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CONSTRUCTIVE DIMENSION THEORY

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Abstract

The uniform dimension theory of totally bounded metric spaces is developed from a constructive point of view, starting from a modification of the Lebesgue definition of covering dimension. The subspace theorem and the (finite) sum theorem are proved. It is shown that $\dim X \leq n$ if and only if for every map from X into the $(n+1)$-cell I^{n+1}, every point of I^{n+1} is unstable. An example shows that $\dim X \geq n$ is not constructively equivalent to the existence of a map from X into I^n with a stable value. However it is shown that these latter two properties are equivalent if $\dim X \leq n$. The product theorem is proved by a direct construction of a cover. Also proved are the (almost) fixed point theorem for I^n, the Pflastersatz, and the equivalence of $\operatorname{int} X \neq \phi$ and $\dim X = n$ for closed subsets X of I^n.

1. Introduction

Since the publication of Bishop's book [3] on constructive analysis, there has been a resurgence of interest in the constructive approach to mathematics. Dimension theory provides an historical testing ground for this approach. In 1911 Lebesgue [8] introduced the notion of *covering dimension* and gave a somewhat faulty proof that Euclidean n-space has covering dimension n. Two years later Brouwer [4] gave a correct proof of Lebesgue's claim, introduced the idea of *inductive dimension*, and proved that Euclidean n-space also has inductive dimension n. In 1926 Brouwer [5] formulated the notion of inductive dimension in intuitionistic terms and proved, in a constructive way, that Euclidean n-space has inductive dimension n. The purpose of this paper is to develop the theory of covering dimension from a constructive point of view. Berg *et al.* [2] investigate the relationships between Brouwer's constructive theory of inductive dimension and our constructive theory of covering dimension.

Compact metric spaces form the natural category for a constructive theory (e.g., Brouwer's *katalogisiertkompakte Spezies*). In attempting to constructivize Lebesgue's definition of covering dimension we were forced to adopt a modification of this definition that sidesteps the problems of constructing a point that is in $n+1$ sets. This definition is classically equivalent to the original for compact spaces. However, the completeness of the space is irrelevant in most applications. Hence we state our definitions and theorems for totally bounded spaces, although they agree with the classical ones only for compact spaces. Our basic definition turns out to be classically equivalent to Isbell's uniform dimension δd as defined in [7; p. 64].

We assume that the reader is acquainted with the constructive approach of Bishop [3]. All spaces will be totally bounded metric spaces. A set A is *located* if $d(x, A) = \inf\{d(x, a)\colon a \in A\}$ exists (in the extended reals) for each x. Note that the empty set is located and $d(x, \phi) = \infty$ for each x. A subset A is *bilocated* if both A and its *metric complement* $\sim A = \{x : d(x, A) > 0\}$ are located. A slight modification of the proof of [3; Thm 8, p. 101] shows

THEOREM 0: *Let f be a continuous real valued function on a totally bounded metric space X, then for all except countably many real numbers α, the sets*

$$\{x : f(x) < \alpha\} \quad \textit{and} \quad \{x : f(x) > \alpha\}$$

are bilocated and are metric complements of each other.

We denote by $S_\epsilon(A)$, the ϵ-neighborhood $\{x : d(x, A) < \epsilon\}$ of A. By Theorem 0 given $0 < \alpha < \beta$, there exists $\epsilon \in (\alpha, \beta)$ so that $S_\epsilon(x)$ is bilocated and $\sim\sim S_\epsilon(x) = S_\epsilon(x)$.

2. Covering dimension

The central notion in the development of covering dimension is that of the order of a cover. Classically, the order of a cover is at least n if there is a point common to n sets of the cover. This definition poses problems from the constructive point of view. For one thing, it is generally difficult to tell if a given point is in a given set. Also, while total boundedness assures us of an ample supply of points that are *near* things, we are often hard put to construct points that are *in* things, even in a complete space. It is thus more natural to demand points that are arbitrarily close to n sets. In a compact space these notions are classically equivalent. Since we will be measuring distances to the sets in the cover we require these sets to be located. It will be convenient to have the notion of order relativized to a subspace.

DEFINITION: Let X be a totally bounded metric space, n a non-negative integer, L a subset of X, and $\mathscr{F}$ a finite family of located subsets of X.

(1) $o(\mathscr{F}) \geq n$ *on* L if, for every $\delta > 0$, there is an x in L such that $d(x, F) < \delta$ for n distinct elements F in $\mathscr{F}$. If $L = X$ we write simply $o(\mathscr{F}) \geq n$.

(2) $o(\mathscr{F}) \leq n$ *on* L *with separation* δ if, for each x in L, we have $d(x, F) \geq \delta$ for all but at most n elements F in $\mathscr{F}$. If $L = X$ we write simply $o(\mathscr{F}) \leq n$ *with separation* δ.

(3) $o(\mathscr{F}) \leq n$ (*on* L) if $o(\mathscr{F}) \leq n$ (on L) with separation δ for some $\delta > 0$.

The term *finite family* denotes what Bishop calls a *subfinite set*, that is, a family indexed by the first k positive integers for some k. It is convenient to allow a finite family to be empty. Two elements F_i and F_j of a finite family are said to be *distinct* if $i \neq j$ even though possibly $F_i = F_j$. This allows you to construct covers, and compute their orders, without having to worry whether two elements in the cover are equal, an undecidable question in general. For simplicity we shall suppress reference to the index set and use set terminology when dealing with a finite family. For example we use the notations $F \in \mathscr{F}$ and $\mathscr{F} = \{F : P(F)\}$ if the latter collection is naturally indexed.

Classically either $o(\mathscr{F}) \leq n$ or $o(\mathscr{F}) \geq n + 1$. Constructively this is not so. For example let $\{f_m\}$ be a fugitive sequence and $a = \Sigma\,(f_m/m)$, let $\mathscr{F} = \{[-1, 0], [a, 1]\}$ and $n = 0$. However, the following weaker statement holds.

THEOREM 1: *Let X be a totally bounded metric space and $\mathscr{F}$ a finite family of located subsets of X. If $o(\mathscr{F}) \leq n$ is impossible, then $o(\mathscr{F}) \geq n + 1$.*

PROOF: Let $\delta > 0$. Choose a δ-approximation Y to X. If, for each y in Y we have $d(y, F) \geq 2\delta$ for all but at most n elements F of $\mathscr{F}$, then

$o(\mathscr{F}) \leq n$, which is impossible. Hence there is a y in Y such that $d(y, F) < 3\delta$ for $n+1$ distinct element F in $\mathscr{F}$. Thus $o(\mathscr{F}) \geq n+1$.

Since the order of a cover $\mathscr{F}$ of X is defined in terms of points being *near* subsets, it is natural to require only that $\cup\mathscr{F}$ be dense in X, rather than demanding $\cup\mathscr{F} = X$. This is convenient since quite often $\cup\mathscr{F} = X$ in the classical sense but not in the constructive sense, while $\cup\mathscr{F}$ is dense constructively. For example $X = [0, 2]$ and $\mathscr{F} = \{[0, 1], [1, 2]\}$.

Classically, one need only define what $\dim X \leq n$ means. Constructively we need also to define $\dim X \geq n$. These correspond to Brouwer's *oberer Dimensionsgrad* and *unterer Dimensionsgrad* [5].

DEFINITION: Let X be a totally bounded metric space.

(1) A *cover* $\mathscr{F}$ of X is a finite family of located subsets of X such that $\cup\mathscr{F}$ is dense in X.

(2) A cover $\mathscr{F}$ is an *ϵ-cover* if $\operatorname{diam} F < \epsilon$ for every F in $\mathscr{F}$.

(3) $\dim X \leq n$ if, for every $\epsilon > 0$, there is an ϵ-cover $\mathscr{F}$ of X such that $o(\mathscr{F}) \leq n+1$.

(4) $\dim X \geq n$ if there is an $\epsilon > 0$ such that if $\mathscr{F}$ is an ϵ-cover of X, then $o(\mathscr{F}) \geq n+1$.

(5) $\dim X = n$ if $\dim X \leq n$ and $\dim X \geq n$.

Our setting is the category of totally bounded metric spaces and uniformly continuous functions. Hence a *homeomorphism* comes equipped with moduli of uniform continuity in both directions. With this in mind it is easily seen that these notions of dimension are invariant under homeomorphism. A space X need not have a dimension even when $\dim X \leq 1$. To see this, let $\{f_n\}$ be a fugitive sequence and let $a = \Sigma f_n/n$. Consider the closed interval $X = [0, a]$. It is easily verified that $\dim X \leq 1$, that $\dim X = 0$ if $a = 0$, and that $\dim X = 1$ if $a \neq 0$. However we cannot tell whether a is 0 or not, so we cannot assign a dimension to X.

Even when the classical dimension of the space is known to be n we may be unable to verify that $\dim X = n$. The problem is that $\dim X \geq n$ is not merely the negation of $\dim X < n$, but entails the construction of a number ϵ. An example illustrating this distinction is constructed as follows. Let $\{f_n\}$ be a fugitive sequence and let $K(n, N) = \{j/2^n : 1 \leq j \leq 2^{n-N}\}$. Let $H_n = K(n, 0)$ if $f_N = 0$ for all $N < n$, and let $H_n = K(n, N)$ if $f_N = 1$ for $N < n$. Let X be the closure of the union of the H_n. Then X is a compact metric space which, classically, is either $[0, 1]$ or $[0, 1/2^N] \cup \{j/2^N : 1 \leq j \leq 2^N\}$ for some N. Hence X

has classical dimension 1, but to show $\dim X \geq 1$ you would need to find an ϵ that would, essentially, measure the size of the non-degenerate interval contained in X. This ϵ would provide unobtainable information about $\{f_n\}$.

The subspace theorem takes two forms.

THEOREM 2: *Let X be a totally bounded metric space, L a located subset of X, and n a positive integer. Then*

(1) *If* $\dim X \leq n-1$, *then* $\dim L \leq n-1$.

(2) *If* $\dim L \geq n$, *then there is an* $\epsilon > 0$ *such that if* $\mathcal{F}$ *is an* ϵ*-cover of X, then* $o(\mathcal{F}) \geq n+1$ *on L. Hence* $\dim X \geq n$.

PROOF: Suppose we have positive numbers ϵ and δ, and an ϵ-cover $\mathcal{F}$ of X. By Theorem 0 we can pick $0 < \eta < \delta/2$ such that $\{x \in L: d(x, F) < \eta\} = S_\eta(F) \cap L$ is located and has diameter less than ϵ for each F in $\mathcal{F}$. Then $\mathcal{F}_L = \{S_\eta(F) \cap L: F \in \mathcal{F}\}$ is an ϵ-cover of L. If $o(\mathcal{F}) \leq n$ with separation δ, then $o(\mathcal{F}_L) \leq n$ with separation $\delta/2$, so (1) is verified. If $\dim L \geq n$, choose ϵ so that $o(\mathcal{F}_L) \geq n+1$ for any ϵ-cover $\mathcal{F}_L$ of L. Then there is an x in L that comes within $\delta/2$ of $n+1$ distinct elements of ϕ_L, and hence within δ of $n+1$ distinct elements of $\mathcal{F}$. Since this is true for any δ, part (2) follows.

If L is a *dense* subset of X, then any ϵ-cover $\mathcal{F}$ of L is an ϵ-cover of X, so $o(\mathcal{F}) \leq n+1$ on L if and only if $o(\mathcal{F}) \leq n+1$ on X, and $o(\mathcal{F}) \geq n+1$ on L if and only if $o(\mathcal{F}) \geq n+1$ on X. Hence, if $\dim L \leq n$, then $\dim X \leq n$, and if $\dim X \geq n$, then $\dim L \geq n$. Thus dense subsets have the same dimension properties as their containing spaces. This is because we are dealing with a dimension theory that is invariant only under uniform equivalence, (see [7; cor. 23, p. 66]).

It is a classical theorem that if a space is the union of countably many closed subspaces, then its dimension is the supremum of the dimensions of those subspaces. Thus, for example, the rational points in the square form a space of classical dimension zero. One might hope to get a constructive version of this theorem by restricting attention to compact spaces for which the topological and the uniform theories are classically equivalent. However, such a theorem might prove to have very little interest since it is probably impossible to construct a compact space that is a union of countably many closed subspaces in an interesting way. To appreciate the difficulties involved, consider the space consisting of the points 0, 1, 1/2, 1/3, ... This space is certainly the union of countably many closed subspaces, but it is not constructively compact because it is not constructively complete.

As in the uniform theory [7; cor. 8, p. 80], we can get the sum theorem for finite sums. Constructively we must distinguish two classically equivalent statements in this theorem.

THEOREM 3: *Let* $\{A, B\}$ *be a cover of the totally bounded metric space* X. *Let* $\dim A \le n$. *Then*
(1) *if* $\dim B \le n$, *then* $\dim X \le n$, *and*
(2) *if* $\dim X \ge n+1$, *then* $\dim B \ge n+1$.

PROOF: Since $\dim A \le n$ we can find, for any $\epsilon > 0$, an $(\epsilon/2)$-cover $\mathscr{F}$ of A such that $o(\mathscr{F}) \le n+1$ on X with separation $\delta < \epsilon/4$. Suppose $\mathscr{G}$ is a $(\delta/16)$-cover of B such that $o(\mathscr{G}) \le k$ on X with separation $n < \delta/16$. We shall construct an ϵ-cover $\mathscr{E}$ of X such that $o(\mathscr{E}) \le \max(n+1, k)$ with separation η. Partition $\mathscr{G}$ into disjoint subfamilies $\hat{\mathscr{G}}$ and $\mathscr{H}_F$, where F ranges over $\mathscr{F}$, such that if $G \in \mathscr{H}_F$, then $d(G, F) < \delta/2$ while if $G \in \hat{\mathscr{G}}$, then $d(G, F) > \delta/4$ for all F in $\mathscr{F}$. For each F in $\mathscr{F}$ let $\hat{F} = F \cup \bigcup \mathscr{H}_F$, and let $\hat{\mathscr{F}} = \{\hat{F} : F \in \mathscr{F}\}$. If $d(x, F) \ge \delta$ and $G \in \mathscr{H}_F$, then $d(x, G) \ge d(x, F) - d(G, F) - \operatorname{diam}(G)$, so $d(x, \hat{F}) \ge \delta - \delta/2 - \delta/16 \ge \eta$. Hence $o(\hat{\mathscr{F}}) \le n+1$ with separation n.

Clearly $\mathscr{E} = \hat{\mathscr{F}} \cup \hat{\mathscr{G}}$ is an ϵ-cover of X. We shall show that $o(\mathscr{E}) \le \max(n+1, k)$ with separation η. For x in X, either $d(x, G) > \eta$ for all G in $\hat{\mathscr{G}}$, or $d(x, G) < 2\eta$ for some G in $\hat{\mathscr{G}}$. In the first case the result follows from the fact that $o(\hat{\mathscr{F}}) \le n+1$ with separation η. In the second case $d(x, F) > \delta/4 - \delta/16 - 2\eta > \eta$ for each F in $\mathscr{F}$. Thus if $d(x, E) < \eta$ for E in $\mathscr{E}$, then either $E \in \hat{\mathscr{G}}$, or $E = \hat{F}$ and $d(E, G) < \eta$ for some G in $\mathscr{H}_F$. For each G in $\mathscr{G}$ let $t(G) = G$ if $G \in \hat{\mathscr{G}}$, and $t(G) = \hat{F}$ if $G \in \mathscr{H}_F$. Since $o(\mathscr{G}) \le k$ with separation η, we may choose a subfamily $\mathscr{J}$ of $\mathscr{G}$ of cardinality k such that $d(x, G) \ge \eta$ for G in $\mathscr{G} \backslash \mathscr{J}$. It follows that if $E \in \mathscr{E} \backslash t(\mathscr{J})$, then $d(x, E) > \eta$. Hence $o(\mathscr{E}) \le k$ with separation η.

Statement (1) now follows by choosing $\epsilon > 0$ arbitrarily and using $\dim B \le n$ to get $k = n+1$. To prove statement (2) choose $\epsilon > 0$ so that if $\mathscr{C}$ is an ϵ-cover of X, then $o(\mathscr{C}) \ge n+2$. Then the assumption that $o(\mathscr{G}) \le n+1$ is absurd, so $o(\mathscr{G}) \ge n+2$ by Theorem 1. But $\mathscr{G}$ can be any $(\delta/16)$-cover of B. Hence $\dim B \ge n+1$.

Note that we did not prove that if $\dim X \ge n+1$, then $\dim A \ge n+1$ or $\dim B \ge n+1$. Indeed, we cannot hope to show this constructively. To see this, let $\{a_i\}$ be an enumeration of the rational numbers in $[0, 1]$ and $\{b_i\}$ an enumeration of the rational numbers in $[1, 2]$. Let $\{f_n\}$ be a fugitive sequence and define $\{x_k\}$ as follows. Let $x_{2j} = b_j$ if $f_{2n} = 0$ for all $n \le j$, otherwise let $x_{2j} = 0$. Let $x_{2i-1} = a_i$ if

$f_{2n-1} = 0$ for all $n \leq i$, otherwise let $x_{2i-1} = 0$. Let X be the closure of the set $\{x_k\}$. Then X is a compact metric space. Let $A = X \cap [0, 1]$ and $B = X \cap [1, 2]$. It is easily verified that $\dim X \geq 1$, but we have no way of knowing whether $\dim A \geq 1$ or whether $\dim B \geq 1$.

3. Stable values

Let X and Y be totally bounded metric spaces and f a uniformly continuous function from X to Y. A point a in Y is said to be a *stable value* of f if there is an $\epsilon > 0$ such that if g is a uniformly continuous function from X to Y, and $\|f - g\| < \epsilon$, then $d(a, g(X)) = 0$. A point a in Y is an *unstable value* of f if for every $\epsilon > 0$, there is a uniformly continuous function g from X to Y such that $\|f - g\| < \epsilon$ and $d(a, g(X)) > 0$. In the sequel $I = [0, 1]$.

THEOREM 4: *If X is a totally bounded metric space and $f: X \to I^n$ is a uniformly continuous function, then the set of stable values of f is open.*

PROOF: Suppose a is a stable value of f, so that if $\|f - g\| < \epsilon$, then $d(a, g(X)) = 0$. Note that $S_\epsilon(a) \subseteq I^n$. We shall show that if $|b - a| < \epsilon/2$, then b is a stable value of f. Suppose $\|f - g\| < \epsilon/2$. We want to show that $d(b, g(X)) = 0$. Let ϕ be a homeomorphism of I^n such that $\phi(b) = a$ and $\|id - \phi\| < \epsilon/2$, where id is the identity map. Then $\|f - \phi \circ g\| < \epsilon$ so $d(a, \phi(g(X))) = 0$. Thus for any $\delta > 0$ there is an x in X such that $|a - \phi(g(X))| < \omega(\delta)$, where ω is a modulus of continuity for ϕ^{-1}. Hence $|b - g(x)| < \delta$, and we are done.

We need here a few elementary facts about general position. These facts will also play a prominent role in the proof of the imbedding theorem in section 6. The constructive point of view requires a slight modification of the usual definitions and proofs. If $v_1, \ldots, v_k$ are elements of $\mathbb{R}^n$ we say they are *linearly independent* if for each $\epsilon > 0$ we can find a $\delta > 0$ such that $\Sigma |r_i| < \epsilon$ whenever $\Sigma |r_i v_i| < \delta$. It is not difficult to show that $v_1, \ldots, v_k$ are linearly independent if and only if there is an invertible $n \times n$ matrix T such that $Tv_i = e_i$ for $1 \leq i \leq k$, where e_i is the standard ith basis vector of $\mathbb{R}^n$. Elements $v_0, v_1, \ldots, v_k$ of $\mathbb{R}^n$ are *affine independent* if $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent. A finite subset V of $\mathbb{R}^n$ is in *general position* if every finite subset of V of cardinality not exceeding $n + 1$ is affine independent.

If V is in general position in $\mathbb{R}^n$, then there is a $\delta > 0$ such that for any two disjoint finite subsets S and T of V, if card S + card $T \leq n+1$, then the distance between the convex hulls of S and T is at least δ. Finally, if $x_1, \ldots, x_m$ are any points in $\mathbb{R}^n$, and $\epsilon > 0$, then there exist points $y_1, \ldots, y_m$ in general position in $\mathbb{R}^n$ such that $x_1 = y_1$ and $|x_j - y_j| < \epsilon$ for $2 \leq j \leq m$. This can be proved by induction on m using the characterization of linear independence stated in the preceding paragraph.

THEOREM 5: *Let X be a totally bounded metric space and f a uniformly continuous function X to I^{n+1}. Then*

(1) *if* $\dim X \leq n$, *then each point in I^{n+1} is an unstable value of f, and*

(2) *if f has a stable value, then* $\dim X \geq n+1$.

PROOF: Let ω be a modulus of continuity for f. Fix a point a in I^{n+1}, a number $\epsilon > 0$, and an $\omega(\epsilon/3)$-cover $\mathcal{F}$ of X. Choose δ such that $0 < \delta < \omega(\epsilon/3)$. For each F in $\mathcal{F}$, choose a point b_F in I^{n+1} so that $d(b_F, f(F)) < \epsilon/3$ and the b_F, together with a, are in general position. Let $\lambda_F(x) = \sup(0, 1 - d(x, F)/\delta)$. Define g by

$$g(x) = \frac{\Sigma_{\mathcal{F}} \lambda_F(x) b_F}{\Sigma_{\mathcal{F}} \lambda_F(x)}.$$

Since $\mathcal{F}$ is a cover of X, the denominator $\Sigma\, \lambda_F(x)$ is at least 1. Note that if $d(x, F) \geq \delta$, then $\lambda_F(x) = 0$, whereas if $d(x, F) < \omega(\epsilon/3)$, then $|b_F - f(x)| < \epsilon$. Hence $\|f - g\| < \epsilon$.

To prove statement (1), suppose $o(\mathcal{F}) \leq n+1$. Then we can choose δ such that for every x in X, there is a subset $\mathcal{J}$ of $\mathcal{F}$ of cardinality $n+1$ so that $d(x, F) \geq \delta$ if F is in $\mathcal{F} \backslash \mathcal{J}$. Since a and the b_F are in general position, and since no more than $n+1$ of the numbers $\lambda_F(x)$ are different from 0, we have $d(a, g(X)) > 0$. To prove statement (2), suppose $d(a, g(X)) = 0$ for all δ (note that g depends on δ). Then $o(\mathcal{F}) \leq n+1$ is absurd, and hence $o(\mathcal{F}) \geq n+2$ by Theorem 1.

The converse of the first part of Theorem 5 is proved with the aid of an intermediate notion that is classically equivalent to [7; Th. 24, p. 87], and, for compact spaces, to [6; Prop. C, p. 35]. We say that B is an *ϵ-enlargement* of A if $A \subseteq B$ and $d(b, A) < \epsilon$ for each b in B. If K and L are any subsets of a metric space, then $d(K, L) > 0$ means that there is an $\eta > 0$ such that if $x \in K$ and $y \in L$ then $d(x, y) \geq \eta$.

DEFINITION: Let X be a totally bounded metric space.

(1) *X satisfies condition D_n* if for any located subsets $A_0, \ldots, A_n$ of

X, and $\epsilon > 0$, there exist bilocated ϵ-enlargements $B_0, \ldots, B_n$, and $\delta > 0$, such that $d(\cap S_\delta(B_i), \cap S_\delta(\sim B_i)) > 0$.

(2) *X satisfies condition D'_n* if there exist located subsets $A_0, \ldots, A_n$ of X, and $\epsilon > 0$, such that for any bilocated ϵ-enlargements $B_0, \ldots, B_n$ we have $d(\cap S_\delta(B_i), \cap S_\delta(\sim B_i)) = 0$ for all $\delta > 0$.

We shall show that if each point in I^{n+1} is an unstable value of each map $f: X \to I^{n+1}$, then X satisfies D_n (Theorem 6), and if X satisfies D_n then $\dim X \leq n$ (Theorem 7). Together with Theorem 5 this will show that these three conditions are equivalent. Condition D_n provides a characterization of $\dim X \leq n$ that is new even from the classical point of view. The situation is not quite so simple for $\dim X \geq n+1$. We have shown that if there is a map $f: X \to I^{n+1}$ with a stable value, then $\dim X \geq n+1$ (Theorem 5). We shall show (Theorem 6) that if X satisfies D'_n then such a map f exists. We shall also show (Theorem 8) that if such a map f exists, then X satisfies D'_n. However an example shows that we cannot show (constructively) that if $\dim X > n$ then X satisfies D'_n. In the next section we will develop more machinery (Theorem 9) that will enable us to show that if $\dim X = n+1$ then X satisfies D'_n.

THEOREM 6: *If each point a in I^{n+1} is an unstable value of each uniformly continuous function $f: X \to I^{n+1}$, then X satisfies condition D_n. If X satisfies condition D'_n then there is a uniformly continuous function $f: X \to I^{n+1}$ with a stable value a.*

PROOF: Let $A_0, \ldots, A_n$ be located subsets of X, and $1 > \epsilon > 0$. Define $f: X \to I^{n+1}$ by $f_i(x) = d(x, A_i) \wedge 1$. Let $g: X \to I^{n+1}$ be such that $\|f - g\| < \epsilon/4$. Let a be a point in I^{n+1} such that $\epsilon/2 < a_i < 3\epsilon/4$ for $1 \leq i \leq n+1$. To prove the first statement, choose g so that $d(a, g(X)) > 0$. Now adjust a slightly so that $B_i = g_i^{-1}([0, a_i])$ is bilocated for $1 \leq i \leq n+1$. The $\{B_i\}$ show that X satisfies condition D_n with $\delta = \omega(d(a, g(X))/2)$, where ω is a modulus of continuity for g. To prove the second statement, note that $d(a, g(X)) > 0$ is absurd, for it implies that X satisfies condition D_n. Hence $d(a, g(X)) = 0$, so a is a stable value of f.

We say that the bilocated sets $B_0, \ldots, B_n$ are *well placed* if $d(\cap S_\delta(B_i), \cap S_\delta(\sim B_i)) > 0$ for some $\delta > 0$.

LEMMA 1: *If $B_0, \ldots, B_n$ are well placed, then there is an $\alpha > 0$ such that any bilocated α-enlargements $E_0, \ldots, E_n$ are also well placed.*

PROOF: Suppose $d(\cap S_\delta(B_i), \cap S_\delta(\sim B_i)) > 0$ for $\delta > 0$. Let $\alpha = \delta/2$, and let $E_0, \ldots, E_n$ be bilocated α-enlargements of $B_0, \ldots, B_n$. Then $S_\alpha(E_i) \subseteq S_\delta(B_i)$ and $\sim E_i \subseteq \sim B_i$, whence the lemma follows.

LEMMA 2: *If X satisfies condition D_n and L is a located subset of X, then L satisfies condition D_n.*

PROOF: Suppose $\epsilon > 0$ and $A_0, \ldots, A_n$ are located subsets of L. Then there exist well placed, bilocated $(\epsilon/2)$-enlargements $B_0, \ldots, B_n$ of $A_0, \ldots, A_n$ in X. By Lemma 1 we can choose α in the interval $(0, \epsilon/2)$ so that $S_\alpha(B_0), \ldots, S_\alpha(B_n)$ are well placed, and $C_i = S_\alpha(B_i) \cap L$ is bilocated in L for $0 \leq i \leq n$. Then $C_0, \ldots, C_n$ are well placed, bilocated ϵ-enlargements of $A_0, \ldots, A_n$ in L.

THEOREM 7: *If X satisfies condition D_n, then* $\dim X \leq n$.

PROOF: We shall show, for fixed $\epsilon > 0$ and all positive integers m, that if X satisfies D_n, and X has an ϵ-cover of cardinality m, then X has an ϵ-cover $\mathcal{F}$ such that $o(\mathcal{F}) \leq n+1$. If $m \leq n+1$, there is no problem. Suppose the conclusion is true for spaces having ϵ-covers of cardinality not exceeding m, and let $C_0, \ldots, C_m$ be an ϵ-cover of X. Then, by Lemma 2, $L = C_1 \cup \cdots \cup C_m$ has an ϵ-cover $\mathcal{E}$ such that $o(\mathcal{E}) \leq n+1$. If $\operatorname{card}(\mathcal{E}) \leq n$ we are done. If not, modify $\mathcal{E}$ by successively enlarging $n+1$ tuples of elements of $\mathcal{E}$ so that E is bilocated for each E in $\mathcal{E}$ and such that every $n+1$ tuple of sets of $\mathcal{E}$ is well placed, keeping $o(\mathcal{E}) \leq n+1$. Choose $\delta > 0$ such that $o(\mathcal{E}) \leq n+1$ with separation δ, and such that if $E_0, \ldots, E_n$ are in $\mathcal{E}$, then $d(\cap S_\delta(E_i), \cap S_\delta(\sim E_i)) > 0$. Let $F = \cap_{E \in \mathcal{E}} S_{\delta/2}(\sim E) \cap S_{\delta/2}(C_0)$. We can decrease δ so that F is located and $\operatorname{diam} F < \epsilon$. Let $\mathcal{F} = \mathcal{E} \cup \{F\}$. To see that $\mathcal{F}$ is a cover of X, note that if $x \in X$, then either (i) $x \in F$, or (ii) $d(x, \sim E) > 0$ for some E in $\mathcal{E}$, and hence $d(x, E) = 0$, or (iii) $d(x, C_0) > 0$ and so $d(x, L) = 0$. It is readily seen that $o(\mathcal{F}) \leq n+1$ with separation $\delta/2$.

THEOREM 8: *If f is a uniformly continuous function from X to I^{n+1} with stable value a, then X satisfies D'_n.*

PROOF: Since the set of stable values of f is open, we may choose $a = (a_1, \ldots, a_{n+1})$ so that $A_j = f_j^{-1}([0, a_j])$ is located for $1 \leq j \leq n+1$. Since a is a stable value of f, there is an $\epsilon_0 > 0$ such that if $g: X \to I^{n+1}$ is uniformly continuous and $\|f - g\| < \epsilon_0$, then $d(a, g(X)) = 0$. We can choose ϵ_0 so that $S_{\epsilon_0}(a) \subseteq I^{n+1}$ (in fact, we cannot avoid it). Let

$\epsilon = \omega(\epsilon_0/2)$, where ω is a modulus of continuity for f, and let B_j be any bilocated ϵ-enlargement of A_j for $1 \le j \le n+1$. Define $h : X \to I^{n+1}$ by

$$h_j(x) = [(f_j(x) - \epsilon_0/2) \vee (a_j \wedge f_j(x) + d(x, B_j))] \wedge (f_j(x) + \epsilon_0/2) \wedge 1.$$

Note that $\|f - h\| \le \epsilon_0/2$. If $x \in B_j$, then $h_j(x) \le (f_j(x) - \epsilon_0/2) \vee a_j = a_j$ since $d(x, A_j) < \omega(\epsilon_0/2)$ and $f(y) \le a_j$ for y in A_j. If $x \in \sim B_j$, then $d(x, B_j) > 0$ and $f_j(x) \ge a_j$, so $h_j(x) > a_j$. Define $g : X \to I^{n+1}$ by

$$g_j(x) = 0 \vee (h_j(x) + (\epsilon_0/3)(-1 \vee (d(x, B_j) - d(x, \sim B_j)) \wedge 1)) \wedge 1.$$

Clearly $\|g - h\| \le \epsilon_0/3$, so $\|f - g\| < \epsilon_0$. Given $0 < \delta < 1$, choose x in X such that $d(a, g(x)) < \epsilon_0\delta/6$. If $d(x, \sim B_j) \ge \delta/2$ for some j, then $d(x, B_j) = 0$ so $g_j(x) \le a_j - \epsilon_0\delta/6$ which is impossible. On the other hand, if $d(x, B_j) \ge \delta/2$ for some j, then $x \in \sim B_j$, so $g_j(x) > a_j + \epsilon_0\delta/6$, also impossible. Hence $x \in S_\delta(B_j)$ and $x \in S_\delta(\sim B_j)$ for all j, so $d(\cap S_\delta(B_j), \cap S_\delta(\sim B_j)) = 0$.

We shall now exhibit a space X such that $\dim X \ge 1$, but we cannot assert that X satisfies condition D_0'. Let $s_1, s_2, \ldots$ be a sequence of points in I^2 such that for each positive integer k, if $N \ge 4^k$, then $\{s_1, \ldots, s_N\}$ is a 2^{1-k}-approximation to I^2. Let $p_1, p_2, \ldots$ be an enumeration of the polygonal paths in I^2, with rational endpoints, that join opposite sides of I^2. Let $a_{n1}, a_{n2}, \ldots$ be a sequence of points in P_n such that for each positive integer k, if $N \ge 4^k$, then $\{a_{n1}, \ldots, a_{nN}\}$ is a 2^{1-k}-approximation to P_n. Let $\{f_n\}$ be a fugitive sequence. Define x_i to be s_i if $f_n = 0$ for all $n \le i$, and $x_i = a_{ni}$ if $f_n = 1$ for some (unique) $n \le i$. Let X be the set of all such x_i. Then X is a located subset of I^2, since for each positive integer k, if $N \ge 4^k$, then $\{x_1, \ldots, x_N\}$ is a 2^{1-k}-approximation to X.

We shall show that $\dim X \ge 1$. In fact, we shall show that $o(\mathcal{F}) \ge 2$ for any $1/2$-cover $\mathcal{F}$ of X. Suppose $\mathcal{F}$ is a $1/2$-cover of X. If $o(\mathcal{F}) \le 1$ with separation $\delta < 1/2$, let $A = \{x_1, \ldots, x_N\}$ be a $(\delta/3)$-approximation to X. Then A contains a $(\delta/3)$-approximation A_0 to a polygonal path joining opposite sides of I^2. Hence, for any pair of points a, b in A_0, we can find a sequence of points $a = a_0, a_1, \ldots, a_n = b$ in A_0 such that $d(a_{i-1}, a_i) < \delta$ for $1 \le i \le n$. Since $o(\mathcal{F}) \le 1$ with separation δ, we must have $A_0 \subseteq \bar{F}$ for some F in $\mathcal{F}$. But diam $A_0 > 1/2$ and $\mathcal{F}$ is a $1/2$-cover. This is absurd, so $o(\mathcal{F}) \ge 2$ by Theorem 1.

On the other hand, there is no hope of showing that X satisfies D_0'. To do so, we would have to exhibit a located subset A of X and a number $\epsilon > 0$, so that if B is any bilocated ϵ-enlargement of A, then $d(B, \sim B) = 0$. In particular, we would have to produce an $(\epsilon/2)$-

approximation T to A. Then we could find a δ between $\epsilon/2$ and ϵ such that $2\delta \neq |x-y|$ for all x and y in T, and $\delta \neq |x - s_i|$ for all x in T and all i. Let $K = \cup_{x\in T} S_\delta(x)$ in I^2. It is readily seen that there are paths P_n, with n arbitrarily large, that are either bounded away from K, or bounded away from $\sim K$. Clearly $B = K \cap X$ is a bilocated ϵ-enlargement of A. However, if $f_n = 1$, and P_n is bounded away from K or $\sim K$, then $d(B, \sim B) > 0$. Hence, knowing that $d(B, \sim B) = 0$ gives us information about the sequence f_n that we cannot necessarily obtain by examining a finite number of its terms.

Alexandrov gave a classical characterization of dimension for separable metric spaces in terms of extensions of continuous functions into spheres [6; Theorem VI 4]. As usual the constructive version of this theorem must take two forms. The proofs are modifications of the proof in [6] and are omitted.

THEOREM A: *Let X be a totally bounded space. Then* $\dim X \leq n$ *if and only if for every located subset C of X, each map $f: C \to S^n$ can be extended to X.*

THEOREM A′: *Let X be a totally bounded metric space. Then the following are equivalent*:

(1) *There exists a located subset C of X, a map $f: C \to S^{n-1} \subset I^n$, and a positive number δ such that if $g: X \to I^n$ with $\|f-g\|_C < \delta$, then $g(x)$ is dense in I^n.*
(2) *There exists a located subset C of X, a map $f: C \to S^{n-1}$, and a positive number δ such that if $g: X \to S^{n-1}$, then $\|f-g\|_C > \delta$.*
(3) *There is a map of X onto I^n with a stable value. The constructive Tietze extension theorem* [3; *page* 107] *is required in the proofs of these theorems.*

4. Complementary covers

It is a classical theorem that $\dim X \leq n-1$ if and only if $X = Y_1 \cup \cdots \cup Y_n$ where $\dim Y_j = 0$ for $1 \leq j \leq n$. This is false in the uniform theory (see Theorem 3). At the level of ϵ-covers, however, there is a (constructive) uniform analog. Given an ϵ-cover of order n we can find an ϵ-cover that is the union of n families of order 1. Moreover, we can do this in such a way that the union of the two ϵ-covers has order $n+1$ (see Theorem 9).

LEMMA 3: *If $\mathcal{F}$ is an ϵ-cover of X, and E_k is a located subset of X such that $o(\mathcal{F}) \leq k$ on E_k, then there is a located subset E_{k-1} of E_k such that $o(\mathcal{F}) \leq k-1$ on E_{k-1}, and a finite family $\mathcal{C}_k$ of located subsets of E_k, of diameters less than ϵ, such that $o(\mathcal{C}_k) \leq 1$ and $\mathcal{C}_k \cup \{E_{k-1}\}$ is a cover of E_k.*

PROOF: Choose δ such that $o(\mathcal{F}) \leq k$ on E_k with separation δ, and $\mathcal{F}$ is an $(\epsilon - \delta)$-cover of X. Choose η between $\delta/4$ and $\delta/2$ so that $C(\mathcal{J}) = \{x \in E_k : d(x, F) < \eta$ for all F in $\mathcal{J}\}$ is bilocated in E_k for each finite subset $\mathcal{J}$ of $\mathcal{F}$. Let $\mathcal{C}_k = \{C(\mathcal{J}) : \text{card}(\mathcal{J}) = k\}$. Note that if $C(\mathcal{J}_1)$ and $C(\mathcal{J}_2)$ are in $\mathcal{C}_k$, and $\mathcal{J}_1 \neq \mathcal{J}_2$, then $d(C(\mathcal{J}_1), C(\mathcal{J}_2)) > \delta/4$, for otherwise there would exist an x in $C(\mathcal{J}_1)$ and a y in $C(\mathcal{J}_2)$ such that $d(x, y) < \delta/2$, so x would be within δ of every F in $\mathcal{J}_1 \cup \mathcal{J}_2$, which contains at least $k+1$ sets. Thus $o(\mathcal{C}_k) \leq 1$. Also, if $C \in \mathcal{C}_k$, then $\text{diam}\,(C) < \epsilon$.

Now choose α in the interval $(0, \delta/8)$ so that $E_{k-1} = \{x \in E_k : d(x, \sim C) < \alpha$ for all C in $\mathcal{C}_k\}$ is located. Then $\mathcal{C}_k \cup \{E_{k-1}\}$ is a cover of E_k since, if $x \in E_k$, then either $x \in E_{k-1}$, or $d(x, \sim C) > 0$ for some C in $\mathcal{C}_k$, in which case $d(x, C) = 0$. Moreover, $o(\mathcal{J}) \leq k-1$ on E_{k-1} with separation $\delta/16$. Otherwise there would be an x in E_{k-1} such that $d(x, F) < \delta/8$ for all F in a subset $\mathcal{J}$ of $\mathcal{F}$ of cardinality k. Since $x \in E_{k-1}$, there is a y in $\sim C(\mathcal{J})$ such that $d(x, y) < \delta/8$. But then $d(y, F) < \delta/4$ for all F in $\mathcal{J}$, so $y \in C(\mathcal{J})$, which is absurd.

THEOREM 9: *If $\mathcal{F}$ is an ϵ-cover of X such that $o(\mathcal{F}) \leq n$, then there is an ϵ-cover $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$ of X such that $o(\mathcal{C}_k) \leq 1$ for $1 \leq k \leq n$ and $o(\mathcal{F} \cup \mathcal{C}) \leq n+1$.*

PROOF: By repeated application of Lemma 3, we construct located subsets $X = E_n \supseteq E_{n-1} \supseteq \cdots \supseteq E_0 = \emptyset$ and finite families $\mathcal{C}_k$ of located subsets of diameters less than ϵ, such that $o(\mathcal{F}) \leq k$ on E_k, $\mathcal{C}_k \cup \{E_{k-1}\}$ is a cover for E_k, and $o(\mathcal{C}_k) \leq 1$, for $1 \leq k \leq n$. Hence $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$ is an ϵ-cover of X with $o(\mathcal{C}_k) \leq 1$ for $1 \leq k \leq n$. It remains to show that $o(\mathcal{F} \cup \mathcal{C}) \leq n+1$. Choose δ such that $o(\mathcal{F}) \leq k$ on E_k with separation 3δ, and $o(\mathcal{C}_k) \leq 1$ with separation δ, for $1 \leq k \leq n$. For any $x \in X$, there exists $\mathcal{J} \subseteq \mathcal{F}$ such that $d(x, F) < 2\delta$ for all F in $\mathcal{J}$, and $d(x, F) \geq \delta$ for all F not in $\mathcal{J}$. Let $\mathcal{J}$ have cardinality $j+1$. Then $d(x, E_j) \geq \delta$, since $o(\mathcal{F}) \leq j$ on E_j with separation 3δ. But E_j contains every set C in $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_j$, and, since $o(\mathcal{C}_k) \leq 1$ with separation δ, we have $d(x, C) \geq \delta$ for all but at most $n-j$ sets C in $\mathcal{C}$. Hence $o(\mathcal{F} \cup \mathcal{C}) \leq n+1$ with separation δ.

Armed with Theorem 9, we can settle the remaining question concerning the converse of Theorem 5.

COROLLARY: *If* $\dim X = n$, *then* X *satisfies* D'_{n-1}.

PROOF: Choose $\eta > 0$ so that if $\mathcal{F}$ is a 3η-cover of X, then $o(\mathcal{F}) \geq n+1$. Let $\mathcal{C}_0 \cup \cdots \cup \mathcal{C}_n$ be an η-cover of X such that $o(\mathcal{C}_i) \leq 1$ with separation 3ϵ, where $0 < \epsilon < \eta$, for $0 \leq i \leq n$. Let $A_j = \cup \mathcal{C}_j$ for $1 \leq j \leq n$ and suppose that $B_1, \ldots, B_n$ are bilocated ϵ-enlargements of $A_1, \ldots, A_n$. Let $\mathcal{B}_i = \{B_i \cap S_\epsilon(C) : C \in \mathcal{C}_i\}$, for $1 \leq i \leq n$. Note that $B_i = \cup \mathcal{B}_i$, that $\mathcal{C}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is a 3η-cover of X, and $o(\mathcal{B}_i) \leq 1$ for $1 \leq i \leq n$. If $\delta > 0$, choose α in the interval $(0, \delta)$ so that $C \cap \bigcap_{i=1}^n S_\alpha(\sim B_i)$ is located for each C in $\mathcal{C}_0$. Let $\mathcal{D}$ be the collection of all such sets. Note that $o(\mathcal{D}) \leq 1$. Then $\mathcal{F} = \mathcal{D} \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is a 3η-cover of X, for if $x \in \cup \mathcal{C}_0$, then either $x \in \cup \mathcal{D}$ or $d(x, B_i) = 0$ for some $1 \leq i \leq n$. Thus, since $o(\mathcal{F}) \geq n+1$, there must be points arbitrarily close to $\cup \mathcal{D}$ and all the B_i. But $\cup \mathcal{D} \subseteq \cap S_\delta(\sim B_i)$, so X satisfies D'_{n-1}.

Theorem 9 also allows us to construct a cover of $X \times Y$ from covers of X and Y in such a way as to prove the product theorem.

THEOREM 10: *If* $\dim X \leq m$ *and* $\dim Y \leq n$, *then* $\dim X \times Y \leq m+n$.

PROOF: By repeated application of Theorem 9 we can construct a sequence of ϵ-covers $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ of X such that $o(\mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n) \leq m+n+1$ with separation δ. Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$ be an ϵ-cover of Y such that $o(\mathcal{C}_i) \leq 1$ with separation δ, for $0 \leq i \leq n$. Let $\mathcal{D} = \{F \times C : F \in \mathcal{F}_i$ and $C \in \mathcal{C}_i$ for some $1 \leq i \leq n\}$. Then $\mathcal{D}$ is an ϵ-cover of $X \times Y$. We shall show that $o(\mathcal{D}) \leq m+n+1$ with separation δ. If $(x, y) \in X \times Y$, then there exist sets $C_i \in \mathcal{C}_i$ such that $d(y, C) \geq \delta$ if $C \in \mathcal{C}$ and $C \neq C_i$ for $0 \leq i \leq n$. Also there is a subset $\mathcal{J}$ of $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$ of cardinality $m+n+1$ such that $d(x, F) \geq \delta$ for all F in $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$ outside of $\mathcal{J}$. Then $d((x, y), F \times C) \geq \delta$ unless $C = C_i$ for some i, and $F \in \mathcal{J}$. But each F in $\mathcal{J}$ is paired with a unique C_i, so there are only $m+n+1$ excluded sets $F \times C$.

5. Dimension in the n-cell

Any ordinary notion of dimension of totally bounded spaces must assign dimension n to I^n. That $\dim I^n \leq n$ is fairly straightforward. Clearly $\mathcal{F} = \{[(k-1)/m, k/m]: 1 \leq k \leq m\}$ is a $(1/m)$-cover of I such that $o(\mathcal{F}) \leq 2$ with separation $1/4m$. Hence $\dim I \leq 1$, and repeated application of Theorem 10 shows that $\dim I^n \leq n$. That $\dim I^n \geq n$ is the *Pflastersatz*. We prove it via the (almost) fixed point theorem. The argument here follows classical lines.

LEMMA 4: *Let Δ be an n-simplex with vertices $e_0, \ldots, e_n$. Let $\{A_0, \ldots, A_n\}$ be a cover of Δ such that each face $[e_{i(0)}, \ldots, e_{i(r)}]$ of Δ is contained in $A_{i(0)} \cup \cdots \cup A_{i(r)}$. Then $o(A_0, \ldots, A_n) = n+1$.*

PROOF: We must show that $o(A_0, \ldots, A_n) \geq n+1$. For $\delta > 0$ let T be a triangulation (by successive barycentric subdivision) of the simplex Δ of mesh less than δ. Assign to each vertex v in T a vertex $S(v) = e_i$ of Δ such that $v \in A_i$ and the ith barycentric coordinate of v is nonzero. This is possible by virtue of the hypothesis of the lemma. By Sperner's Lemma [1; p. 160], there exists an n-simplex $U = [v_0, \ldots, v_n]$ of T such that $S(v_0), \ldots, S(v_n)$ are all distinct. But this means that U intersects all the sets $A_0, \ldots, A_n$. Since the diameter of U is less than δ, we conclude that the point v_0 is within δ of each set A_j.

REMARK: We give no proof of Sperner's Lemma because it is a statement of a purely combinatorial, finitistic nature.

THEOREM 11: (*Brouwer Fixed Point Theorem*). *Given a uniformly continuous function $g: I^n \to I^n$ and a $\delta > 0$, we can find a point x in I^n such that $|g(x) - x| < \delta$.*

PROOF: We will identify I^n with the (closed) n-simplex $\Delta = [e_0, \ldots, e_n]$. If $x \in \Delta$, then $x = \Sigma_{i=0}^{n} x(i) e_i$ where $x(i) \geq 0$ and $\Sigma\, x(i) = 1$. Equip Δ with the metric $d(x, y) = \sup |x(i) - y(i)|$. Let ω be a modulus of continuity for g such that $\omega(\delta) \leq \delta$ for all $\delta > 0$. Let x' denote $g(x)$. For arbitrary $\delta > 0$, choose $0 < \epsilon < \omega(\delta/3n)$ so that $A_i = \{x \in \Delta: x'(i) \leq x(i) + \epsilon\}$ is located for $0 \leq i \leq n$. The condition $\Sigma\, x(i) = \Sigma\, x'(i) = 1$ implies that $A_0, \ldots, A_n$ is a cover of Δ. To show that the sets $A_0, \ldots, A_n$ satisfy the hypothesis of Lemma 4, let x be an arbitrary point on the face $[e_{i(0)}, \ldots, e_{i(r)}]$ of Δ. Then $x(i(0)) + \cdots + x(i(r)) = 1 \geq x'(i(0)) + \cdots + x'(i(r))$. Hence we can find a k such that $x'(i(k)) \leq x(i(k)) + \epsilon$. Therefore $x \in A_{i(k)}$. By Lemma 4 we have $o(A_0, \ldots, A_n) = n + 1$. This means that we can find an x in Δ such that $d(x, A_i) < \omega(\delta/3n)$ for $0 \leq i \leq n$. Hence we can find points y_i in A_i such that $d(x, y_i) < \omega(\delta/3n)$, so $d(x', y'_i) < \delta/3n$. Since $y'_i(i) \leq y_i(i) + \delta/3n$, we conclude that $x'(i) < x(i) + \delta/n$ for $0 \geq i \leq n$. Now observe that $1 - x'(i) = \sum_{j \neq i} x'(j) < \sum_{j = i} x(j) + \delta = 1 - x(i) + \delta$. Hence $|x(i) - x'(i)| < \delta$ for $0 \leq i \leq n$. So $d(x, g(x)) < \delta$.

THEOREM 12: *The identity map on I^n has a stable value. Hence* $\dim I^n = n$.

PROOF: Let *id* denote the identity function on I^n and a the point in I^n all of whose components are 1/2. To show that a is a stable value of the map *id*, let $g: I^n \to I^n$ by any uniformly continuous function such that $\|id - g\| < 1/2$. Then the function $h(x) = x - g(x) + a$ is clearly a uniformly continuous function from I^n into I^n. By Theorem 11, for each $\delta > 0$ we can find an x_0 in I^n such that $|h(x_0) - x_0| < \delta$. Hence $|g(x_0) - a| < \delta$. Since we can do this for each $\delta > 0$, it follows that $d(a, g(I^n)) = 0$.

By Theorem 12, any subset of I^n with a nonempty interior has dimension n since it contains a homeomorph of I^n. Conversely, it is a classical result that a closed subset X of I^n of dimension n must have a nonempty interior. However, the example following Theorem 3 shows that even thugh X is located we may not be able to find a point x in X and a $\delta > 0$ so that $S_\delta(x) \subseteq X$. But if X is *bilocated*, then we can find the required x and δ.

THEOREM 13: *If X is a bilocated closed subset of I^n, and* $\dim X \geq n$, *then X contains a nonempty open subset of I^n.*

PROOF: Choose $\epsilon > 0$ so that if $\mathscr{F}$ is any ϵ-cover of I^n then $o(\mathscr{F}) \geq n + 1$ on X. Let T be a triangulation of I^n of mesh less than $\epsilon/2$, and let Σ be the set of open n-simplices of T. Choose a point t_σ in each open n-simplex σ, and choose a $\delta > 0$ so that $S_{2\delta}(t_\sigma) \subseteq \sigma$ for each σ in Σ. Since $\sim X$ is located either, for each $\sigma \in \Sigma$, we can find a point C_σ in $\sigma \cap \sim X$, or no element of $S_\delta(t_\sigma)$ is in $\sim X$. We shall show that the former alternative is impossible. It will follow that if $x \in S_\delta(t_\sigma)$, then we cannot have $d(x, X) > 0$, so $d(x, X) < r$ for any $r > 0$. Thus $x \in X$, since X is closed, so X contains the non-empty open set $S_\delta(t_\sigma)$.

Suppose that for each $\sigma \in \Sigma$ we have a point $C_\sigma \in \Sigma \cap \sim X$. Let T_0

be the barycentric subdivision of T using the points c_σ as the barycenters of the n-simplices σ. If $x \in I^n$, and u is a vertex of T_0, let $x(u)$ denote the barycentric coordinate of x with respect to u. If $x, y \in I^n$, let $d(x, y) = \sup |x(u) - y(u)|$ where u ranges over the vertices of T_0. This gives a metric on I^n that is equivalent to the usual one. Since $c_\sigma \in \sim X$ we can find an $\eta > 0$ such that, for each σ in Σ, if $x(c_\sigma) > 1-(n+1)\eta$, then $d(x, X) > \eta$. Let $F_v = \{x : x(v) \neq 0\}$ for each vertex v in the original triangulation T, and set $\mathcal{F} = \{F_v : v$ is a vertex of $T\}$. Now each n-simplex of T_0 has a vertex in T, so $\mathcal{F}$ is a cover of I^n. The mesh of T is less than $\epsilon/2$, so $\mathcal{F}$ is an ϵ-cover of I^n. We shall show that $o(\mathcal{F}) \leq n$ on X with separation η, a contradiction.

Suppose that $d(x, F_v) < \eta$ for all v in a set V of cardinality $n+1$. Then, for each v in V, there is an (abstract) n-simplex τ_v in T_0 such that $v \in \tau_v$ and $x(u) < \eta$ for each vertex u not in τ_v. So $x(u) < \eta$ for each vertex u not in $\bigcap_{v \in V} \tau_v$. Since the cardinality of V is $n+1$, either $\bigcap_{v \in V} \tau_v = \emptyset$, or $\bigcap_{v \in V} \tau_v = \{c_\sigma\}$ for some σ in Σ. Since $\Sigma_u\, x(u) = 1$, and $x(u)$ can be bounded arbitrarily closely to 0 off-sets of cardinality $n+1$, we must have $x(c_\sigma) > 1-(n+1)\eta$, and hence $d(x, X) > \eta$.

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