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ARITHMETIC ON SINGULAR CUBIC SURFACES

Daniel F. Coray*

In this note we shall give a proof of the following:

PROPOSITION 1:

(a) *Let $V \subset \mathbb{P}^3$ be a cubic surface, defined over the infinite perfect field k , and having exactly 3 singular points Q_1, Q_2, Q_3 . Then V is k -birationally equivalent to a non-singular cubic surface W containing a k -rational set of 3 skew lines.*

(b) *Conversely, every non-singular cubic surface $W \subset \mathbb{P}^3$ containing a k -rational set of 3 skew lines is k -birationally equivalent to a cubic surface V with exactly 3 double points.*

A proof of this result was already outlined in a little known paper of B. Segre [10], but the crucial fact that W is non-singular receives no justification there. Moreover, Segre states the converse with an additional assumption, which is actually not needed, as we shall see (lemma 3).

This proposition has a number of applications to arithmetical questions; we begin by discussing a few of them:

(i) When k is a number field, it was shown by Skolem [12] that singular cubic surfaces satisfy the *Hasse principle* (i.e. if V has \mathfrak{p} -adic solutions for every prime \mathfrak{p} , then V also has a point with coordinates in k). We may clearly assume that there are exactly 3 conjugated double points (cf. Segre [9] for the other cases). Skolem remarked that the equation of V can then be written in the form

$$f(x, y, z) = N_{K_1/k}(\tau) + a \operatorname{Tr}_{K_1/k}(\tau) + b = 0,$$

where τ is a linear combination of x, y and z , with coefficients in the

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cubic extension $K_1 = k(Q_1)$. Let D be the discriminant of a primitive element of K_1/k . Then $K_1(\sqrt{-3D})$ is a purely cubic extension of $k^* = k(\sqrt{-3D})$, by Cardano's formula. This fact enabled Skolem to reduce the equation further to the form $N_{K^*/k^*}(\tau^*) = c^*$, where the cubic extension K^*/k^* is generated by one of the roots of the polynomial $t^3 + 3at + b$. That the Hasse principle holds for this class of cubic surfaces now follows readily from class-field theory.

By a remark of F. Châtelet [2, pp. 70–71], a cubic surface of type $N\tau^* = c^*$ is birationally equivalent to a non-singular cubic surface containing a k^* -rational set of 3 skew lines, for which the Hasse principle is also known to hold [13, theorem 7]. One could combine this remark with Skolem's argument to obtain a slightly weaker form of proposition 1(a). However, it seems more natural to use a purely geometric argument, which avoids many of the rather painful computations of [12] and yields a birational equivalence defined over the groundfield k . As a corollary we get—of course—that *it suffices to give one proof of the Hasse principle*, e.g. [13, p. 15] or [3, p. 19]; that it also holds for singular cubic surfaces follows immediately from proposition 1.

(ii) Another application of this result is to the following:

COROLLARY 1: *Let $V \subset \mathbb{P}^3$ be a singular cubic surface, defined over the field k . Let K/k be an algebraic extension of k , with degree d prime to 3, and suppose that $V(K) \neq \emptyset$. Then $V(k) \neq \emptyset$.*

In view of proposition 1, this is a direct consequence of [4, prop. 8.1], where we proved the analogous statement for a non-singular cubic surface with a k -rational set of 3 or 6 skew lines. The technique we used in [4, prop. 8.2] to establish this corollary was rather more complicated.

(iii) Proposition 1(b) implies the following classical result:

COROLLARY 2: *Every non-singular cubic surface $W \subset \mathbb{P}^3$, containing a k -rational set of 3 skew lines and at least one k -rational point, is k -birationally equivalent to a plane.*

Indeed we can map W birationally onto a cubic surface V with 3 double points Q_1, Q_2, Q_3 . Let us choose a k -rational pair of points Q_4, Q_5 on V . The family of twisted cubics passing through the 5 points Q_1, \dots, Q_5 determines a one-to-one correspondence between the points of V and those of a plane through Q_4 and Q_5 . Indeed a twisted cubic is uniquely determined by 6 points in general position (see e.g.

[14, §11, ex. 4]). If these points form a k -rational set, the twisted cubic is therefore also defined over k (even though the construction given in [14] is irrational!). Now a general cubic through Q_1, \dots, Q_5 meets V in one other point (since Q_1, Q_2 and Q_3 each count as 2 intersections); and it also meets any plane containing Q_4 and Q_5 in one other point.

The description can be made even more explicit if we note that the twisted cubics through Q_1, \dots, Q_5 can be put in one-to-one correspondence with the lines going through a fixed point $T_{33}^{tet}(Q_5)$, by means of a Cremona transformation T_{33}^{tet} of \mathbb{P}^3 [11, p. 179 & ex. 11, p. 186], whose fundamental tetrahedron contains the points Q_1, \dots, Q_4 as vertices.¹ V itself is mapped onto a quadric containing $T_{33}^{tet}(Q_5)$, and this transformation is defined over k provided both Q_4 and Q_5 are k -rational (which we may assume, since the preceding argument shows that $V(k)$ is infinite²).

We now proceed to the proof of proposition 1. The technique is the same as in [10], but we supply a few more details:

(a) If Q_1 has coordinates in k , projecting from Q_1 yields a k -birational equivalence of V onto a plane, which is certainly equivalent to some non-singular cubic surface W , e.g. one containing 3 k -rational skew lines. *We may therefore assume that the Q_i form a complete set of conjugates.*

Let ρ denote the plane $\langle Q_1, Q_2, Q_3 \rangle$ spanned by Q_1, Q_2 and Q_3 . Let l be a k -rational line in \mathbb{P}^3 , which meets V in three distinct points $P_1, P_2, P_3 \notin \rho$ and is not incident with any of the finitely many lines of V passing through Q_1, Q_2 or Q_3 . Consider the 3-dimensional family \mathfrak{M} of quadrics containing the base set $\Sigma = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$. If

¹ This is one of the simplest space transformations. With a suitable choice of coordinates, a possible expression is

$$T_{33}^{tet}: (x, y, z, t) \mapsto (1/x, 1/y, 1/z, 1/t).$$

A more complicated type of cubo-cubic transformation will be introduced later (see footnote ⁶).

² Of course this is a special case of a theorem of Segre, for which a very nice geometric proof was also found by Skolem [12, pp. 308–309]. The end of the proof («*Andererseits ist klar: . . .*») is based on a continuity argument, but it can be replaced by the following remark, which applies to a field of arbitrary characteristic (we retain Skolem's notation and assume the reader is familiar with the first part of his proof): *There are only finitely many ways of obtaining each point $Q \neq P$ with Skolem's construction.* Indeed, let C_i be the intersection of the cubic surface F with the tangent plane T_i ($i = 1, 2$). We may clearly assume that C_i is irreducible (it suffices to avoid hitting one of the lines of F when choosing l). Then Q_2 belongs to the cone with vertex Q and base C_1 . If C_2 meets this cone in finitely many (and therefore ≤ 9) points, our assertion is proved. Otherwise C_2 is contained in that cone; but this implies that the two double points P_1 and P_2 are collinear with the vertex Q , and this is absurd because $Q \neq P$. Since we can use infinitely many pairs (Q_1, Q_2) and no more than a finite number of them give rise to the same point Q , there must be infinitely many such points.

$\varphi_0, \dots, \varphi_3$ are k -rational generators for this linear system, let $\Phi : V \rightarrow \mathbb{P}^3$ be the rational transformation associated with them, viz. $\Phi = (\varphi_0 : \dots : \varphi_3)$. This is the restriction to V of a well-known Cremona transformation T_{23} (see [5, pp. 560–561]); we shall examine its effect on the surface V . We claim that the image $W = \Phi[V]$ is a non-singular cubic surface, is k -birationally equivalent to V , and contains a k -rational set of 3 skew lines. It is clear that W is defined over k , since the generators \mathfrak{M} have been chosen k -rational; and Φ is a birational equivalence, since V is not contained in any exceptional set of T_{23} . In fact, let $V_0 = V - \rho - \{P_1, P_2, P_3\}$; then $\Phi|_{V_0}$ is an embedding, in virtue of the following lemma (see e.g. [8, chap. 2, §5.5, lemma]):

LEMMA 1: \mathfrak{M} separates points and infinitely near points on V_0 . It also separates the points infinitely near P_i ($i = 1, 2, 3$).

PROOF: Let R and S be two distinct points of V_0 . We denote by $\sigma_i = \langle l, Q_i \rangle$ the plane spanned by l and Q_i ; and if $D_i = \langle Q_j, Q_k \rangle$ is the line joining Q_j to Q_k ($i \neq j, k$), we let $\tau_i = \langle D_i, R \rangle$. Of course, $R \in \sigma_i \cup \tau_i \in \mathfrak{M}$ for all $i = 1, 2, 3$; we want to show that $S \notin \sigma_i \cup \tau_i$ for some i . Now $\sigma_i \cap \sigma_j = l \ \forall i \neq j$, and $\tau_1 \cap \tau_2 \cap \tau_3 = \{R\}$. So if $S \in \sigma_i \cup \tau_i \ \forall i$, then $S \in \sigma_1 \cap \tau_2 \cap \tau_3$ (say). But the line $\tau_2 \cap \tau_3 = \langle R, Q_1 \rangle$ must then be contained in σ_1 (and therefore meet l); and it lies on V , since it contains the three points R, S and Q_1 (which is double). This contradicts our choice of l .

The proof for infinitely near points is practically identical: let $\vec{s} \in T_R V$ be a tangent vector at R . At least one of the planes τ_i is transversal to \vec{s} , say τ_1 . Then $\sigma_1 \cup \tau_1$ is transversal to \vec{s} , unless $R \in \sigma_1$. In the latter case $R \notin \sigma_2 \cup \sigma_3$ (since $\sigma_1 \cap \sigma_j = l \ \forall j \neq 1$), and if τ_j ($j = 2$ or 3) is transversal to \vec{s} , we may then use $\sigma_j \cup \tau_j$. If not, then $\vec{s} \in \tau_2 \cap \tau_3 = \langle R, Q_1 \rangle$ and we conclude as before, since the line $\langle R, Q_1 \rangle$ meets V with multiplicity 2 both at R and at Q_1 .

Finally, given any two tangent vectors at P_i , they can be separated by means of pairs of planes of type $\rho \cup \sigma$, where σ is a suitably chosen plane that contains l (l is transversal to V at P_i , since it meets V in 3 distinct points). q.e.d.

The degree of W is the number of its intersections with a generic line of \mathbb{P}^3 . The inverse image of such a line is the trace on V of the intersection of two quadrics containing the base set Σ , and therefore consists of 12 points, 3 of which are movable. (The fixed points Q_i each count as 2 intersections; to see that the 3 remaining points are indeed movable, one can consider degenerate cases, like the two quadrics

$\sigma_1 \cup \tau_1$ and $\sigma_2 \cup \tau_2$, where the τ_i are determined by an arbitrary point $R \in V_0$.) Hence W is a cubic surface.

Let \tilde{V} be the variety obtained by blowing up the three non-singular points P_1, P_2, P_3 . We have the following commutative diagram

$$\begin{array}{ccc} \tilde{V} & & \\ \pi \downarrow & \searrow \tilde{\Phi} & \\ V & \xrightarrow{\Phi} & W \subset \mathbb{P}^3 \end{array}$$

Let $\tilde{L}_i = \pi^{-1}(P_i)$ and $L_i = \tilde{\Phi}[\tilde{L}_i]$. One more lemma and our search for skew lines will be complete:

LEMMA 2: (i) $\tilde{\Phi}$ is regular at every point of \tilde{L}_i ($i = 1, 2, 3$); (ii) the L_i form a set of 3 skew lines; (iii) the points of L_i are non-singular on W .

PROOF: All these assertions can be checked by explicit calculations or constructions; but it seems preferable to use some general properties of correspondences, from which they derive almost formally. We first recall a few facts about blowings-up (cf. [7, prop. 20.4])³:

Suppose that \mathfrak{M} is generated by $\varphi_0, \dots, \varphi_n$ ($n = 3$ in our case, but we now consider a somewhat more general situation) and that the general member φ of \mathfrak{M} cuts out on the surface V a divisor (φ) , on which the isolated fixed point P (which will be one of our P_i) is a k -fold point. Then it is known [8, chap. 4, §3.1, thm. 1] that the proper transform of (φ) , in a neighbourhood of P , is $\pi^{-1}[\varphi] = \pi^*(\varphi) - k\tilde{L}$, where $\tilde{L} = \pi^{-1}(P)$. Hence the map $\tilde{\Phi} = (\pi^*\varphi_0 : \dots : \pi^*\varphi_n)$ is non-regular at a point \tilde{x} of \tilde{L} if and only if \tilde{x} belongs to all the divisors $\Delta_j = (\pi^*\varphi_j) - k\tilde{L}$ (see e.g. [8, chap. 3, §1.4, thm. 2]).

Now there is a 1-1 correspondence between the points $\tilde{x} \in \tilde{L}$ and the tangent vectors \tilde{x} at P . In fact, by a local isomorphism, we can even think of V as being a piece of a plane and of \tilde{x} as being represented by a straight line X through P . Thus we have a 1-1 correspondence between the points $\tilde{x} \in \tilde{L}$ and the 'lines' X through P (see [8, chap. 2, §4.1]). It is clear that $\tilde{x} \in \Delta_j$ if and only if, locally above P , the intersection number $(\pi^{-1}[X], \Delta_j)_{\text{loc}}$ is positive. Since this number is equal to $(\pi^*X - \tilde{L}, (\pi^*\varphi_j) - k\tilde{L})_{\text{loc}} = (\pi^*X, \pi^*(\varphi_j))_{\text{loc}} - k$, we therefore derive, using [8, chap. 4, §3.2, thm. 2], that

$$\tilde{x} \in \Delta_j \Leftrightarrow (X, (\varphi_j))_{\text{loc}} \geq k + 1$$

³ A particularly nice exposition of this theory can be found in Šafarevič's lectures at the Tata Institute (Bombay, 1966), especially in lectures 2& 3, pp. 18–30.

Thus we have established, in particular, that $\tilde{\Phi}$ is non-regular at $\tilde{x} \in \tilde{L}$ if and only if $(X, (\varphi_j))_{\text{loc}} \geq k + 1 \forall j$. With these preliminaries, our lemma should appear almost trivial:

(i) In our case, $k = 1$ at each P_i . By the above discussion, it follows that if $\tilde{\Phi}$ were non-regular at $\tilde{x} \in \tilde{L}_i$, there would exist a vector $\tilde{x} \in T_{P_i}V$ such that $\tilde{x} \in T_{P_i}\varphi_j$ for every j . But we know already that there is no such \tilde{x} (lemma 1).

(ii) Of course L_i has dimension 1; in fact, lemma 1 shows that $\tilde{\Phi}$ is even injective on \tilde{L}_i . The degree of L_i is equal to the number of its (distinct) intersections with a generic plane in \mathbb{P}^3 ; since $\tilde{\Phi}$ is regular and injective on \tilde{L}_i , it is therefore equal to the multiplicity k with which a general quadric of \mathfrak{M} intersects V at the point P_i , namely 1.

Suppose now $y \in L_1 \cap L_2$; then y is the image by $\tilde{\Phi}$ of 2 points, $\tilde{x}_1 \in \tilde{L}_1$ and $\tilde{x}_2 \in \tilde{L}_2$. But this is absurd, because \mathfrak{M} separates any two tangent vectors, \tilde{x}_1 at P_1 and \tilde{x}_2 at P_2 . Indeed, let $\tau_i = \langle D_i, P_i \rangle$; then P_1 is double on $\sigma_i \cup \tau_i$, which is of course transversal to \tilde{x}_2 for some i .

(iii) By the Zariski Main Theorem, $\tilde{\Phi}|_{\pi^{-1}(V - \rho)}$ is an isomorphism, providing W is normal at every point of each L_i . Since W is a hypersurface, this is equivalent to saying that W has only finitely many double points on L_i , i.e. that L_i is not a double line of W . And this can be seen in various ways; for instance: if L_i were double, then any plane through L_i would meet W in *one* residual line. But such a plane corresponds to a quadric with a double point at P_i , e.g. $\varphi = \sigma_1 \cup \tau_1$, where $\tau_1 = \langle D_1, P_1 \rangle$. Now $\varphi \cap V$ consists of D_1 (which is blown down) and of *two* other components; a contradiction. q.e.d.

REMARK 1: The sets of 3 skew lines on a non-singular cubic surface W always occur in pairs. Indeed, given any such set, there is a unique non-singular quadric Q that contains all three lines; and $W \cap Q$ contains 3 lines from each ruling of Q . Instead of choosing the lines L_i as above, we could therefore (like Segre) have used this other set of 3 lines, which are simply the proper transforms of the curves $\sigma_i \cap V$. But our present choice shows a little more: if $V(k) \neq \emptyset$, it is possible to select the P_i k -rational, and then each of the 3 skew lines L_i is defined over k . In particular this reproves corollary 2: after mapping W onto V , we can map V onto W' , where W' is a non-singular cubic surface containing 3 k -rational skew lines; and there is a very obvious birational correspondence between W' and a plane (see e.g. [7, ex. 12.6]).

It remains to prove that W is non-singular. By the foregoing lemmas, a singularity of W comes either from one of the double points of V , or

from one of the lines D_i joining these points. Each of these lines is blown down to a single point $S_i \in W$. To see that S_i is non-singular, it is enough to check that a generic line through S_i meets W residually in 2 movable points. Such a line corresponds to the intersection of 2 quadrics of \mathfrak{M} containing D_i . We define two special quadrics as follows: let τ_i be a plane through D_i , whose residual intersection with V is a non-degenerate conic C , and which meets l in a point $R \notin V$. A general line d of τ_i , passing through R , will then meet C in two⁴ movable points T_1, T_2 . If σ denotes the plane $\langle l, d \rangle$ spanned by l and d , we see that the two quadrics $\sigma_i \cup \tau_i$ and $\rho \cup \sigma$ contain T_1 and T_2 , whence our assertion follows.

Finally we must examine the image of a double point Q_i and show that it does not contain any singularity of W . This part of the argument is easier to make rigorous if we use explicit equations. At this stage we can forget about the ground field⁵ and assume that the coordinates (x, y, z, t) of the double points are $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$. The equation of V is then simply:

$$(1) \quad t(ax^2 + bxy + cxz + dyz) + x(ax^2 + \beta xy + \gamma xz + \delta yz) = 0,$$

where $d \neq 0$, because V is irreducible. The tangent cone at the point $(0, 0, 0, 1)$ is given by

$$(2) \quad ax^2 + bxy + cxz + dyz = 0.$$

It is degenerate if and only if $ad = bc$, in which case its equation writes $(bx + dz)(cx + dy) = 0$. Since $d \neq 0$, this represents two *distinct* planes. The condition for all three tangent cones to be degenerate is as follows:

$$(3) \quad ad = bc; \quad \beta d = b\delta; \quad \gamma d = c\delta.$$

A simple example where this situation occurs is the cubic $x^3 - yzt = 0$.

It is easy to convince oneself that the result does not depend on the particular line l chosen, as long as it satisfies the conditions imposed at the beginning of the proof. Indeed, given any two lines l and l' satisfying those conditions, there is a projective transformation that fixes Q_1, Q_2 and Q_3 , and carries l into l' (cf. [1, chap. III, n° 13]). *We are*

⁴ Not necessarily distinct in characteristic 2, since R might be the intersection of all the lines tangent to C ! But we can always avoid this situation by selecting another plane τ_i . (If this were not possible, l would be contained in the tangent cone to V at each Q_j ($j \neq i$), but then the line $\langle P_i, Q_j \rangle$ would lie on V .)

⁵ Almost! See the remark at the end (remark 2).

therefore free to assume that l is the line $y = z = t$. The Cremona transformation and its inverse can then be written as follows:

$$(4) \quad \begin{cases} (x, y, z, t) \xrightarrow{\Phi} (x(y-t), x(y-z), z(y-t), t(y-z)) \\ (\xi, \eta, \zeta, \tau) \xrightarrow{\Phi^{-1}} (\xi\eta(\eta-\xi), \xi\eta(\tau-\zeta), \eta\zeta(\eta-\xi), \xi\tau(\eta-\xi)) \end{cases}$$

and the equation of W reads:

$$(5) \quad g(\xi, \eta, \zeta, \tau) = \tau((a\xi + c\zeta)(\eta - \xi) + (b\xi + d\zeta)(\tau - \zeta)) + \eta((\alpha\xi + \gamma\zeta)(\eta - \xi) + (\beta\xi + \delta\zeta)(\tau - \zeta)) = 0.$$

The image of the point $(0, 0, 0, 1)$ is the curve $C = \{\eta = 0; (a\xi + c\zeta)\xi - (b\xi + d\zeta)(\tau - \zeta) = 0\}$. We want to show that none of these points is singular. To this effect, we compute the derivatives of g in the plane $\eta = 0$ and get the following equations:

$$(6) \quad C: \quad (a\xi + c\zeta)\xi - (b\xi + d\zeta)(\tau - \zeta) = 0$$

$$(7) \quad \partial g / \partial \eta: (a\xi + c\zeta)\tau - (\alpha\xi + \gamma\zeta)\xi + (\beta\xi + \delta\zeta)(\tau - \zeta) = 0$$

$$(8) \quad \partial g / \partial \zeta: ((b + c)\xi + 2d\zeta - d\tau)\tau = 0$$

$$(9) \quad \partial g / \partial \tau: (a\xi + c\zeta)\xi - (b\xi + d\zeta)(2\tau - \zeta) = 0$$

We distinguish two cases:

First case: $\tau = 0$

$$\text{Then} \quad (6) \Rightarrow a\xi^2 + (b + c)\xi\zeta + d\zeta^2 = 0,$$

$$\text{and} \quad (7) \Rightarrow \alpha\xi^2 + (\beta + \gamma)\xi\zeta + \delta\zeta^2 = 0.$$

But if (ξ_0, ζ_0) is a common solution of these two equations, then the line $y = z = x \zeta_0 / \xi_0$ lies on V and is incident with l . This contradicts the choice of l .

Second case: $\tau \neq 0$

Then, in view of (6), (9) $\Rightarrow b\xi + d\zeta = 0$, and (8) $\Rightarrow d\tau = (b + c)\xi - 2b\xi = (c - b)\xi$. Therefore $(\xi, \zeta, \tau) = (d, -b, c - b)$. Now (6) $\Rightarrow a\xi + c\zeta = 0 \Rightarrow ad = bc$. Hence the tangent cone at $(0, 0, 0, 1)$ is

degenerate. But at the beginning we made the assumption that the Q_i formed a complete set of conjugates. Thus if one of the tangent cones is degenerate, so are the other two. Hence (3) holds. If we insert these relations in (7), we get: $\alpha d = b\gamma = bc\delta/d = a\delta$. But then (1) splits as $(t + x\delta/d)(ax^2 + bxy + cxz + dyz) = 0$, in contradiction with the irreducibility of V . This completes the proof that W is non-singular.

REMARK 2: It may be of some interest to note that the proof does not carry through without the condition that the Q_i form a complete set of conjugates. For instance, the surface $tz(x + y) + x^2(x + y + z) = 0$ has 3 double points and only one degenerate tangent cone, which corresponds to a *binode* (cf. [15, chap. 1, §5]). Its image by Φ is the surface $(\tau\xi + \eta\xi)(\eta - \xi + \tau - \zeta) + \eta\xi(\eta - \xi) = 0$, on which the point $(1, 0, 0, 1)$ is double, although l is not in special position with respect to the lines of V .

(b) The converse is easier to prove. Segre [10] makes use of the inverse Cremona transformation T_{23}^{-1} , but he introduces the additional hypothesis that the 3 skew lines should have a common k -rational transversal in \mathbb{P}^3 . As a matter of fact, this condition is automatically fulfilled:

LEMMA 3: *Any three skew lines L_1, L_2, L_3 , forming a k -rational set, are incident with infinitely many k -rational lines $E \subset \mathbb{P}^3$.*

PROOF: The three lines are contained in a unique quadric Q , which is necessarily non-singular and defined over k . Furthermore $Q(k) \neq \emptyset$, for if K_1 is a cubic extension of k over which L_1 is defined, then $Q(K_1) \neq \emptyset$, and hence $Q(k) \neq \emptyset$. (This is a special case of proposition 2.1 of [4] and is quite easy to prove: let $S_1 \in Q(K_1)$, and consider its conjugates, S_2 and S_3 ; there are ∞^2 k -rational conics through these 3 points; they meet Q in 4 points, leaving a residual k -rational point.) Let S be any k -rational point of Q . The tangent plane at S contains 2 lines of Q , one of which meets L_1, L_2 and L_3 . This line E , which is uniquely determined by rational conditions, is k -rational! (Alternatively: E is the residual intersection of L_1 in $\pi \cap Q$, where π denotes the plane spanned by L_1 and S . Hence it is defined over K_1 . But since it lies in the tangent plane at S , it is also defined over a quadratic extension l/k . Hence it is defined over $K_1 \cap l = k$.) q.e.d.

With the help of this lemma we could complete the proof as in [10]. But it may be of more interest to introduce another birational map,

which can also be described⁶ as the restriction of a Cremona transformation T_{33} of \mathbb{P}^3 :

Let Q be a k -rational point of \mathbb{P}^3 , chosen such that (i) it does not lie on W , and (ii) the planes ρ_i spanned by L_i and Q meet W in a non-degenerate conic C_i (it is well-known [14, p. 151] that there are only 5 planes through L_i for which the conic is degenerate!). We consider the family \mathfrak{R} of cubic surfaces passing through the following 15 points: $\Sigma = \{L_i \cap C_i$ (2 points for each i); $L_i \cap C_j$ (1 point for each pair $i \neq j$); $C_i \cap C_j$ (1 point for each pair $i < j$)}. Σ imposes 15 independent conditions on the surfaces of \mathfrak{R} . (If we adjoin the point Q and one point on each conic C_i as additional constraints, the only cubic

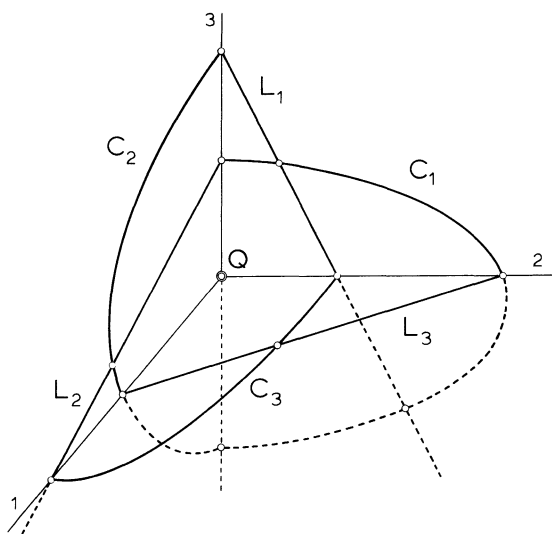


Figure 1

⁶ This transformation (of type $B1$ in the classification of [6]) is similar to T_{33}^{tet} , but the fundamental tetrahedron is replaced by the configuration of the six straight lines of fig. 1. The associated linear system of cubic surfaces is the subfamily \mathfrak{R}_Q of \mathfrak{R} with base set $\Sigma \cup \{Q\}$. The fundamental set of the inverse transformation consists of a twisted cubic through Q_1, Q_2, Q_3 and of the 3 lines D_1, D_2, D_3 joining these points. The twisted cubic – which blows up in \mathbb{P}^3 to the unique quadric containing the L_i (and hence the E_i) – meets V residually in R_1, R_2 and R_3 . Each line D_i of V (although it is a fixed component of the linear system) is blown down on W to the point $C_j \cap C_k$. This is because a cubic surface with two nodes Q_i, Q_k has the property that all the non-singular points on $D_i = \langle Q_i, Q_k \rangle$ have one and the same tangent plane. Hence if any element of the linear system defining Ω^{-1} (which automatically inherits a node at each Q_i) intersects V with multiplicity 2 at some non-singular point of D_i , the intersection must contain the whole line D_i with multiplicity 2. Some further details will be found in [6], but it is much simpler to use none of these global properties and to work directly on W with the map Ω . As an exercise, the reader may try to reprove proposition 1(a), using the transformation Ω^{-1} instead of Φ .

containing all these 19 points is $\rho_1 \cup \rho_2 \cup \rho_3$.) Therefore \mathfrak{N} defines a rational transformation $\Omega : W \rightarrow V \subset \mathbb{P}^3$, where V is a cubic surface. Indeed the degree of V is equal to the number of its intersections with a generic line of \mathbb{P}^3 ; and such a line corresponds to the intersection of 2 cubic surfaces containing Σ . Now, one such cubic surface meets W in a sextic curve Γ_6 , residual to L_1, L_2 and L_3 , and having 4 points on each L_i . Hence another cubic surface meets Γ_6 in 18 points, 12 of which are on a fundamental line. The other 6 are the intersections $C_i \cap C_j$ and 3 movable points, so that the degree of V is equal to 3.

Concerning the separation of points by \mathfrak{N} , we first note that the 3 skew lines E_i , forming the complementary set to the L_i (see remark 1 above), are blown down to 3 points R_i . Each conic C_i is also blown down to a single point Q_i . Let now $W_0 = W - \bigcup C_i - \bigcup E_i$. We claim that Ω separates points and infinitely near points on W_0 . Indeed the lines L_i are fixed for the surfaces of \mathfrak{N} , but – by general theory (see e.g. [16, §I.3]) – this does not affect the transformation Ω , since we can delete any fixed components of the family of divisors cut out by \mathfrak{N} . Now, to show that one such reduced divisor does *not* go through a point R of L_i , it suffices to show that R is *simple* on the intersection of W with a cubic surface of \mathfrak{N} defining that divisor. Similarly for infinitely near points. It is now a simple matter to check that all points and infinitely near points on W_0 can be separated by means of triplets of planes containing L_1, L_2 and L_3 respectively⁷.

Finally we check that $Q_1 = \Omega[C_1]$ is a double point of V : a line through Q_1 is the image of the intersection of 2 cubic surfaces containing C_1 . The trace on W of one such cubic surface consists of the 3 lines L_i , the conic C_1 and a quartic curve Γ_4 going through $C_2 \cap C_3$. It is easy to see that $\Gamma_4 \cdot (L_1 + L_2 + L_3 + C_1) = 10$, so that the second cubic surface meets Γ_4 in $12 - 10 - 1 = 1$ movable point. Consequently, an arbitrary line through Q_1 meets the surface V in only one other point, and Q_1 is a double point of V , as asserted. We further note that the Q_i define a unique plane, which corresponds to the unique divisor $(\omega) = (\rho_1 \cup \rho_2 \cup \rho_3)$ that contains all the C_i . Hence they are not

⁷ For completeness we give a brief sketch of the argument: let $R, S \in W_0$, with $R \neq S$. First assume $R \notin L_i$ for any i . Then, if $\tau_i = \langle L_i, R \rangle$, we see that $\bigcap \tau_i = \{R\}$. Otherwise, $\bigcap \tau_i$ would be a line E meeting all three L_i . But E also contains the point R ; hence $E \subset W$ and E is one of the E_i ; a contradiction, since $R \notin E_i$. Therefore we may assume without loss of generality that $S \notin \tau_1$. Consider $\omega = \tau_1 \cup \rho_2 \cup \rho_3$. Then $R \in \tau_1 \subset \omega$, but $S \notin \omega$, unless $S \in L_2 \cup L_3$ ($S \in \rho_2 \Rightarrow S \in L_2 \cup C_2 \Rightarrow S \in L_2$). In the latter case, however, (ω) contains S as a *simple* point, which is all we need to prove! Finally, if $R \in L_1$ (say), we can use exactly the same argument, but τ_1 is replaced by the tangent plane to W at R , and we must take care not to forget multiplicities. The proof for infinitely near points is almost identical.

collinear, and this implies that the cubic surface V is normal! It follows that the points $R_i = \Omega[E_i]$ are simple, because the E_i are exceptional divisors of the first kind and we may apply theorem II.2.3 of [16]. The points $S_i = C_j \cap C_k$ are blown up to the 3 lines $D_i = \langle Q_j, Q_k \rangle$, as may be seen – for instance – by repeating the argument of lemma 2. The only difference is that, this time, the D_i are not skew, since we cannot separate the two tangent vectors $\vec{x}_i \in T_{S_j}C_k$ and $\vec{x}_j \in T_{S_j}C_k$, which both correspond to the point Q_k . The Zariski Main Theorem (used to prove the third assertion of the lemma) still applies, but of course it does not imply that the Q_i are non-singular, since the fibres above these points are infinite!

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REFERENCES

- [1] E. BERTINI: *Introduzione alla Geometria Proiettiva degli Iperspazi*. (G. Principato, Messina; 1923).
- [2] F. CHÂTELET: Points rationnels sur certaines surfaces cubiques. *Coll. Int. CNRS, Clermont-Ferrand* (1964) 67–75.
- [3] J-L. COLLIOT-THÉLÈNE: Quelques propriétés arithmétiques des surfaces rationnelles. *Sém. de Théorie des Nombres, Bordeaux* (1971–72) exposé 13; 22p.
- [4] D. F. CORAY: Algebraic points on cubic hypersurfaces. *Acta Arithmetica*, Vol. 30 (to appear).
- [5] F. ENRIQUES & O. CHISINI: *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, vol. II. (N. Zanichelli, Bologna; 1918).
- [6] H. P. HUDSON: On the 3-3 birational transformation in three dimensions. *Proc. London Math. Soc.*, 9 (1911) 51–66.
- [7] Y. MANIN: *Cubic forms*. (North-Holland, Amsterdam; 1974).
- [8] I. R. ŠAFAREVIČ: *Basic Algebraic Geometry*. (Springer-Verlag; 1974).
- [9] B. SEGRE: On arithmetical properties of singular cubic surfaces. *J. London Math. Soc.*, 19 (1944) 84–91.
- [10] B. SEGRE: Alcune osservazioni sulle superficie cubiche nel campo razionale. *Rendiconti Accad. Naz. Lincei*, 20 (1956) 147–149.
- [11] J. G. SEMPLE & L. ROTH: *Introduction to Algebraic Geometry* (Clarendon Press, Oxford; 1949).
- [12] TH. SKOLEM: Einige Bemerkungen über die Auffindung der rationalen Punkte auf gewissen algebraischen Gebilden. *Math. Zeitschrift*, 63 (1955) 295–312.
- [13] H. P. F. SWINNERTON-DYER: Applications of Algebraic Geometry to Number Theory. *Proc. Symp. AMS*, 20 (1971) 1–52.
- [14] B. L. VAN DER WAERDEN: *Einführung in die algebraische Geometrie*. (Springer-Verlag; 1939).

[15] O. ZARISKI: *Algebraic Surfaces*. (Springer-Verlag; 1935, 1971).

[16] O. ZARISKI: Introduction to the problem of minimal models in the theory of algebraic surfaces. (*Publ. Math. Soc. Japan* ; 1958 / *Collected Papers II*, p. 277–369).

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