quad \text{for } m = 1, 2, \ldots \tag{3.4}$$

Let $\epsilon > 0$ be given. Assuming that $\sup_m \|T_m\| > 0$ we fix a positive sequence (ϵ_k) with $(\sum_{k=1}^\infty 4\epsilon_k^2) \le \epsilon \sup_m \|T_m\|$. Now using (3.3) and (3.4) we define inductively increasing sequences of indices $(m_k)_{k \ge 1}$ and $(n_k)_{k \ge 0}$ with $m_1 = 1$ and $n_0 = 0$ so that (admitting $P_0 = 0$)

$$\|P_{n_{k-1}} T_{m_k} P_{n_{k-1}}\| \le \epsilon_k \quad \text{for } k = 1, 2, \ldots \tag{3.5}$$

$$\|T_{m_k} - P_{n_k} T_{m_k} P_{n_k}\| \le \epsilon_k \quad \text{for } k = 1, 2, \ldots \tag{3.6}$$

Let us put, for $k = 1, 2, \ldots$,

$$B_k = (P_{n_k} - P_{n_{k-1}}) T_{m_k} P_{n_k}, \quad C_k = P_{n_{k-1}} T_{m_k} (P_{n_k} - P_{n_{k-1}}).$$

Clearly (3.5) and (3.6) yield

$$\|T_{m_k} - B_k - C_k\| = \|T_{m_k} - P_{n_k} T_{m_k} P_{n_k} + P_{n_{k-1}} T_{m_k} P_{n_{k-1}}\| \le 2\epsilon_k.$$

Let (t_k) be a fixed finite sequence of scalars. Since the projections $P_{n_k} - P_{n_{k-1}}$ $(k = 1, 2, \dots)$ are orthogonal and mutually disjoint, for every $x \in \hbar$, we have

$$\left\|\sum t_k B_k(x)\right\|^2 = \left\|\sum t_k (P_{n_k} - P_{n_{k-1}})[(T_{m_k} P_{n_k})(x)]\right\|^2$$
$$= \sum |t_k|^2 \|(P_{n_k} - P_{n_{k-1}})[(T_{m_k} P_{n_k})(x)]\|^2$$
$$\le \sum |t_k|^2 \|P_{n_k} - P_{n_{k-1}}\|^2 \|P_{n_k}\|^2 \|T_{m_k}\|^2 \|x\|^2$$
$$\le \sum |t_k|^2 \sup_m \|T_m\|^2 \|x\|^2.$$

Hence

$$\left\|\sum t_k B_k\right\| \le \Big(\sum_k |t_k|^2\Big)^{1/2} \sup_m \|T_m\|.$$

Similarly

$$\left\|\sum_k t_k C_k\right\| = \left\|\sum_k \bar{t}_k C_k^*\right\| = \left\|\sum_k \bar{t}_k (P_{n_k} - P_{n_{k-1}}) T_{m_k}^* P_{n_{k-1}}\right\|$$
$$\le \Big(\sum_k |t_k|^2\Big)^{1/2} \sup_m \|T_m\|.$$

Thus

$$\left\|\sum_k t_k T_{n_k}\right\| \le \sum_k |t_k| \|T_{n_k} - B_k - C_k\| + \left\|\sum_k t_k B_k\right\| + \left\|\sum_k t_k C_k\right\|$$
$$\le \Big(\sum_k |t_k|^2\Big)^{1/2} \Big(\Big(\sum_{k=1}^{\infty} 4\epsilon_k^2\Big)^{1/2} + 2 \sup_m \|T_m\|\Big)$$
$$\le (2+\epsilon) \sup_m \|T_m\| \Big(\sum_k |t_k|^2\Big)^{1/2}.$$

This completes the proof of Proposition 3.3 and therefore of Theorem 3.1.

REMARKS: (1) Let us sketch a proof of Paley's result (P) which uses the technique of the proof of Theorem 3.1.

Assume first that (m_k) is a sequence of positive integers such that

$$m_{k+1} \ge 2m_k \quad \text{for } k = 1, 2, \dots \tag{3.7}$$

Let $T_m = T_{\chi_{-m}}$ for $m = 0, 1, \dots$ be the compact operator on H^2 which is the image of the coset $\{\chi_{-m} + A_0\}$ by the isometry $C_{2\pi}/A_0 \to K(H^2)$ defined in the proof of Proposition 3.2(ii). Then $\langle T_m \chi_j, \chi_k \rangle = 0$ for $j + k \ne m$ and $\langle T_m \chi_j, \chi_k \rangle = 1$ for $j + k = m$. Let $P_m : H^2 \xrightarrow[\text{onto}]{} \text{span}$ $(\chi_0, \chi_1, \dots, \chi_{m-1})$ be the orthogonal projection. It follows from (3.7) that $P_{m_{k-1}} T_{m_k} P_{m_{k-1}} = 0$ and $T_{m_k} = P_{m_k} T_{m_k} P_{m_k}$ for $k = 1, 2, \dots$ (i.e. the sequences (P_{n_k}) and (T_{m_k}) satisfy (3.5) and (3.6) with $n_k = m_k$ and $\epsilon_k = 0$ for all k). Thus the argument used in the proof of Proposition

3.3 yields

$$\left\| \sum t_k T_{m_k} \right\| \leq 2 \left(\sum |t_k|^2 \right)^{1/2}$$

for every finite sequence of scalars (t_k). Obviously $(\Sigma\, t_k T_{m_k})(\Sigma\, \bar{t}_k \chi_{m_k}) = \Sigma_k\, |t_k|^2$. Hence

$$\left\| \sum t_k T_{m_k} \right\| \geq \left(\sum |t_k|^2 \right)^{1/2}.$$

Thus the subspace $[T_{m_k}]$ is isomorphic to ℓ^2. Moreover Q defined by $Q(S) = \Sigma_k\, \langle S(x_0), \chi_{m_k} \rangle T_{m_k}$ for $S \in K(H^2)$ is a projection onto $[T_{m_k}]$ with $\|Q\| \leq 2$. Let us regard Q as an operator from $[T_m]$ (= the isometric image of $C_{2\pi}/A_0$) into itself and let P be the adjoint of Q. Then, by Proposition 3.1(ii), P can be regarded as an operator from H^1 into itself. Obviously $\|P\| = \|Q\| \leq 2$. A direct computation shows that P is the orthogonal projection of H^1 onto $[\chi_{m_k}]$. To complete the proof of (P) in the general case observe that every lacunary sequence admits a decomposition into a finite number of sequences satisfying (3.7).

(2) A similar argument gives also the following relative result.

Let (f_n) be a sequence in H^1. Assume that $+\infty > \sup_n \|f_n\|_\infty \geq \inf_n \|f_n\|_1 > 0$ and

$$\lim_n \hat{f}_n(j) = 0 \quad \text{for every } j = 0, 1, \ldots$$

Then there exists an infinite subsequence (n_k) and a $1 \leq K < \infty$ such that the sequence (f_{n_k}) is K-equivalent to the unit vector basis of ℓ^2 and the orthogonal projection from H^1 onto $[f_{n_k}]$ is a bounded operator.

Our next aim is to give a quantitative generalization of Theorem 3.1 to the case of H^p spaces $(1 < p \leq 2)$.

THEOREM 3.2: *Let $1 < p \leq 2$ and let $K \geq 1$. Then there exists an absolute constant c (independent of K and p) such that if (f_n) is a sequence in H^p which is K-equivalent to the unit vector basis of ℓ^2, then there exists a subsequence (n_k) such that there exists a projection P from H^p onto $[f_{n_k}]$—the closed linear span of (f_{n_k}) with $\|P\| \leq cK^2$.*

PROOF: Let $X = [f_n]$. By the assumption, there exists an isomorphism $T : \ell^2 \xrightarrow[\text{onto}]{} X$ with $\|T\|\|T^{-1}\| \leq K$. Hence, by Proposition 2.3, there exists a $\varphi \in H^1$ which satisfies an outer (2.2), (2.3), (2.4). Let us set $\|f\|_{\varphi,q} = (1/(2\pi)\int_0^{2\pi} |f(t)|^q |\varphi(t)| dt)^{1/q}$ for f measurable and for $1 \leq q < \infty$. It follows from (2.2) that there exists in the open unit disc a holomorphic function, say $\tilde{g}$, such that $\tilde{\varphi} = e^{p\tilde{g}}$. Let us set

$$\varphi^{-1/p}(t) = \lim_{r=1} e^{-\tilde{g}(re^{it})} \quad \text{for } t \in [0, 2\pi].$$

Since $0 \neq \varphi \in H^1$, the limit exists for almost all t and $\varphi^{1/p} \underset{\text{df}}{=} 1/\varphi^{-1/p \in H^p}$. Furthermore observe that (2.4) is equivalent to

$$\|f\varphi^{-1/p}\|_{\varphi,2} \leq \gamma K \|f\varphi^{-1/p}\|_{\varphi,p} \quad \text{for } f \in X, \tag{3.8}$$

where γ is the absolute constant appearing in Proposition 2.2. On the other hand, by the logarithmic convexity of the function $q \to \|f\varphi^{-1/p}\|_q^q$, we get

$$\|f\varphi^{-1/p}\|_{\varphi,p} \leq \|f\varphi^{-1/p}\|_{\varphi,1}^{(2/p)-1} \|f\varphi^{-1/p}\|_{\varphi,2}^{2-(2/p)} \quad \text{for } f \in X.$$

Thus

$$\|f\varphi^{-1/p}\|_{\varphi,p} \leq (\gamma K)^{(2p-2)/(2-p)} \|f\varphi^{-1/p}\|_{\varphi,1} \quad \text{for } f \in X. \tag{3.9}$$

Now, let H^1_φ denote the Banach space being the completion of the trigonometric polynomials $\Sigma_{n\geq 0} c_n \chi_n$ in the norm $\|\cdot\|_{1,\varphi}$. It easily follows from (2.2) and (2.3) that H^1_q is isometrically isomorphic to H^1. The desired isometry is defined by $f \to f\varphi$ for $f \in H^1_\varphi$. Next (3.9) and the obvious relation

$$\|f\|_p = \|f\varphi^{-1/p}\|_{\varphi,p} \geq \|f\varphi^{-1/p}\|_{\varphi,1} \quad \text{for } f \in H^p$$

imply that the sequence $(f_n \varphi^{-1/p})$ belongs to H^1_φ and in H^1_φ is $K^{(2p-2)/(2-p)+1}\gamma^{(2p-2)/(2-p)}$—equivalent to the unit vector basis of ℓ^2. Hence, by Theorem 3.1 which we apply to H^1_φ—the isometric image of H^1, there exists a subsequence (n_k) and a projection

$$Q : H^1_\varphi \xrightarrow[\text{onto}]{} [f_{n_k}\varphi^{-1/p}] \text{ with } \|Q\| \leq 5\gamma^{(2p-2)/(2-p)} K^{p/(2-p)}.$$

Let us set

$$P(f) = \varphi^{1/p} Q(f\varphi^{-1/p}) \quad \text{for } f \in H^p.$$

To see that P is well defined observe first that if $f \in H^p$, then, by the Hölder inequality and by (2.3),

$$\|f\varphi^{-1/p}\|_{\varphi,1} = \|f|\varphi|^{(p-1)/p}\|_1 \leq \|f\|_p \|\varphi\|_1^{(p-1)/p} = \|f\|_p.$$

Thus, by (3.9), for every $f \in H^p$, we have

$$\begin{aligned} \|P(f)\|_p &= \|\varphi^{1/p} Q(f\varphi^{-1/p})\|_p = \|Q(f\varphi^{-1/p})\|_{\varphi,p} \\ &\leq (\gamma K)^{(2p-2)/(2-p)} \|Q(f\varphi^{-1/p})\|_{\varphi,1} \\ &\leq 5[\gamma^{(2p-2)/(2-p)}]^2 K^{(3p-2)/(2-p)} \|f\varphi^{-1/p}\|_{\varphi,1} \\ &\leq 5\gamma^{(4p-4)/(2-p)} K^{(3p-2)/(2-p)} \|f\|_p. \end{aligned}$$

Thus P is bounded. Obviously $P(H^p) \subset X$ and $P(f) = f$ for $f \in [f_{n_k}]$. Hence P is a projection. Now, for $p \le \frac{6}{5}$ we get (remembering that $\gamma \ge 1$ and $K \ge 1$)

$$\|P\| \le 5\gamma^{(4p-4)/(2-p)} K^{(3p-2)/(2-p)} \le 5\gamma K^2.$$

If $p > \frac{6}{5}$, then an inspection of the proof of Proposition 2.1 shows that there exists an isomorphism T from L^p onto a subspace of H^1 with $\|T\| \|T^{-1}\| \le k = \gamma_{11/10,6/5} \cdot 2\beta_{11/10}$ (we put in (2.1) and further $p_2 = \frac{6}{5}$, $p_1 = \frac{11}{10}$). Thus, by Theorem 3.1, we infer that every sequence in L^p ($p > \frac{6}{5}$) (particularly in H^p) which is K-equivalent to the unit vector basis of ℓ^2 contains an infinite subsequence whose closed linear span is the range of a projection from L^p of norm $\le 5k \cdot K$. This completes the proof.

Corollary 3.1: *There exists an absolute constant $c \ge 1$ such that, for $1 \le p \le 2$, every infinite-dimensional hilbertian subspace of H^p contains an infinite dimensional subspace which is the range of a projection from H^p of norm $\le c$ and which is a range of an isomorphism from ℓ^2, say T, with $\|T\| \|T^{-1}\| \le c$.*

Proof: Combine Theorems 3.1 and 3.2 with the recent result of Dacunha–Castelle and Krivine [5] from which, in particular, follows that every infinite-dimensional hilbertian subspace of L^p contains, for every $\epsilon > 0$, a subspace which is $(1+\epsilon)$—isomorphic to ℓ^2.

Since the argument of Dacunha–Castelle and Krivine is quite involved, to make the paper self contained we include a proof of a slightly weaker Proposition 3.4 (which suffices for the proof of Corollary 3.1). This result and the argument below is due to H. P. Rosenthal[1] and is published here with his permission.

Proposition 3.4: *There exists an absolute constant c such that every infinite-dimensional hilbertian subspace X of L^p $(1 \le p \le 2)$ contains an infinite dimensional subspace E such that there exists an isomorphism $T : \ell^2 \xrightarrow[\text{onto}]{} E$ with $\|T\| \|T^{-1}\| \le c$.*

Proof: Since L^p is isometrically isomorphic to a subspace of L^1 $(1 < p \le 2)$, it is enough to consider the case $p = 1$. For $X \subset L^1$ and X

[1]It was presented at the Functional Analysis Seminar in Warsaw in October 1973.

isomorphic to ℓ^2 we put

$$d(X, \ell^2) = \inf \{\|S\| \|S^{-1}\| : S : \ell^2 \xrightarrow[\text{onto}]{} X \text{ isomorphism}\}$$

$$I_2(X) = \inf \left\{ \sup_{x \in X, \|x\|_1 = 1} \|T(x)\|_2 : T : L^1 \xrightarrow[\text{onto}]{} L^1 \text{ positive isometry} \right\}.$$

$$\tilde{I}_2(X) = \inf \{I_2(Y) : Y \subset X, \dim X/Y < \infty\}.$$

Recall that, for the complex L^1, if $Z \subset L^1$ and Z is isomorphic to ℓ^2, then

$$I_2(Z) \leq \frac{2}{\sqrt{\pi}} d(Z, \ell^2). \tag{3.10}$$

(This is a result of Grothendieck [12], cf. also Rosenthal [27]. It can be easily deduced from a result of Maurey [20], cf. the proof of our Proposition 2.3). Clearly

$$I_2(Z) = \inf \left\{ \sup_{x \in Z : \|x\|_1 = 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |x(t)|^2 g^{-1}(t) dt \right)^{1/2} : g > 0, \|g\|_1 = 1 \right\}.$$

Now fix X isomorphic to ℓ^2 and pick $Y \subset X$ with $\dim X/Y < \infty$ so that $I_2(Y) < 2\tilde{I}_2(X)$. Replacing, if necessary X by $T(X)$ for an appropriate positive isometry T (depending only on Y but not on subspaces of Y of finite codimension), one may assume without loss of generality that

$$I_2(Z) = \sup_{y \in Z : \|y\|_1 = 1} \|y\|_2 < 2\tilde{I}_2(X) \text{ for every } Z \subset Y \text{ with } \dim Y/Z < \infty. \tag{3.11}$$

We claim that (3.11) implies

(3.12) for every $Z \subset Y$ with $\dim Y/Z < \infty$ there exists a $y \in Z$ such that

$$1 = \|y\|_1 \leq \|y\|_2 < \frac{4}{\sqrt{\pi}}.$$

Indeed, let $m = \inf \{\|y\|_2 : y \in Z \text{ and } \|y\|_1 = 1\}$. Then, by (3.11), $m\|y\|_1 \leq \|y\|_2 < 2\tilde{I}_2(X)\|y\|_1$ for every $y \in Z$. Thus

$$\frac{2\tilde{I}_2(X)}{m} > d(Z, \ell^2).$$

Hence, by (3.10),

$$\frac{2\tilde{I}_2(X)}{m} > \frac{\sqrt{\pi}}{2} I_2(Z) \geq \frac{\sqrt{\pi}}{2} \tilde{I}_2(X).$$

Hence $m < 4/\sqrt{\pi}$ and this proves (3.12).

Let (h_j) denote the Haar orthonormal basis. It follows from (3.12) that one can define inductively a sequence (y_n) in Y so that, for all n,

$$1 = \|y_n\|_2 \geq \|y_n\|_1 > \frac{\sqrt{\pi}}{4},$$

y_n is orthogonal to $y_1, y_2, \ldots, y_{n-1}$ and $h_1, h_2, \ldots, h_{n-1}$.

By a result of [2], passing again to a subsequence (if necessary) we may also assume that (y_n) is equivalent to a block basic sequence with respect to the Haar basis regarded as a basis in $L^{3/2}$. Now using the Orlicz inequality (cf. e.g. [25], p. 283), for arbitrary finite sequence of scalars (t_n) we get

$$\begin{aligned}\left\|\sum t_n y_n\right\|_2 &\geq \left\|\sum t_n y_n\right\|_{3/2} \geq a\left(\sum |t_n|^2\|y_n\|^2\right)^{1/2} \\ &\geq a\left(\sum |t_n|^2\|y_n\|_1^2\right)^{1/2} \geq \frac{a\sqrt{\pi}}{4}\left(\sum |t_n|^2\right)^{1/2} \\ &= \frac{a\sqrt{\pi}}{4}\left\|\sum t_n y_n\right\|_2.\end{aligned}$$

where a is an absolute constant depending only on the unconditional constant of the Haar basis in $L^{3/2}$ and the constant in the Orlicz inequality for $L^{3/2}$. Thus, for every $f \in \operatorname{span}(y_n)$,

$$\|f\|_2 \geq \|f\|_{3/2} \geq \frac{a\sqrt{\pi}}{4}\|f\|_2.$$

Hence by the logarithmic convexity of the function $r \to \|f\|_r^r$

$$\|f\|_2 \geq \|f\|_1 \geq \left(\frac{a\sqrt{\pi}}{4}\right)^3 \|f\|_2 \quad \text{for } f \in \operatorname{span}(y_n).$$

Thus the same inequality holds for $f \in [y_n]$. Therefore $[y_n]$ is a subspace of X with $d([y_n], \ell^2) \leq (4/(a\sqrt{\pi}))^3$. This completes the proof.

It is interesting to compare Corollary 2.1 with the following fact

PROPOSITION 3.5: *Let $1 \leq p < 2$, let Y be a hilbertian subspace of H^p. Then there exists a non complemented hilbertian subspace X of H^1 which contains Y.*

PROOF: Observe first that there exists a non complemented hilbertian subspace of H^p $(1 \leq p < 2)$. This follows from Proposition 1.2(iii) and from the corresponding fact for L^p $(1 < p < 2)$ (If $1 < p \leq \frac{4}{3}$ then, by an observation of Rosenthal [26], p. 52, a result of Rudin [30] yields the existence of a non-complemented hilbertian subspace. If $\frac{4}{3} < p < 2$, then the same fact for L^p was very recently observed by several mathematicians (cf. Bennet, Dor, Goodman, Johnson and

Newman [9]), finally H^1 contains an uncomplemented hilbertian subspace because, by Proposition 2.1, H^1 contains H^p isomorphically for $2 > p > 1$.

Now Proposition 3.5 is an immediate consequence of the following general fact

PROPOSITION 3.6: *If a Banach space Z contains a non complemented hilbertian subspace, say E, then every hilbertian subspace of Z is contained in a non complemented hilbertian subspace.*

PROOF: Let Y be a hilbertian subspace of Z. If Y is finite dimensional, then the desired subspace is $Y + E$. If Y is uncomplemented then there is nothing to prove. In the sequel suppose that Y is infinite dimensional and that there exists a projection $P : Z \xrightarrow[\text{onto}]{} Y$. Let E_1 denote any subspace of E with $\dim E/E_1 < \infty$. Let P_{E_1} denote the restriction of P to E_1. If P_{E_1} were an isomorphic embedding, then the formula SQP would define a projection from Z onto E_1 where Q is a projection from a hilbertian subspace Y onto its closed subspace $P_{E_1}(E_1)$ and $S : P_{E_1}(E_1) \to E_1$—the inverse of P_{E_1}. Since E is uncomplemented in Z, so is E_1. Hence the restriction of P to no subspace of E of finite codimension is an isomorphism. Combining this fact with the standard gliding hump procedure and the block homogeneity of the unit vector basis in ℓ^2 (cf. [2]) we define a sequence (e_n) in E which is equivalent to the unit vector basis of ℓ^2 and satisfies the condition $\|P(e_n)\| < 2^{-n}\|e_n\|$ for $n = 1, 2, \ldots$. This implies that, for some n_0, the perturbed sequence $(e_n - P(e_n))_{n > n_0}$ is equivalent to the unit vector basis of ℓ^2; hence the space $F = [e_n - P(e_n)] \subset \ker P$ is hilbertian. If F is not complemented in Z, then the desired subspace is $F + Y$. If F is complemented in Z and therefore in $\ker P$, then the standard decomposition method (cf. [22]) yields that $\ker P$ is isomorphic to Z. Thus $\ker P$ contains a non complemented hilbertian subspace, say F_1. The desired subspace can be defined now as $F_1 + Y$.

A modification of the above argument gives

PROPOSITION 3.7: *Let Z be a separable Banach space such that* (i) *there exists a non complemented hilbertian subspace of Z,* (ii) *every infinite dimensional hilbertian subspace of Z contains an infinite dimensional subspace which is complemented in Z. Then*

(*) *given infinite dimensional complemented hilbertian subspaces of Z, say Y_1 and Y_2, there exists an isomorphism of Z onto itself which carries Y_1 onto Y_2.*

In particular H^p satisfies (*) *for* $1 \le p < 2$.

PROOF: Let P_j be a projection from Z onto Y_j $(j = 1, 2)$. Using (i) we construct similarly as in the proof of Proposition 3.6 subspaces F_j of $\ker P_j$ which are isomorphic to ℓ^2. By (ii) we may assume without loss of generality that F_j are complemented in Z and therefore in $\ker P_j$ $(j = 1, 2)$. Now the decomposition technique gives that $\ker P_j$ is isomorphic to Z for $j = 1, 2$. This allows to construct an isomorphism of Z onto itself which carries $\ker P_1$ onto $\ker P_2$ and $P_1(Z)$ onto $P_2(Z)$.

4. Remarks and open problems

We begin this section with a discussion of the behavior of the Banach Mazur distances $d(L^p, H^p)$, $d(L^p, L^p/H_0^p)$, $d(H^p, L^p/H_0^p)$ for $p \to \infty$ and for $p \to 1$.

Recall that if X and Y are isomorphic Banach spaces, then $d(X, Y) = \inf\{\|T\|\,\|T^{-1}\| : T : X \xrightarrow[\text{onto}]{} Y,\ T - \text{isomorphism}\}$; if X and Y are not isomorphic, then $d(X, Y) = \infty$. Let $p^* = p(p-1)^{-1}$. Then

$$(H^p)^{\perp} = \left\{ f \in L^{p^*} : \int_0^{2\pi} f(t)g(t)dt = 0 \text{ for } g \in H^p \right\} = H_0^{p^*}$$

Hence the map $\{f + H_0^{p^*}\} \to x_f^*$ where $x_f^*(g) = 1/(2\pi) \int_0^{2\pi} f(t)g(t)dt$ for $g \in H^p$ is a natural isometric isomorphism from $L^{p^*}/H_0^{p^*}$ onto the conjugate $(H^p)^*$. Thus, for $1 < p < \infty$,

$$d(L^p, H^p) = d(L^{p^*}, L^{p^*}/H_0^{p^*});\ d(H^p, L^p/H_0^p) = d(H^{p^*}, L^{p^*}/H_0^{p^*}). \tag{4.1}$$

The formulae (4.1) allow us to restrict our attention to the case where $p \to 1$. In the sequel we assume that $1 \le p \le 2$.

The results enlisted in section 1 give upper estimates for the distances in question. We have

PROPOSITION 4.1: *There exists an absolute constant K such that*

$$\max\,(d(L^p, H^p),\ d(L^p, L^p/H_0^p),\ d(H^p, L^p/H_0^p)) \le K \frac{p}{p-1}\ (1 < p \le 2).$$

PROOF: By Proposition 1.1(iii) and 1.2(iii), $d(L^p, H^p) \le K(p/p-1)$ and $d(L^{p^*}, H^{p^*}) \le Kp^* = Kp/(p-1)$. Hence, by (4.1), $d(L^p, L^p/H_0^p) \le Kp/(p-1)$. Let $\bar{H}^p = f \in L^p : \bar{f} \in H^p$ and let V denote the restriction to $\bar{H}^p$ of the quotient map $L^p \to L^p/H_0^p$. Clearly

$\|V(f)\|_{L^p/H_0^p} \le \|f\|_p$ for $f \in \bar{H}^p$. Since $Q(\bar{g})$ for $g \in H_0^p$ (cf. section 1 for the definition of Q), we have, for $f \in \bar{H}^p$ and $g \in H_0^p$, $\|f\|_p = \|\bar{f}\|_p = \|Q(\bar{f})\|_p = \|Q(\overline{f-g})\|_p \le \|Q\|_p \|f-g\|_p$. Thus $\|V(f)\|_{L^p/H_0^p} = \inf_{g \in H_0^p} \|f-g\|_p \ge \|Q\|_p^{-1}\|f\|_p$ for $f \in \bar{H}^p$. Therefore the range of V is closed in L^p/H_0^p and since $\bar{H}^p + H_0^p$ is dense in L^p, $V(\bar{H}^p)$ maps $\bar{H}^p$ onto L^p/H_0^p. Since $\bar{H}^p \cap H_0^p = \{0\}$, we infer that V is one to one. Thus $d(\bar{H}^p, L^p/H_0^p) \le \|V\|\|V^{-1}\| \le \|Q\|_p \le \|B\|_p \le Kp/(p-1)$. To complete the proof observe that $\bar{H}^p$ is isometrically isomorphic to H^p via the map $f \to f^*$ where $f^*(t) = f(-t)$.

PROBLEM 4.1: Does there exist an absolute constant $k > 0$ such that, for $1 < p < 2$,

$$\min(d(L^p, H^p), d(L^p, L^p/H_0^p), d(H^p, L^p/H_0^p)) > k\frac{p}{p-1}.$$

We are able to prove only

PROPOSITION 4.2: *There exists an absolute constant $k > 0$ such that*

(a) $d(L^p, H^p) \ge k\sqrt{\dfrac{p}{p-1}}$ $(1 < p \le 2)$,

(b) $d(H^p, L^p/H_0^p) \ge k\sqrt{\dfrac{p}{p-1}}$ $(1 < p \le 2)$,

(c) $\lim\limits_{p=1} d(L^p, L^p/H_0^p) = \infty$.

PROOF: (a) is an immediate consequence of the following stronger result.

(a′) There exists an absolute constant $k > 0$ such that if X is a subspace of H^p $(1 < p \le 2)$, if X contains a subspace isomorphic to ℓ^2, and if $X \xrightarrow{S} L^p \xrightarrow{T} X$ is a factorization of identity (i.e. $TS =$ the identity on X), then $\|T\|\|S\| \ge k\sqrt{p/(p-1)}$.

PROOF Applying Corollary 3.1: we can choose a subspace $E \subset X$ an isomorphism $U: E \xrightarrow[\text{onto}]{} \ell^2$ and a projection $P: X \xrightarrow[\text{onto}]{} E$ so that $\|U\|\|U^{-1}\| \le c$ and $\|P\| \le c$ where c is an absolute constant. Let $S_1 = SU^{-1}$ and $T_1 = UPT$. Then $\ell^2 \xrightarrow{S_1} L^p \xrightarrow{T_1} \ell^2$ is a factorization of identity with $\|S_1\|\|T_1\| \le \|S\|\|T\| \cdot c^2$. Now the desired conclusion follows from a result of Gordon, Lewis and Retherford [11], Remark (1) to Corollary 5.7 which asserts that there exists an absolute constant k_1 such that if $\ell^2 \xrightarrow{S_1} L^p \xrightarrow{T_1} \ell^2$ is any factorization of identity, then $\|T_1\|\|S_1\| \ge k_1\sqrt{p/(p-1)}$ $(1 < p \le 2)$. This completes the proof of (a′).

(b) is an immediate consequence of a slightly stronger result.

(b′) There exists an absolute constant $k>0$ such that if U is an isomorphism from L^p/H^p_0 onto a subspace X of H^p $(1<p\le 2)$ then $\|U\|\|U^{-1}\|\ge k\sqrt{p/(p-1)}$.

PROOF: Let X_p denote the closed linear subspace of L^p $(1<p\le 2)$ generated by the sequence (χ_{-2^k}). Let $I_p: L^p\to L^1$ and $j_p: L^p/H^p_0\to L^1/H^p_0$ denote natural embeddings (i.e. $j_p(\{f+H^p_0\})=\{f+H^1_0\}$) and let $q_p: L^p\to L^p/H^1_0$ denote the quotient map. Clearly $\|q_p\|\le 1$ and, we have $j_pq_p=q_1I_p$. A direct computation shows that $\|f\|_4\le 2^{1/4}\|f\|_2$ for $f\in X_2$. Thus the logarithmic convexity of the function $p\to\|f\|_p$ yields

$$\|f\|_2\ge\|f\|_p\ge\|f\|_1\ge 2^{-1/2}\|f\|_2 \quad \text{for } f\in X_p.$$

It follows from the above inequality and from the proof of Proposition 2.4 that the operator V_p – the restriction of q_p to X_p is invertible and $\|V_p^{-1}\|\le c$ where c is an absolute constant independent of p. Since X_p is isomorphic to ℓ^2, so is $UV_p(X_p)$. Hence, by Corollary 3.1, there exist a subspace E of $UV_p(X_p)$ an isomorphism $T: E\xrightarrow[\text{onto}]{}\ell^2$ and a projection $P: X\xrightarrow[\text{onto}]{}E$ with $\|T\|\,\|T^{-1}\|\le c_1$ and $\|P\|\le c_1$ where c_1 is an absolute constant. Now we consider the factorization of identity.

$$\ell^2\xrightarrow{T^{-1}}E\xrightarrow{U^{-1}}V_p(X_p)\xrightarrow{V_p^{-1}}L^p\xrightarrow{q_p}L^p/H^p_0\xrightarrow{U}X\xrightarrow{P}E\xrightarrow{T}\ell^2$$

By a result of [11], Remark (1) to Corollary 5.7, there exists an absolute constant $k_1>0$ such that

$$\begin{aligned} k_1\sqrt{\frac{p}{p-1}} &\le \|V_p^{-1}U^{-1}T^{-1}\|\,\|TPUq_p\| \\ &\le \|T\|\,\|T^{-1}\|\,\|V_p^{-1}\|\,\|q_p\|\,\|P\|\,\|U\|\,\|U^{-1}\| \\ &\le c_1^2c\|U\|\,\|U^{-1}\|. \end{aligned}$$

Thus $\|U\|\,\|U^{-1}\|\ge k\sqrt{(p/p-1)}$ for $k=k_1c_1^{-2}c^{-1}$. This completes the proof of (b′).

To prove (c), in view of the fact that, for $1<p\le 2$ $H^p\subset L^p$ is isometrically isomorphic to a subspace of L^1 (cf. e.g. [27], p. 354), it is enough to show

(c′) Let $d_p=\inf\{d(L^p/H^p_0,X): X\subset L^1\}$ $(1<p\le 2)$. Then $\lim_{p=1}d_p=\infty$.

PROOF of (c′): Fix $\epsilon>0$ and a finite-dimensional subspace B of L^1/H^1_0. Since the continuous 2π-periodic functions are dense in L^1, the standard perturbation argument (cf. e.g. [2]) yields the existence of a (dim B)-dimensional subspace G of $C_{2\pi}$ with $G \cap H_0^1 = \{0\}$ such that

$$d(B, (G + H_0^1)/H_0^1) < (1 + \epsilon)^{1/2}$$

($G + H_0^1$ is regarded as a subspace of L). Let us put

$$|||g|||_p = \inf \{\|g + h\|_p : h \in H_0^p\} \quad (g \in G, p \geq 1)$$

and let G_p stand for G equipped with the norm $|||\cdot|||_p$. We claim that

$$\text{(4.2)} \qquad \text{If } g \downarrow p, \text{then } |||g|||_q \downarrow |||g|||_p \quad (g \in G, p \geq 1).$$

To see (4.2) observe first that

$$|||g|||_p = \inf \{\|g + h\|_p : h \in A_0\} \quad (g \in G, p \geq 1),$$

because A_0 is dense in each H_0^p. Next note that, for every $g \in G$ and $h \in A_0$, the function $p \to \|g + h\|_p$ is (finite) continuous and non decreasing. Thus

$$\overline{\lim_{q \downarrow p}} |||g|||_q \leq |||g|||_p \text{ and } |||g|||_q \geq |||g|||_p \quad (g \in G, 1 \leq p < q)$$

which yield (4.2).

Let $S_G^1 = \{g \in G : |||g|||_1 = 1\}$. Since G is finite-dimensional, S_G^1 is compact. Hence Dini's Theorem combined with (4.2) implies that $|||g|||_p \to |||g|||_1 = 1$ uniformly on S_G^1 as $p \to 1$. Therefore there exists a $p_0 = p_0(B, \epsilon) > 1$ such that

$$(1 + \epsilon)^{1/2} \geq |||g|||_p \geq 1 \text{ for } g \in S_G^1 \text{ and for } 1 < p < p_0.$$

Equivalently the formal identity map $j_p : G_p \to G_1$ is an isomorphism with $\|j_p\| \|j_p^{-1}\| \leq (1 + \epsilon)^{1/2}$. Clearly G_p is isometrically isomorphic to the subspace $(G + H_p)/H_p$ of L^p/H_0^p. Using this fact for $p = 1$ we get

$$\text{(4.3)} \qquad d(B, G_p) \leq 1 + \epsilon \quad (1 < p < p_0).$$

Now suppose to the contrary that there exist a sequence $(p(n))$ with $\lim_n p(n) = 1$, a constant $\lambda > 0$ and a sequence $(\mathcal{X}_n)$ of subspaces of L^1 such that

$$d(L^{p(n)}/H_0^{p(n)}, \mathcal{X}_n) < \lambda \quad \text{for all } n.$$

Then (4.3) would imply that for every finite-dimensional subspace B of L^1/H_0^1 there exists a subspace B_1 in L^1 with $d(B, B_1) < \lambda$. Hence, by [16], Proposition 7.1, L^1/H_0^1 would be isomorphic to a subspace of some $L^1(\mu)$-space which contradicts [24]. This completes the proof of (c′) and therefore of Proposition 4.2.

There are several problems related to Proposition 2.1.

PROBLEM 4.2: Does there exist an absolute constant $\lambda \geq 1$ such that, for every p and q with $1 \leq q < p < 2$, there exists a subspace $X_{p,q}$ of H^q such that $d(H^p, X_{p,q}) \leq \lambda$? In particular is H^p isometrically isomorphic to a subspace of H^q?

The recent result of Dacunha-Castelle and Krivine [5] yields that, for every p with $1 \leq p < \infty$ and for every $\lambda > 1$, there exists a subspace X of H^p such that $d(X, \ell^2) < \lambda$. In fact a subspace X with the above property can be defined as the closed linear span of a sequence $(\sum_{j=mk+1}^{(m+1)k} \chi_{n_j})_{m=1,2\dots}$ where k and the "lacunary" sequence (n_j) depend on p and q. We do not know, however, whether ℓ^2 is isometrically isomorphic to a subspace of H^p for any $p \neq 2$? On the other hand there is no subspace of H^p which is isometrically isomorphic to the 2-dimensional space ℓ_2^p $(p \neq 2)$. Otherwise there would exist in H^p functions f_1 and f_2 of norm one such that $\|f_1 + f_2\|^p + \|f_1 - f_2\|^p = 2(\|f_1\|^p + \|f_2\|^p)$. Then (cf. e.g. [22]) $f_1 \cdot f_2 = 0$. Thus the analyticity of the f_i's would imply that either f_1 or f_2 is zero, a contradiction. This remark answers negatively a question of Boas [4] who asked whether H^p is isometrically isomorphic to L^p for some $p \neq 2$.

Finally we would like to mention the well known open problems concerning the existence of unconditional structures in H^1.

PROBLEM 4.3: (a) Does H^1 have an unconditional basis?

(b) Is H^1 isomorphic to a subspace of a Banach space with an unconditional basis? (c) Does H^1 have a local unconditional structure either in the sense of [6] or of [10]?

Let us mention that the basis for H^1 which has been constructed by Billard [3] is conditional.

Let us recall briefly Billard's construction. Let H_R^1 denote the real Banach space of functions $f \in L_R^1$ such that $\mathcal{H}(f) \in L_R^1$ equipped with the norm $|||f_1|||_1 = \sqrt{\|f\|^2 + \|\mathcal{H}(f)\|_1^2}$. It is easy to see that the complexification of H_R^1 is isomorphic to H^1. Therefore every basis for H_R^1 induces a basis for H^1. Billard [3] has proved that the classical Haar system $(h_k)_{0 \leq k < \infty}$ is a basis for H_R^1. (In our convention the h_k's are defined on the whole real line, are 2π-periodic, and restricted to $[0, 2\pi)$ consist the Haar orthonormal system i.e. $h_0 \equiv 1$ and for $j = 0, 1, \dots, r = 0, 1, \dots, 2^j - 1$,

$$h_{2^j+r}(t) = 2^{j/2}(I_{\Delta(j+1,2r)} - I_{\Delta(j+1,2r+1)})(t) \quad \text{for } 0 \leq t < 2\pi$$

where $\Delta(j+1, k) = \{t \in R : 2\pi k 2^{-j-1} < t < 2\pi(k+1)2^{-j-1}$ and I_A denotes the characteristic function of a set $A \subset R$.)

PROPOSITION: *The sequence* $(h_k)_{0 \leq k < \infty}$ *is a conditional basis for* H_R^1.

PROOF: Let us set $g_0 = 2h_1$, $g_0^* = 2h_1$,

$$g_n = 2h_1 + \sum_{j=1}^{n} 2^{j/2}(h_{2^j} + h_{2^{j+1}-1}),$$

$$g_n^* = 2h_1 + \sum_{k=1}^{k \le n/2} 2^{(2k+1)/2}(h_{2^{2k+1}} + h_{2^{2k+2}-1}).$$

Since $|||g_n^*|||_1 \ge \|g_n^*\|_1 \ge n/4$ for all n (an easy computation), to complete the proof it suffices to show that $\sup_n |||g_n|||_1 < \infty$. Observe that, for all n,

$$g_n(t) = 2^{n+1}(I_{\Delta(n+1,0)} - I_{\Delta(n+1,2^{n+1}-1)})(t) \quad \text{for } 0 \le t < 2\pi.$$

Thus $\|g_n\|_1 = 2$ for all n. Therefore our task is to show that $\sup_n \|\mathcal{H}(g_n)\|_1 < \infty$.

We have almost everywhere (cf. [33], [7])

$$\mathcal{H}(g_n)(t) = \frac{1}{2\pi} \lim_{\epsilon=0} \int_\epsilon^\pi ctg\left(\frac{s}{2}\right) [g_n(t-s) - g_n(t+s)]ds$$

$$= \frac{1}{2\pi} \lim_{\epsilon=0} \int_\epsilon^\pi \left[ctg\left(\frac{s}{2}\right) - \frac{2}{s}\right] [g_n(t-s) - g_n(t+s)]ds$$

$$+ \frac{1}{2\pi} \lim_{\epsilon=0} \int_\epsilon^\pi \frac{2}{s} [g_n(t-s) - g_n(t+s)]ds.$$

Since

$$\left| ctg\frac{s}{2} - \frac{2}{s} \right| < \frac{2}{\pi} \quad \text{for } 0 < s < \pi \text{ and } \|g_n\|_1 = 2,$$

we infer that

$$\frac{1}{2\pi} \lim_{\epsilon=0} \int_\epsilon^\pi \left[ctg\left(\frac{s}{2}\right) - \frac{2}{s}\right] [g_n(t-s) - g_n(t+s)]ds\|_1 \le c_1$$

for some constant c_1 independent of n. On the other hand, evaluating the second integral, we get

$$\frac{1}{2\pi} \lim_{\epsilon=0} \int_\epsilon^\pi \frac{2}{s} [g_n(t-s) - g_n(t+s)] = \frac{2^n}{\pi} \ln \left| \frac{(t - 2^{-n}\pi)(t + 2^{-n}\pi)}{t^2} \right|$$

$$= \frac{2^n}{\pi} \ln \left| 1 - \frac{\pi^2}{(2^n t)^2} \right|.$$

Since

$$2^n \int_0^{2\pi} \ln \left| 1 - \frac{\pi^2}{(2^n t)^2} \right| dt = c_2 < +\infty,$$

we infer that $\|\mathcal{H}(g_n)\|_1 \le c_1 + c_2$ for all n. This completes the proof.

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