COMPOSITIO MATHEMATICA

C. H. HOUGHTON

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Compositio Mathematica, tome 33, nº 2 (1976), p. 209-225

<http://www.numdam.org/item?id=CM_1976__33_2_209_0>

COMPOSITIO MATHEMATICA, Vol. 33, Fasc. 2, 1976, pag. 209–225
Noordhoff International Publishing
Printed in the Netherlands

RESTRICTED SUBGROUPS OF WREATH PRODUCTS OF GROUPS

C. H. Houghton

1. Introduction

Hartley [5] investigated the conjugacy classes of baseless subgroups of wreath products of groups, that is, subgroups which intersect the base group trivially. In [8], it was shown that his results are related to the theory of ends. Here we consider the conjugacy classes of those subgroups of a wreath product whose intersection with the base group consists of functions with support of size less than some fixed infinite cardinal.

The wreath product $W = A \text{ Wr } B$ of groups A and B may be taken as the split extension by B of the left B-group $F = A^B$ of functions from B to A, with $x(fg) = (xf)(xg)$ and $x({}^b f) = (xb)f$, for $f, g \in F$, x, $b \in B$. Thus W consists of all pairs fb, with $f \in F$, $b \in B$, and $(fb)(gc) = (f\,{}^b g)bc$, for $f, g \in F$ and $b, c \in B$; we shall assume throughout that A and B are non-trivial. Let $\sigma(f)$ denote the support of $f \in F$. For an infinite cardinal α, we define F_α to consist of those $f \in F$ such that $|\sigma(f)| < \alpha$ and we put $W_\alpha = BF_\alpha \leq W$. When $\alpha = \aleph_0$, F_α consists of the functions with finite support and W_α is the restricted wreath product $A \text{ wr } B$ of A and B.

A subgroup L of W will be called α-restricted if $L \cap F \leq F_\alpha$; in the case $\alpha = \aleph_0$, we simply say that L is restricted. Clearly all subgroups of W_α are α-restricted and the question we consider is when an α-restricted subgroup L of W is conjugate in W to a subgroup of W_α.

We define the B-image of a subgroup L of W to be the image of L under the natural map from W to B. Our first result shows that if the B-image C of an α-restricted subgroup L is sufficiently small, then L is conjugate to a subgroup of W_α. Let β be the least cardinal such that α is the sum of β cardinals each $< \alpha$.

THEOREM (A): *If $\alpha > \aleph_0$ and $|C| \leq \beta$ or if $\alpha = \aleph_0$ and C is countable and locally finite, then all α-restricted subgroups of $W = A$ Wr B with B-image C are conjugate in W to subgroups of W_α.*

The remaining results are concerned with the case $\alpha = \aleph_0$ and are related to work of Farrell [3, 4] and Bieri [1]. We show that if C has a normal finitely presented infinite subgroup N of infinite index, then, in most cases, the problem reduces to finding the number of ends of N and C/N. We summarise these results, using $e(G)$ to denote the number of ends of the group G.

THEOREM (B): *Every restricted subgroup of $W = A$ Wr B with B-image C is conjugate to a subgroup of A wr B if C has a finitely generated free subgroup of finite index or C has a finitely presented normal subgroup N of infinite index such that either $e(N) = 1$, or $e(N) = 2$ and $e(C/N) = 1$, or $e(N) = \infty$, $e(C/N) = 1$ and C is finitely generated. There exist restricted subgroups with B-image C which are not conjugate to subgroups of A wr B if C has a finitely presented normal subgroup N with $e(N) > 1$ and $e(C/N) > 1$. For a polycyclic by finite group C, every restricted subgroup with B-image C is conjugate to a subgroup of A wr B if and only if C has Hirsch number 2.*

If $A^{(B)}$ denotes the B-group of functions from B to A with finite support, then A wr B is the split extension of $A^{(B)}$ by B. The previous results imply the following theorem.

THEOREM (C): *Let A be any non-trivial group. All extensions of $A^{(B)}$ by B split if B is countable and locally finite or is finitely generated free by finite. If B is polycyclic by finite, all extensions of $A^{(B)}$ by B split if and only if B has Hirsch number different from 2.*

We make use of the theory of groupoids, details of which may be found in Higgins [6]. Our definition of the wreath product has been chosen to correspond to the natural multiplication in the covering groupoid associated with a permutation representation of a given group.

2. The general case

Suppose C acts as a group of permutations of a set X. The associated covering groupoid of C is the set $X \times C$ with vertex set X and multiplication $(x, c)(xc, d) = (x, cd)$ for $x \in X$, $c, d \in C$. A map θ

from $X \times C$ to a group A will be called an almost homomorphism if, for each pair $c, d \in C$, $(x, c)\theta(xc, d)\theta = (x, cd)\theta$, for almost all $x \in X$; that is, the exceptions form a set of cardinal less than α. All homomorphisms are almost homomorphisms; in particular, this applies to the trivial map. The almost homomorphisms θ and ϕ are defined to be equivalent if there is a map γ from X to A such that, for each $c \in C$, $(x, c)\phi = (x\gamma)^{-1}(x, c)\theta(xc\gamma)$, for almost all $x \in X$. We note that if θ is any almost homomorphism and γ is any map from X to A, then the corresponding ϕ will be an almost homomorphism. Also, for any almost homomorphism θ, we have $(x, 1)\theta = 1$, for almost all $x \in X$, and so θ is equivalent to an almost homomorphism ϕ such that $(x, 1)\phi = 1$, for all $x \in X$.

Let W be the wreath product of A and C relative to the action of C on X; that is, W consists of all pairs fc, with f in the group F of functions from X to A and $c \in C$, and the multiplication is given by $(fc)(gd) = (f\,{}^{c}g)cd$, where $x({}^{c}g) = (xc)g$. As before, F_α denotes the subgroup of F consisting of those f such that $|\sigma(f)| < \alpha$ and a subgroup L of W is α-restricted if $L \cap F \leq F_\alpha$. If $f, g \in F$ are congruent modulo F_α then they differ on a set of cardinal $< \alpha$. We say f is almost equal to g and write $f =^{a} g$. We shall consider the case where C acts semiregularly on X, that is, the stabiliser of each point is trivial and so the representation can be thought of as a sum of regular representations.

THEOREM (1): *Let C act semiregularly on the set X and let W be the wreath product of A and C relative to X. The conjugacy classes of α-restricted subgroups containing F_α and having C as image under the projection from W to C correspond bijectively to the equivalence classes of almost homomorphisms from $X \times C$ to A.*

Let $W_1 = A \operatorname{Wr} B$ be the standard wreath product of A and B and suppose C is a subgroup of B. Every α-restricted subgroup of W_1 with B-image C is conjugate to a subgroup of the α-restricted wreath product W_α of A and B if and only if all almost homomorphisms from $B \times C$ to A are equivalent.

PROOF: Given an almost homomorphism θ from $X \times C$ to A, we define $f_c \in F$, for each $c \in C$, by $xf_c = (x, c)\theta$. For $c, d \in C$, we have $x(f_c\,{}^{c}f_d) = (x, c)\theta(xc, d)\theta$ and $xf_{cd} = (x, cd)\theta$ and so $f_c\,{}^{c}f_d =^{a} f_{cd}$. Let $R = R(\theta)$ be the subgroup of W generated by all $f_c c$, with $c \in C$, and by F_α. Now $f_c\,{}^{c}f_d F_\alpha = f_{cd}F_\alpha$ so $(f_c c)(f_d d)F_\alpha = f_{cd}cdF_\alpha$ and $R = \{f_c c : c \in C\}F_\alpha$. Hence $R \cap F = F_\alpha$ and $R = R(\theta)$ is an α-restricted subgroup of W. Suppose ϕ is an almost homomorphism equivalent to θ and so, for each $c \in C$, $(x, c)\phi = (x\gamma)^{-1}(x, c)\theta(xc)\gamma$, for almost all

$x \in X$. Putting $xh_c = (x, c)\phi$, we have $h_c =^a \gamma^{-1} f_c {}^c\gamma$ and $(h_c c)F_\alpha = (\gamma^{-1}(f_c c)\gamma)F_\alpha$ so

$$R(\phi) = \{h_c c : c \in C\}F_\alpha = \gamma^{-1}\{f_c c : c \in C\}F_\alpha \gamma = \gamma^{-1} R(\theta)\gamma.$$

Suppose R is an α-restricted subgroup of W containing F_α and having image C under the projection map from W to C. If T is a transversal of the cosets of F_α in R, then $T = \{f_c c : c \in C\}$ and $f_c {}^c f_d =^a f_{cd}$. Defining $(x, c)\theta = xf_c$ gives an almost homomorphism from $X \times C$ to A. We note that θ depends on the choice of transversal as well as on R. Suppose S is a subgroup of W conjugate to R. We shall show that if ϕ is an almost homomorphism associated with S then ϕ is equivalent to θ. For some $fb \in W$, we have $S = fbRb^{-1}f^{-1}$ and

$$bRb^{-1} = bTb^{-1}F_\alpha = \{{}^b f_c(bcb^{-1}) : c \in C\}F_\alpha = \{{}^b f_c d : d \in C\}F_\alpha,$$

where $d = bcb^{-1}$. Putting $e = b^{-1}$,

$${}^b f_c =^a {}^b f_{edb} =^a {}^b f_e {}^{be} f_d {}^{bed} f_b =^a {}^b f_e f_d {}^d f_b.$$

Also $f_b {}^b f_e =^a 1$, so ${}^b f_c =^a f_b^{-1} f_d {}^d f_b$ and $bRb^{-1} = f_b^{-1}\{f_d d : d \in C\} f_b F_\alpha$. Thus $S = ff_b^{-1} R f_b f^{-1}$. Putting $g = f_b f^{-1}$ and choosing a transversal U for F_α in S, we have $U = \{k_c c : c \in C\}$ with $k_c =^a g^{-1} f_c {}^c g$. Taking $(x, c)\phi = xk_c$, we have, for each $c \in C$, $(x, c)\phi = (xg)^{-1}(x, c)\theta(xc)g$, for almost all $x \in X$, and hence ϕ is equivalent to θ. Thus the conjugacy classes of restricted subgroups containing F_α correspond to the equivalence classes of almost homomorphisms.

Suppose every almost homomorphism from $B \times C$ to A is equivalent to the trivial one and let R be an α-restricted subgroup of W_1 with B-image C. Then RF_α is contained in the wreath product W of A and C with $X = B$ and therefore RF_α is conjugate to a subgroup of W_α and so also is R. Conversely, if θ is an almost homomorphism from $B \times C$ to A, then there is a corresponding α-restricted subgroup R of W containing F_α. Now B normalises W_α so if R is conjugate to a subgroup of W_α, we have $R^{fb} \leq W_\alpha$ and hence $R^f \leq W_\alpha$, for some $f \in F$, $b \in B$. Since R^f is in the conjugacy class of R in W, the first part implies that θ is equivalent to the trivial homomorphism.

We note that if the subgroup R above is baseless, that is, $R \cap F = 1$, then $f_c {}^c f_d = f_{cd}$, for all $c, d \in C$, and then the corresponding θ is a homomorphism. The next result shows that Theorem C is a consequence of Theorems A and B.

THEOREM (2): *Let F_α be the B-group of functions f from B to A with $|\sigma(f)| < \alpha$ and $x({}^b f) = (xb)f$, for all $x, b \in B$. All extensions of F_α by B split if and only if all α-restricted subgroups of $W = A$ Wr B which contain F_α and have B-image B are conjugate.*

PROOF: Let K be an extension of F_α by B and let ρ be the natural map from K to B. For each $b \in B$, we may choose $b\tau \in K$ such that $b\tau\rho = b$ and $(b\tau)k(b\tau)^{-1} = {}^bk$, for all $k \in K$. We define $\omega : K \to W$ by $k\omega = f_k k\rho$, where $f_k \in F$ and $bf_k = 1((b\tau)k((b \cdot k\rho)\tau)^{-1})$, for $b \in B$; we note that the last expression is the value at 1 of some element of F_α. If $m \in K$ then

$$k\omega m\omega = f_k{}^{k\rho}f_m(km)\rho$$

and

$$b(f_k{}^{k\rho}f_m) = 1((b\tau)k((b \cdot k\rho)\tau)^{-1}1((b \cdot k\rho)\tau)m((b \cdot k\rho \cdot m\rho)\tau)^{-1} = bf_{km}.$$

Thus ω is a homomorphism. If $k \in F_\alpha$ then $k\omega = f_k$ with $bf_k = 1((b\tau)k(b\tau)^{-1}) = 1({}^bk) = bk$, for all $b \in B$, so $k\omega = k$. Then ω is injective and K is isomorphic to a subgroup L of W with B-image B and $L \cap F = F_\alpha$. We call such a subgroup a full α-restricted subgroup of W and note that we have shown that any extension of F_α by B is isomorphic to one of these.

A full α-restricted subgroup L of W which is conjugate to $W_\alpha = BF_\alpha$ is a split extension of F_α. Suppose conversely that L splits as an extension of F_α by B. Then L has a subgroup $M = \{f_b b : b \in B\}$, with $f_b \in F$, and M is baseless, that is, $M \cap F = 1$. From Lemma 3.2(i) of [5], M is conjugate in W to B and so L is conjugate to W_α. (This also follows from Lemma 3 below, since the θ corresponding to M is a homomorphism.) Thus all extensions split if and only if all full α-restricted subgroups are conjugate.

We now explain the relation between our work and that of Farrell [3, 4] and Bieri [5]. For abelian A, all extensions of F_α by B split if and only if $H^2(B, F_\alpha) = 0$. Suppose R is a commutative ring with identity, A is a free R-module and $\alpha = \aleph_0$. Then the B-group F_α of functions from B to A with finite support is B-isomorphic to $A \otimes_R RB$. Thus $H^2(B, F_\alpha)$ is isomorphic to $H^2(B, A \otimes_R RB)$, which is the group studied by Farrell and Bieri. Farrell considers the case where R is a field and obtains results which are not included in the results proved here for general A.

We now consider the classification of the equivalence classes of almost homomorphisms from $X \times C$ to A, under the assumption that C acts semiregularly on X.

LEMMA (3): *Suppose θ is an almost homomorphism from $X \times C$ to A and G is a subgroupoid of $X \times C$. If the restriction of θ to G is a homomorphism, there exists $\gamma : X \to A$ such that $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$, for $(x, c) \in G$, and θ is equivalent to an almost homomorphism trivial on G.*

PROOF: Let U be a subset of X containing one vertex from each connected component of G. Since C acts semiregularly, all vertex groups of G are trivial and there is a unique element joining one vertex to another in the same component of G. If x is a vertex of G then $x = ud$ for a unique $(u, d) \in G$ with $u \in U$. For $c \in C$, with $(x, c) \in G$, we have $(x, c) = (x, d^{-1})(u, dc) = (u, d)^{-1}(u, dc)$, so $(u, dc) \in G$ and $(x, c)\theta = ((u, d)\theta)^{-1}(u, dc)\theta$. Taking $x\gamma = (u, d)\theta$ gives $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$, for $(x, c) \in G$. Let $x\gamma$ be defined in this way for all vertices of G and let $x\gamma = 1$ for all other $x \in X$. If $(x, c)\phi = (x\gamma)(x, c)\theta((xc)\gamma)^{-1}$, then ϕ is an almost homomorphism equivalent to θ and trivial on G.

If D is a subgroup of C, we shall refer to a connected component of $X \times D$ as a D-sheet of $X \times C$. Thus each D-sheet consists of all (xd, e), with $d, e \in D$ and x a fixed vertex of X. We recall that β was defined as the least cardinal such that α is the sum of β cardinals each $< \alpha$.

LEMMA (4): *Suppose* $|C| < \beta$ *and* θ *is an almost homomorphism from* $X \times C$ *to* A. *Then* θ *is a homomorphism on almost all* C*-sheets of* $X \times C$, *that is, there exists a subset* T *of* X *with* $|T| < \alpha$, *such that the restriction of* θ *to* $xC \times C$ *is a homomorphism for all* $x \in X \backslash TC$.

PROOF: For $c, d \in C$, let $X(c, d)$ be the set of all $x \in X$ such that $(x, c)\theta(xc, d)\theta \neq (x, cd)\theta$. Then $|X(c, d)| < \alpha$ and if T is the union of all $X(c, d)$, with $c, d \in C$, then $|T| < \alpha$. Clearly the restriction of θ to $(X \backslash TC) \times C$ is a homomorphism.

Theorem 1 shows that the next result implies Theorem A.

THEOREM (5): *Suppose* $\alpha > \aleph_0$ *and* $|C| \leq \beta$ *or* $\alpha = \aleph_0$ *and* C *is countable and locally finite. If* C *acts semiregularly on* X, *all almost homomorphisms from* $X \times C$ *to* A *are equivalent.*

PROOF: Considering α as an ordinal which is not equivalent to any of its predecessors, our assumption implies that we can express C as $\cup_{i<\beta} C_i$, with $C_i \leq C_j$ for $i \leq j$, $C_\lambda = \cup_{i<\lambda} C_i$ for λ a limit ordinal, and $|C_i| < \beta$ for all i. Let θ be an almost homomorphism from $X \times C$ to A. For $x \in X$, let $J(x)$ be the set of ordinals $j < \beta$ such that the restriction of θ to the C_j-sheet containing x is not a homomorphism. If $J(x)$ is non-empty, it has a first element $j(x)$. For a limit ordinal λ, if θ is a homomorphism on all C_j-sheets containing x with $j < \lambda$, then θ is a homomorphism on the C_λ-sheet containing x. Thus $j(x)$ is not a limit ordinal and hence has an immediate predecessor. So for each $x \in X$ there is a maximal ordinal i such that θ restricted to the C_i-sheet containing x is a homomorphism; if $J(x)$ is empty, $i = \beta$. Suppose y is another element of X with corresponding maximal ordinal j. If the maximal sheets containing x and y intersect, then $xC_i \cap yC_j \neq \emptyset$ so, assuming $i \leq j$, we have $xC_j = yC_j$ and hence $i = j$. Thus X is partitioned into maximal subsets xC_i such that θ is a homomorphism on the C_i-sheet with vertex set xC_i. If G denotes the subgroupoid which is the union of all these sheets, then θ is a homomorphism on G. Let X_i be the set of all vertices of X not contained in a C_i-sheet of G. From Lemma 4, $|X_i| < \alpha |C_i| = \alpha$. From Lemma 3, we can choose γ so that $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$, for $(x, c) \in G$. Then for $c \in C_i$, $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$ for $x \in X \backslash X_i$ and hence for almost all $x \in X$. So θ is equivalent to the trivial homomorphism.

3. The case $\alpha = \aleph_0$

From now on we restrict our attention to the case $\alpha = \aleph_0$. Although some of the lemmas hold for a general cardinal, we can only make significant deductions in this case. We begin by describing the results we need from the theory of ends. For further details, see Cohen [2].

A subset S of a group G is almost invariant if $Sg \cap (G \backslash S)$ is finite, for all $g \in G$. The number of ends of G, denoted by $e(G)$, is the supremum of the number of parts in a partition of G into infinite almost invariant subsets. Then $e(G) = 0, 1, 2$, or ∞, with $e(G) = 0$ if and only if G is finite and $e(G) = 2$ if and only if G is infinite cyclic by finite. Finitely generated groups G with $e(G) = \infty$ have been characterised and any countable locally finite group G has $e(G) = \infty$. If G has an ascendant subgroup which is not locally finite and has no non-abelian free subgroups, then $e(G) = 1$, unless G is infinite cyclic by finite. Further results may be found in [2] and [9].

Let G act semiregularly on X and let θ be a homomorphism from $X \times G$ to A. We say θ is almost trivial if, for each $g \in G$, $(x, g)\theta = 1$, for almost all $x \in X$. Two such homomorphisms θ and ϕ will be called equivalent if, for some function f from X to A, almost equal to a function constant on all xG, we have $(x, g)\phi = (xf)^{-1}(x, g)\theta(xg)f$, for all $(x, g) \in X \times G$. We note that this definition of equivalence is much weaker than equivalence between almost homomorphisms. From Lemma 3, there is a function h from X to A such that $(x, g)\theta = (xh)^{-1}(xg)h$, for all $(x, g) \in X \times G$. Since θ is almost trivial, $h =^{a} {}^{g}h$, for all $g \in G$. Such a function h is said to be almost G-invariant and we note that any such function defines an almost trivial homomorphism from $X \times G$ to A by putting $(x, g)\theta = (xh)^{-1}(xg)h = x(h^{-1\,g}h)$. Then each almost trivial homomorphism θ is given by $(x, g)\theta = x(h^{-1\,g}h)$, for some almost invariant function h, and if $(x, g)\phi = x(k^{-1\,g}k)$, with k almost invariant, then θ and ϕ are equivalent if and only if $k^{-1\,g}k = f^{-1}h^{-1\,g}h\,{}^{g}f$, for some f almost equal to a function constant on all xG and for all $g \in G$.

For any function f from X to A, let $\{S_i : i \in I\}$ be the decomposition of X into constancy sets of f, that is, $x, y \in S_i$ if and only if $xf = yf$. Then f is almost invariant if and only if $\cup_{i \in I}(S_i g \cap (X \setminus S_i))$ is finite, for all $g \in G$. In the case where $X = G$ and f is almost invariant, the sets S_i are almost invariant subsets of G. Detailed analysis gives the following results.

LEMMA (6): *Let G act semiregularly on X. If* $e(G) = 0$ *or* 1, *every almost invariant function from G to A is almost equal to a constant function and every almost trivial homomorphism from* $X \times G$ *to A is equivalent to the trivial homomorphism. If* $e(G) = 2$ *and* $\{S, T\}$ *is a partition of G into infinite almost invariant subsets, then any almost invariant function from G to A is almost equal to a function constant on S and T. If* $e(G) = \infty$ *and G is finitely generated, for any almost invariant function f from G to A, there is a finite partition* $\{S_1, \ldots, S_r\}$ *of G into infinite almost invariant subsets such that f is almost equal to a function constant on each* S_i. *If* $e(G) > 1$, *the set Z of equivalence classes of almost trivial homomorphisms from* $X \times G$ *to A contains a subset bijective with the set* $A_0^{(X/G)}$, *consisting of the functions with finite support from* X/G *to the set* A_0 *of conjugacy classes of A; if* $e(G) = 2$, *this subset is the whole of Z.*

PROOF: If $e(G) = 0$, G is finite and the results are immediate. For infinite G, the remarks about almost invariant functions follow from Lemmas 4.3 and 4.4 of [7]. Translation of Theorem 3 of [8] from the language of wreath products to that of groupoids, shows that, if $e(G) = 1$, every almost trivial homomorphism from $X \times G$ to A is equivalent to the trivial homomorphism. Finally, we must consider the set Z when $e(G) > 1$.

Let S be an infinite almost invariant subset of G such that $T = G \setminus S$ is infinite and let U be a transversal of the orbits of X under G. For any function $k \in A^{(U)}$ with support V, we can define an almost invariant function $h = h(k)$ from X to A by taking $(vS)h = vk$, for $v \in V$, and $(X \setminus VS)h = 1$. If m is conjugate to k in $A^{(U)}$ and $t = h(m)$, then for some function f from X to A, constant on all xG, we have $t = fhf^{-1}$. For $g \in G$, $t^{-1g}t = fh^{-1}f^{-1g}f^{g}h^{g}f^{-1} = fh^{-1g}h^{g}f^{-1}$, and thus the homomorphisms corresponding to h and t are equivalent. Conversely, if the homomorphisms corresponding to h and t are equivalent, there are functions e, f from X to A, with f constant on all xG and $e = {}^a f$, such that $t^{-1\,g}t = eh^{-1\,g}h^{\,g}e^{-1}$, for $g \in G$. Then

$$tfh^{-1}f^{-1} = {}^a teh^{-1}f^{-1} = {}^g(teh^{-1})f^{-1} = {}^{ag}(tfh^{-1}f^{-1}),$$

for all $g \in G$, and so $tfh^{-1}f^{-1}$ is constant on all xG. For $u \in U$, $(uT)tfh^{-1}f^{-1} = 1$, so $tfh^{-1}f^{-1} = 1$ and $t = fhf^{-1}$. Then m is conjugate to k and so we have shown that Z has a subset bijective with $A_0^{(U)}$.

In the case where $e(G) = 2$, a standard argument (see Theorem 3 of [8] or Lemma 2.3 of [2]) shows that any almost invariant function f from X to A is constant on almost all uG, $u \in U$. Let V be the finite set consisting of the remaining $u \in U$. Since f is almost invariant on each vG, $v \in V$, it is almost equal to a function d with $(vS)d = a_v$, $(vT)d = b_v$, for $v \in V$, and $(uG)d = a_u$, for $u \in U \setminus V$. Then the almost trivial homomorphism associated with f is equivalent to the almost homomorphism ϕ associated with d and defined by $(x, g)\phi = (xd)^{-1}(xg)d$. Now $(x, g)\phi = 1$, for $x \in X \setminus VG$, and if $x \in vG$ with $v \in V$, $(x, g)\phi = 1$ for $x \in v(S \cap Sg^{-1}) \cup v(T \cap Tg^{-1})$, $(x, g)\phi = a_v^{-1}b_v$ for $x \in v(S \cap Tg^{-1})$, and $(x, g)\phi = b_v^{-1}a_v$ for $x \in v(T \cap Sg^{-1})$. Let $k \in A^{(U)}$ be given by $vk = a_v^{-1}b_v$ and let $h = h(k)$. Then $(x, g)\phi = (xh)^{-1}(xg)h$ and hence every almost trivial homomorphism is equivalent to one associated with $A_0^{(U)}$.

We now return to the analysis of almost homomorphisms.

LEMMA (7): *Suppose N is a finitely presented subgroup of C and let θ be an almost homomorphism from $X \times C$ to A. Then θ is equivalent to an almost homomorphism ψ trivial on almost all N-sheets of $X \times C$. If N is free, ψ may be taken as trivial on all N-sheets.*

PROOF: Let $\langle P : Q \rangle$ be a presentation of N with P and Q finite; if N is free, let Q be empty. A relator r in Q is expressed as a word $y_1 \cdots y_s$ with $y_i \in P \cup P^{-1}$. For almost all $x \in X$,

$$1 = (x, 1)\theta = (x, y_1 \cdots y_s)\theta = (x, y_1)\theta(xy_1, y_2)\theta \cdots (xy_1 \cdots y_{s-1}, y_s)\theta,$$

and also, for $p \in P$, $1 = (x, 1)\theta = (x, p)\theta((x, p)^{-1})\theta$. Now P and Q are finite, so there is a finite subset V of X such that, if $x \in Y = X \setminus VN$, an equation of the previous kind holds for all relators in Q and $(xp, p^{-1})\theta = ((x, p)^{-1})\theta = ((x, p)\theta)^{-1}$, for $p \in P$. If N is free then, replacing θ by an equivalent almost homomorphism, we may assume V is empty.

With $y \in Y$ and a word $w = p_1 \cdots p_t$, where $p_i \in P \cup P^{-1}$, we can associate the product $v(y, w) = (y, p_1)\theta(yp_1, p_2)\theta \cdots (yp_1 \cdots p_{t-1}, p_t)\theta$. Now $(x, p)\theta(xp, p^{-1})\theta = 1 = (xp, p^{-1})\theta(x, p)\theta$, for $x \in Y$, so deletion from w or insertion in w of products pp^{-1} or $p^{-1}p$, with $p \in P$, gives a word w' with $v(y, w') = v(y, w)$. Similarly, if w' is obtained from w by deleting a relator, then $v(y, w') = v(y, w)$. Suppose $n \in N$ has two expressions in terms of $P \cup P^{-1}$, $n = p_1 \cdots p_t = z_1 \cdots z_s$, and let w be the word $p_1 \cdots p_t z_s^{-1} \cdots z_1^{-1}$. After deletion and insertion of products pp^{-1} and $p^{-1}p$, we obtain a word u which is a product of conjugates of relators and for which $v(y, u) = v(y, w)$. Deleting the relators from u gives a word m with $v(y, m) = v(y, u)$ and the invariance of v under deletion of products pp^{-1} and $p^{-1}p$ implies that $v(y, m) = 1$. Hence $v(y, w) = 1$ and $v(y, p_1 \cdots p_t) = v(y, z_1 \cdots z_s)$. For $y \in Y$ and $n \in N$, with $n = p_1 \cdots p_t$, we put $(y, n)\phi = (y, p_1)\theta \cdots (yp_1 \cdots p_{t-1}, p_t)\theta$. This gives a well defined map from $Y \times N$ to A, which is clearly a homomorphism. We extend ϕ to a map from $X \times C$ by taking $(x, c)\phi = (x, c)\theta$ for (x, c) outside $Y \times N$. Now $(x, p)\phi = (x, p)\theta$ for all $p \in P \cup P^{-1}$ and all $x \in X$. Suppose $m, n \in N$ and $(x, m)\phi = (x, m)\theta$, $(x, n)\phi = (x, n)\theta$, for almost all $x \in X$. Then, for almost all $y \in Y$, $(y, mn)\phi = (y, m)\phi(ym, n)\phi = (y, m)\theta(ym, n)\theta = (y, mn)\theta$ and so $(x, mn)\phi = (x, mn)\theta$, for almost all $x \in X$. Thus, for all $n \in N$, we have $(x, n)\phi = (x, n)\theta$, for almost all $x \in X$, and so ϕ is an almost homomorphism equivalent to θ. From Lemma 3, ϕ is equivalent to an almost homomorphism ψ, trivial on $Y \times N$.

LEMMA (8): *Suppose C has a finitely presented normal subgroup N of infinite index. Every almost homomorphism from $X \times C$ to A is equivalent to an almost homomorphism ψ trivial on all N-sheets of $X \times C$. For such a ψ, if $xf_c = (x, c)\psi$, then f_c is almost N-invariant, for all $c \in C$.*

PROOF: From Lemma 7, any almost homomorphism is equivalent to an almost homomorphism θ trivial on almost all N-sheets. Let T be a finite subset of X with one vertex in each exceptional N-sheet. Since C/N is infinite, for any N-sheet with vertex set yN, $y \in T$, there exists $d \in C$ such that θ is trivial on the sheet with vertex set $yNd = ydN$. If $n \in N$, then $d^{-1}nd \in N$ and, for almost all $m \in N$,

$$\begin{aligned}(ym, n)\theta &= (ym, d)\theta(ymd, d^{-1}nd)\theta((ymn, d)\theta)^{-1} \\ &= (ym, d)\theta((ymn, d)\theta)^{-1}.\end{aligned}$$

We now define $\gamma : X \to A$ by putting $(ym)\gamma = (ym, d)\theta$, for $y \in T$, $m \in N$, and $d = d(y)$ chosen as above; we take $x\gamma$ trivial on all other

$x \in X$. If $(x, c)\phi = (x\gamma)^{-1}(x, c)\theta(xc)\gamma$, for $(x, c) \in X \times C$, then ϕ is equivalent to θ. For fixed $n \in N$, $(x, n)\phi$ is non-trivial only for $x = ym$, where $y \in T$, $m \in N$, and for each y the exceptions m form a finite set. Thus $(x, n)\phi$ is trivial for almost all $x \in X$ and so ϕ is equivalent to an almost homomorphism ψ trivial on all N-sheets.

Let $c \in C$ be fixed. For $x \in X$ and $n \in N$, we have $(xn, c) = (xn, n^{-1})(x, c)(xc, c^{-1}nc)$ so, for fixed n, $(xn, c)\psi = (x, c)\psi$, for almost all $x \in X$. Thus ${}^n f_c =^a f_c$, for all $n \in N$, and f_c is almost N-invariant.

LEMMA (9): *Suppose C has a finitely generated normal subgroup N. Let U be a transversal of the orbits of X under C and T a transversal of the cosets of N in C. If $f: X \to A$ is almost N-invariant, there exists a partition $\{N_1, \ldots, N_r\}$ of N into infinite almost invariant subsets such that f is almost equal to a function constant on almost all xN and on all uN_it, for $u \in U$, $t \in T$.*

PROOF: Let P be a finite generating set for N and let V be the intersection of UT with $(\cup_{p \in P} \sigma(f^{-1}\,{}^p f))N$. Then V is finite and, for $x \in X \backslash VN$, $(xp)f = xf$, for $p \in P$, and so f is constant on xN. For $v = ut \in V$ with $u \in U$, $t \in T$, let $h_v = h: N \to A$ be defined by $mh = (umt)f$, for $m \in N$. If $m, n \in N$, then

$$m({}^n h) = (mn)h = (umnt)f = (umt(t^{-1}nt))f = (umt)\,{}^y f,$$

where $y = t^{-1}nt$. But ${}^y f =^a f$ so, for almost all $m \in N$, $m({}^n h) = (umt)f = mh$, and so h is almost invariant. If S_i, $i \in I$, are the constancy sets of h and h is not constant on N, then, for each $i \in I$, $S_iN \neq S_i$ and so $S_ip \cap (N \backslash S_i)$ is non-empty, for some $p \in P$. But $\cup_{i \in I} S_ip \cap (N \backslash S_i)$ is finite for all $p \in P$ and so I is finite. Consider all possible subsets of N of the form $\cap_{v \in V} R_v$, where R_v is a constancy set of h_v. These form a finite partition of N into almost invariant subsets on which all h_v are constant. Incorporating all the finite parts in one of the infinite parts, we have a partition $\{N_1, \ldots, N_r\}$ of N into infinite almost invariant subsets such that, for $v \in V$, $h_v =^a k_v$, where k_v is a function constant on all N_i. Then f is almost equal to a function constant on almost all xN and on all uN_it, with $u \in U$, $t \in T$.

LEMMA (10): *Suppose C has a finitely generated normal subgroup N with a partition $\{N_1, \ldots, N_r\}$ into infinite almost invariant subsets. Let U be a transversal of the orbits of X under C and T a transversal of the cosets of N in C and put $Y = X/N$, $D = C/N$. Let Z be the set of equivalence classes of almost homomorphisms θ from $X \times C$ to A which are trivial on all N-sheets and such that, if $xf_c = (x, c)\theta$, then f_c is* *constant on almost all xN and constant on all uN_it,* with $u \in U$, $t \in T$. *Then Z is trivial if $r = 1$. If $r \geq 2$, then Z has a subset bijective with the set of equivalence classes of almost trivial homomorphisms from $Y \times D$ to A and this is the whole of Z if $r = 2$.*

PROOF: Let θ be an almost homomorphism satisfying the given conditions. If $n \in N$, $c \in C$, then $f_{nc} =^a f_n{}^n f_c =^a f_c$. But f_c and f_{nc} are constant on all the infinite sets uN_it and so $f_c = f_{nc}$.

Let ρ denote the natural maps from X to Y and D to C. For fixed i, there is a well defined map $\phi = \phi_i$ from $Y \times D$ to A given by $((ut)\rho, c\rho)\phi = (uN_it)f_c$, for $u \in U$, $t \in T$, $c \in C$. If c, $e \in C$, then

$$(((ut)\rho, c\rho)((utc)\rho, e\rho))\phi = ((ut)\rho, (ce)\rho)\phi = (uN_it)f_{ce}.$$

Since $f_{ce} =^a f_c{}^c f_e$, we have $(unt)f_{ce} = (unt)f_c(untc)f_e$, for almost all $n \in N_i$. For fixed t, c, we have $tc = ms$, for some $m \in N$, $s \in T$. Since $N_im \cap (N \backslash N_i)$ is finite, $uN_ims \cap u(N \backslash N_i)s = uN_itc \cap u(N \backslash N_i)s$ is finite and hence $(untc)f_e = (uN_is)f_e$, for almost all $n \in N_i$. Thus

$$\begin{aligned}(uN_it)f_{ce} &= (uN_it)f_c(uN_is)f_e = ((ut)\rho, c\rho)\phi((us)\rho, e\rho)\phi \\ &= ((ut)\rho, c\rho)\phi((utc)\rho, e\rho)\phi.\end{aligned}$$

So $\phi = \phi_i$ is a homomorphism from $Y \times D$ to A.

We take $i = 1$ and note from Lemma 3 that there is a map $\gamma : Y \to A$ such that $(y, d)\phi_1 = (y\gamma)^{-1}(yd)\gamma$ for $y \in Y$, $d \in D$. Let $g : X \to A$ be given by $xg = x\rho\gamma$, for $x \in X$, and put $(x, c)\psi = (xg)(x, c)\theta((xc)g)^{-1}$, for $x \in X$, $c \in C$. Then ψ is equivalent to θ and, since g is constant on each xN, ψ satisfies the conditions given for θ. Let $(x, c)\psi = xh_c$ and, for each i, let λ_i denote the homomorphism from $Y \times D$ to A associated with ψ and N_i. For $u \in U$, $t \in T$, $c \in C$, we have

$$\begin{aligned}((ut)\rho, c\rho)\lambda_1 = (uN_1t)h_c &= (uN_1t)g(uN_1t, c)\theta(uN_1tc)g \\ &= (ut)\rho\gamma((ut)\rho, c\rho)\phi_1((utc)\rho\gamma)^{-1} = 1.\end{aligned}$$

Thus λ_1 is trivial and h_c is trivial on UN_1T. Since h_c is constant on almost all xN, it is trivial on almost all xN and so each λ_i is almost trivial. Thus each almost homomorphism θ is equivalent to an almost homomorphism ψ given by $(unt, c)\psi = ((ut)\rho, c\rho)\lambda_i$, for $u \in U$, $t \in T$, $c \in C$ and $n \in N_i$, where $\lambda_1 = 1, \lambda_2, \ldots, \lambda_r$ are almost trivial homomorphisms from $Y \times D$ to A.

Conversely, suppose $\lambda_1 = 1, \lambda_2, \ldots, \lambda_r$ are almost trivial homomorphisms from $Y \times D$ to A and $\psi : X \times C \to A$ is given by $(unt, c)\psi =$

$((ut)\rho, c\rho)\lambda_i$, for $u \in U$, $t \in T$, $c \in C$ and $n \in N_i$. If $e \in C$ and $n \in N_i$,

$$(unt, ce)\psi = ((ut)\rho, (ce)\rho)\lambda_i = ((ut)\rho, c\rho)\lambda_i((utc)\rho, e\rho)\lambda_i.$$

Now $(unt, c)\psi = ((ut)\rho, c\rho)\lambda_i$ and, if $tc = ms$ with $m \in N$, $s \in T$, then $untc = unms \in uN_is$, for almost all $n \in N_i$, and so $(untc, e)\psi = ((utc)\rho, e\rho)\lambda_i$, for almost all $n \in N_i$. So, for fixed u, t, c, e, we have $(unt, ce)\psi = (unt, c)\psi(untc, e)\psi$, for almost all $n \in N$. Since the λ_i are almost trivial, $(x, c)\psi$ is trivial on almost all xN and hence $(x, ce)\psi = (x, c)\psi(xc, e)\psi$, for almost all $x \in X$. Thus ψ is an almost homomorphism.

Next suppose $\phi : X \times C \to A$ corresponds to the almost trivial homomorphisms $\mu_1 = 1, \mu_2, \ldots, \mu_r$ from $Y \times D$ to A. If ϕ is equivalent to ψ there is a map $\gamma : X \to A$ such that, for $c \in C$, $(x, c)\phi = (x\gamma)^{-1}(x, c)\psi(xc)\gamma$, for almost all $x \in X$. Since ϕ and ψ are trivial on all N-sheets, $\gamma =^{a\,n} \gamma$, for $n \in N$, and so γ is almost N-invariant. From Lemma 9, $\gamma =^a \delta$, for some δ constant on almost all xN and on all uSt, where S runs through the sets in a finite partition of N into infinite almost invariant subsets. For each i, there is some such S with $M_i = N_i \cap S$ infinite. Then M_i is almost invariant and both δ and all f_c are constant on all uM_it. Since $\delta =^a \gamma$, $(x, c)\phi = (x\delta)^{-1}(x, c)\psi(xc)\delta$, for almost all $x \in X$. Now $(uN_1t, c)\phi = 1 = (uN_1t, c)\psi$ so, for fixed t, $s \in T$ with $tc = ms$, we have $(unt)\delta = (untc)\delta = (unms)\delta$, for almost all $n \in M_1$. Since $M_1 \cap M_1m$ is infinite, $(uM_1t)\delta = (uM_1s)\delta$, and so δ is constant on each uM_1T. For $u \in U$, $t \in T$, $c \in C$, with $tc \in Ns$, $s \in T$, we have

$$\begin{aligned}((ut)\rho, c\rho)\mu_i &= (uN_it, c)\phi = (uM_it, c)\phi\\ &= ((uM_it)\delta)^{-1}(uM_it, c)\psi(uM_is)\delta\\ &= ((uM_it)\delta)^{-1}((ut)\rho, c\rho)\lambda_i(uM_is)\delta.\end{aligned}$$

Putting $(ut)\rho\epsilon_i = (uM_it)\delta$, we have

$$((ut)\rho, c\rho)\mu_i = ((ut)\rho\epsilon_i)^{-1}((ut)\rho, c\rho)\lambda_i(utc)\rho\epsilon_i.$$

Now δ is constant on each uM_1T and on almost all xN. Thus $(uM_it)\delta = (uM_1t)\delta$, for almost all $ut \in UT$ and ϵ_i is almost equal to the function $\epsilon = \epsilon_1$, which is constant on each $u\rho D$. If $\beta_i = \epsilon_i\epsilon^{-1}$, we have a function ϵ, constant on each yD, and functions $\beta_i =^a 1$ such that $(y, d)\mu_i = (y\beta_i\epsilon)^{-1}(y, d)\lambda_i(yd)\beta_i\epsilon$. In this situation, we say that $(\mu_1, \ldots, \mu_r)$ is equivalent to $(\lambda_1, \ldots, \lambda_r)$. This implies, but is not implied by, the equivalence of μ_i and λ_i for each i.

Finally, we show that if $(\mu_1, \ldots, \mu_r)$ is equivalent to $(\lambda_1, \ldots, \lambda_r)$, then ϕ is equivalent to ψ. We have functions β_i, ϵ from Y to A, with $\beta_i =^a 1$

and ϵ constant on each yD, such that

$$((ut)\rho, c\rho)\mu_i = ((ut)\rho\beta_i\epsilon)^{-1}((ut)\rho, c\rho)\lambda_i(utc)\rho\beta_i\epsilon.$$

Define $\delta, \nu : X \to A$ by $(uN_it)\delta = (ut)\rho\beta_i$, $(uC)\nu = u\rho\epsilon$. Let $c \in C$, $u \in U$, $t \in T$, be fixed. If $tc = ms$, with $m \in M$, $s \in T$, then

$$(uN_it, c)\phi = ((ut)\rho, c\rho)\mu_i = ((uN_it)\delta\nu)^{-1}(uN_it, c)\psi(uN_is)\delta\nu.$$

Now ν is constant on all xC, so $(uns)\nu = (untc)\nu$, for $n \in N$. Also, $(uns)\delta = (untc)\delta$ unless, for some i, we have $(utc)\rho \in \sigma(\beta_i)$ and $n \in N_i$, $nm \notin N_i$, that is, $n \in N_i \cap (N\backslash N_i)m^{-1}$. For each i, β_i has finite support and N_i is almost invariant. Thus, for fixed c, only a finite number of exceptions occur and so, for almost all $x \in X$, $(x, c)\phi = (x\delta\nu)^{-1}(x, c)\psi(xc)\delta\nu$. Hence ϕ is equivalent to ψ.

We have shown that Z is bijective with the set of equivalence classes of r-tuples $(1, \lambda_2, \ldots, \lambda_r)$ and so Z is trivial if $r = 1$. For $r > 1$, $(1, \lambda_2, 1, \ldots, 1)$ is equivalent to $(1, \mu_2, 1, \ldots, 1)$ if and only if λ_2 and μ_2 are equivalent. So Z has a subset bijective with the set of equivalence classes of almost trivial homomorphisms from $Y \times D$ to A, which is the whole of Z when $r = 2$.

THEOREM (11): *Let C act semiregularly on X. If C has a finitely presented normal subgroup N of infinite index such that $e(N) = 1$, then all almost homomorphisms from $X \times C$ to A are equivalent.*

PROOF: Since $e(N) = 1$, the only partition of N into infinite almost invariant subsets is the trivial partition $\{N\}$. From Lemmas 8 and 9, any almost homomorphism is equivalent to an almost homomorphism θ satisfying the conditions of Lemma 10, with $r = 1$. Then θ is equivalent to the trivial homomorphism.

THEOREM (12): *Let C act semiregularly on X and suppose C has a normal subgroup N such that $e(N) = 2$. If $e(C/N) = 1$, all almost homomorphisms from $X \times C$ to A are equivalent. If $e(C/N) > 1$, the set Z of equivalence classes of almost homomorphisms contains a subset bijective with $A_0^{(X/C)}$, where A_0 is the set of conjugacy classes of A; if $e(C/N) = 2$, this subset is the whole of Z.*

PROOF: Let S be an infinite almost invariant subset of N with infinite complement R. From Lemma 8, we need only consider almost homomorphisms ψ trivial on all N-sheets and with f_c almost N-invariant, where $xf_c = (x, c)\psi$. Since $e(N) = 2$, Lemma 9 implies that each f_c is almost equal to a function constant on all uSt and uRt. Thus

ψ is equivalent to an almost homomorphism satisfying the conditions of Lemma 10 with $r = 2$. The result now follows from Lemma 6.

THEOREM (13): *Let C act semiregularly on X. If C has a finitely generated free subgroup of finite index, all almost homomorphisms from $X \times C$ to A are equivalent. If C has a finitely presented normal subgroup N of infinite index with $e(N) = \infty$, then all almost homomorphisms from $X \times C$ to A are equivalent if C is finitely generated and $e(C/N) = 1$. If $e(C/N) > 1$, there are inequivalent almost homomorphisms.*

PROOF: If C is finitely generated free by finite, it has a finitely generated free normal subgroup N of finite index. Using Lemmas 7 and 8, in both cases we need only consider almost homomorphisms ψ such that ψ is trivial on all N-sheets and f_c is almost N-invariant, where $xf_c = (x, c)\psi$. If C is finitely generated, then $C = \langle c_1, \ldots, c_v, N\rangle$ for some $c_1, \ldots, c_v \in C \backslash N$. From Lemma 9, for each c_j there is a finite partition of N into almost invariant subsets S such that f_{c_j} is almost equal to a function constant on almost all xN and on all uSt. Taking the intersections of all such S that arise, over all j, we obtain a finite partition of N into almost invariant subsets. If we incorporate the finite parts in one of the infinite parts, we have a partition $\{N_1, \ldots, N_r\}$ of N into infinite almost invariant subsets so that each f_{c_j} is almost equal to a function g_{c_j} constant on all uN_it and on almost all xN. Now $f_n = 1$, for $n \in N$, and $C = \langle c_1, \ldots, c_v, N\rangle$. Suppose for some $c, d \in C$, we have f_c, f_d almost equal to functions g_c, g_d constant on all uN_it and on almost all xN. Then ${}^cg_d$ is constant on almost all xN. Let $xN = uNt$ be an exception and suppose $tc = ms$, with $m \in N$, $s \in T$. For $n \in N_i$,

$$(unt)\,{}^cg_d = (untc)g_d = (unms)g_d = (uN_is)g_d,$$

unless $n \in N_i \cap (N\backslash N_i)m^{-1}$. Thus ${}^cg_d$ is almost equal to a function constant on all uN_it and on almost all xN. Now $f_{c^{-1}} = {}^a\,({}^{c^{-1}}f_c)^{-1}$ and $f_{cd} = {}^a f_c\,{}^cf_d$, so it follows by induction that, for all $c \in C$, f_c is almost equal to a function g_c, constant on all uN_it and almost all xN. Putting $(x, c)\phi = xg_c$, we obtain an almost homomorphism equivalent to ψ and satisfying the conditions of Lemma 10. The first two statements of the theorem now follow from Lemma 6. If $e(C/N) > 1$, we take an arbitrary partition of N into two infinite almost invariant subsets. Applying Lemmas 10 and 6, we know that there are almost homomorphisms not equivalent to the trivial homomorphism.

Theorem B follows from Theorems 1, 11, 12 and 13, together with the next result.

COROLLARY (14): *Suppose C is a polycyclic by finite group acting semiregularly on X. All almost homomorphisms from $X \times C$ to A are equivalent unless C has Hirsch number* 2.

PROOF: Let h be the Hirsch number of C. The result is trivial for $h = 0$, since C is then finite. Otherwise, C has a non-trivial poly-(infinite cyclic) normal subgroup N of finite index. If $h = 1$, then N is infinite cyclic and the result follows from Theorem 13. If $h > 2$, then C has a normal series $C = C_0 \geq N = C_1 > C_2 > 1$, with all C_i finitely presented, C/N finite, C_1/C_2 infinite, and C_2 poly-(infinite cyclic) with Hirsch number > 1. From the remarks preceding Lemma 6, $e(C_2) = 1$ and so Theorem 11 implies that an almost homomorphism from $X \times C$ to A is equivalent to one θ which is trivial on all N-sheets. Now N also has 1 end and so Lemmas 8 and 9 show that θ is equivalent to an almost homomorphism satisfying the conditions of Lemma 10, with $r = 1$. So θ is equivalent to the trivial homomorphism.

Finally, suppose $h = 2$. Then C has a normal subgroup N of finite index which is infinite cyclic by infinite cyclic. From Theorem 12, given $x \in X$, there is an almost homomorphism ϕ from $xN \times N$ to A which is not equivalent to the trivial homomorphism. If we can extend ϕ to $X \times C$, the extension will not be equivalent to the trivial homomorphism. Let T be a transversal of the cosets of N in C, with $\tau : C \to T$ the transversal map and $1 \in T$. For $t \in T$, $n \in N$, $c \in C$, put $(xnt, c)\theta = (xn, tc((tc)\tau)^{-1})\phi$. Then θ extends ϕ to $xC \times C$ and if $d \in C$,

$$\begin{aligned}&(xnt, c)\theta(xntc, d)\theta \\ &\qquad = (xn, tc((tc)\tau)^{-1})\phi(xntc((tc)\tau)^{-1}, (tc)\tau d((tcd)\tau)^{-1})\phi.\end{aligned}$$

For fixed t, c, d, this equals $(xn, tcd((tcd)\tau)^{-1})\phi = (xnt, cd)\theta$, for almost all $n \in N$. Now T is finite, so $(xe, c)\theta(xec, d)\theta = (xe, cd)\theta$, for almost all $e \in C$. Thus θ is an almost homomorphism from $xC \times C$ to A extending ϕ. Defining θ to be trivial on all other C-sheets, we have an almost homomorphism from $X \times C$ to A which is not equivalent to the trivial one.

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(Oblatum 3-X-1975)

Department of Pure Mathematics
University College
P.O. Box 78
Cardiff CF1 1XL
Wales, U.K.