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1. Introduction

A group G is called a *locally normal group*, if each finite set of elements of G lies in a finite normal subgroup of G. If H is a subgroup of a locally normal group G, we write $\mathrm{Cl}_G H$ and $\mathrm{Lcl}_G H$ for the conjugacy class and local conjugacy class respectively of H in G. A number of authors have proved theorems asserting that, under various hypotheses, $\mathrm{Cl}_G H = \mathrm{Lcl}_G H$ if and only if $\mathrm{Cl}_G H$ is finite (see [4], [6], [7], [8], [9], and [10]). Most, if not all, of these theorems either have hypotheses which visibly imply that G is a homomorphic image of a residually finite locally normal group, or can be readily reduced to that case by arguments like that of [8] Theorem 4.1. Let us for brevity write $\mathfrak{X}$ for the class of all groups which are homomorphic images of residually finite locally normal groups. For $\mathfrak{X}$-groups, Gorčakov has proved the following result:

THEOREM 1: (Gorčakov [1]) *Let H be a subgroup of an $\mathfrak{X}$-group G, and suppose that $|H/H_G|$ is an infinite cardinal α. Then $|\mathrm{Cl}_G H| = \alpha$ and $|\mathrm{Lcl}_G H| = 2^\alpha$.*

Here we have written $H_G = \bigcap_{x \in G} H^x$ for the largest normal subgroup of G contained in H; we also write H^G for the normal closure of H in G. It follows from Theorem 1 that, if $|H/H_G|$ is infinite, then $\mathrm{Cl}_G H \neq \mathrm{Lcl}_G H$. On the other hand, if $|H/H_G|$ is finite, then H^G/H_G is finite, and since the conjugates and local conjugates of H lie between H_G and H^G, $\mathrm{Cl}_G H$ and $\mathrm{Lcl}_G H$ are finite. In fact, it is easy to see that they coincide (cf. [8] Theorem 3.1). Thus Gorčakov's Theorem has the immediate

COROLLARY: *If $G \in \mathfrak{X}$ and $H \leqq G$, then $\mathrm{Cl}_G H = \mathrm{Lcl}_G H$ if and only if $\mathrm{Cl}_G H$ is finite.*

and most of the other known results about the equality of $\mathrm{Cl}_G H$ and $\mathrm{Lcl}_G H$ can be deduced from this as we have explained.

We give in this paper an alternative proof of Gorčakov's Theorem. In fact, Gorčakov actually proves his theorem ([1] Theorem 4) under the hypothesis that G is a homomorphic image of a subgroup of a direct product of finite groups, but also proves ([1] Theorem 2) that every $\mathfrak{X}$-group is such an image. The outcome is Theorem 1 as we have stated it. However, we shall not appeal to Gorčakov's description of $\mathfrak{X}$-groups, but proceed directly to prove a result which on the face of it is more general. Let $\mathfrak{Y}$ denote the class of all locally normal groups L satisfying the following condition:

$$(1) \qquad |L : C_L(T)| \leqq |T| \quad \text{for every infinite subset } T \text{ of } L.$$

Clearly, the same class is obtained if the word 'subset' is replaced by 'normal subgroup'. Also, for a subgroup H of a locally normal group G, write $\alpha = \alpha(H) = |H/H_G|$, $\beta = \beta(H) = |\mathrm{Cl}_G H| = |G : N_G(H)|$, $\bar{\beta} = \bar{\beta}(H) = |\mathrm{Lcl}_G H|$. We prove

THEOREM 1′: *Let $G \in \mathfrak{Y}$ and $H \leqq G$. Suppose that α is infinite. Then $\beta = \alpha$ and $\bar{\beta} = 2^\alpha$.*

Theorem 1 follows from this and

PROPOSITION 1: $\mathfrak{X} \leqq \mathfrak{Y}$,

which is immediate from Gorčakov's result [1] that $\mathfrak{X}$-groups are homomorphic images of subgroups of direct products of finite groups, but for which we give an independent proof in Section 2.

In connection with Theorem 1′, notice the following straightforward and well known fact, which can be established by the considerations of [8] Theorem 3.1:

PROPOSITION 2: *If one of α, β, $\bar{\beta}$ is finite, then all are, and in that case, $\beta = \bar{\beta}$.*

We do not consider this situation further, but go on to consider the relation between α, β and $\bar{\beta}$ in a general locally normal group, when α is assumed infinite. Of course, if G is any locally normal group with center Z, then G/Z is residually finite, and so in a sense G departs only slightly from residual finiteness. But this departure seems to make the behaviour and discussion of α, β and $\bar{\beta}$ quite a bit more complicated. We obtain without much difficulty:

PROPOSITION 3: *If α is infinite, then*
(i) $\alpha \leqq 2^{\beta}$
(ii) $\beta \leqq \bar{\beta} \leqq 2^{\alpha} \leqq 2^{2^{\beta}}$.

This is proved in Section 2.

Thus, if we assume the Generalized Continuum Hypothesis, then α and β must satisfy one of the equations $\alpha = \beta$, $\alpha = 2^{\beta}$, $\beta = 2^{\alpha}$, and β and $\bar{\beta}$ must satisfy one of the equations $\bar{\beta} = \beta, 2^{\beta}, 2^{2^{\beta}}$. We shall show by an example (Section 4) that, for arbitrary infinite α, each of these possibilities occurs. It seems at first rather surprising that $\bar{\beta}$ can be as large as $2^{2^{\beta}}$.

Next we come to the relation between α and $\bar{\beta}$. This is much closer, and is the source of our main theorem:

THEOREM 2: *If H is a subgroup of a locally normal group G and $\alpha(H)$ is infinite, then $\bar{\beta}(H) = 2^{\alpha(H)}$.*

This is an immediate consequence of Theorem B of Tomkinson [9] in the case $\alpha = \aleph_0$, and indeed we shall use his theorem in our proof. However, we have not been able to extend Tomkinson's methods to the general case, and have had to proceed much more indirectly. Let Z be the centre of G. An immediate reduction using Theorem 1 allows us to assume that $HZ \lhd G$. We next deal with the case when HZ/Z is central in G/Z. In that case, we use some of the mildly topological methods and results of [3], the argument depending ultimately on knowing the cardinal of the completion of an abelian group equipped with a Hausdorff topology in which certain subgroups of finite index form a basis of neighborhoods of the identity – an abelian cofinite group in the terminology of [3]. When occasion demands, we shall use freely the notation of [3]. The case when HZ/Z is central in G/Z seems to be the crucial one, and the rest of the argument deduces the general case from it by purely group theoretic means. The proof of Theorem 2 is given in Section 5, and that of Theorem 1′ in Section 3. Section 2 contains some preliminary results, most of which are in essence well known.

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2. Preliminary results

We begin by proving Propositions 1 and 3 of the Introduction.

Proof of Proposition 1: We have a homomorphic image G of a residually finite locally normal group, and have to show that $G \in \mathfrak{Y}$. By (1) of the Introduction, we have to show that $|G: C_G(T)| \leqq |T|$ for each infinite subset T of G. Since G is locally normal, $|T| = |T^G|$, and so we may assume that T is a normal subgroup of G.

We first consider the case when G itself is residually finite and locally normal. The set of all finite normal subgroups of G contained in T is a set of cardinal $|T|$. Since G is residually finite, we may choose, for each such finite normal subgroup F, a normal subgroup $N(F)$ of G, contained as a subgroup of finite index in T, such that $F \cap N(F) = 1$. We may then choose a finite normal subgroup $A(F)$ of G, contained in T, such that $A(F)N(F) = T$.

For each $g \in G$, let $F(g) = g^G \cap T$, where g^G is the normal closure of $\langle g \rangle$ in G, and let $\sigma(g)$ denote the automorphism induced on $A(F(g))$ by conjugation with g. The map $\theta: g \to (F(g), \sigma(g))$ has as image a set of cardinal at most $|T|$, since, for each of the possible subgroups $F(g)$, there are only finitely many possibilities for $\sigma(g)$. But if $g, g' \in G$ and $\theta(g) = \theta(g')$, then $F(g) = F(g')$, $N(F(g)) = N(F(g'))$, $A(F(g)) = A(F(g'))$, and $\sigma(g) = \sigma(g')$. Now $[g^G, N(F(g))] \leqq F(g) \cap N(F(g)) = 1$, and so g and g' centralize $N(F(g)) = N(F(g'))$. The automorphisms induced on T by g and g' are therefore completely determined by their restrictions to $A(F(g)) = A(F(g'))$, that is, by $\sigma(g) = \sigma(g')$. Hence $g'g^{-1} \in C_G(T)$. Thus θ is injective on any set of coset representatives of $C_G(T)$ in G, and $|G: C_G(T)| \leqq |T|$.

To deal with the general case, suppose $G = L/M$, where L is a residually finite locally normal group and $M \lhd L$. Then $T = SM/M$ for some normal subgroup S of L such that $|S| = |T|$. By what we have just seen, $|L: C| \leqq |S|$, where $C = C_L(S)$. Since CM/M centralizes T and has index at most $|S| = |T|$ in G, we have $|G: C_G(T) \leqq |T|$, as required.

Proof of Proposition 3: We have a subgroup H of a locally normal group G, with $\alpha = |H/H_G|$, $\beta = |\mathrm{Cl}_G H|$, $\bar{\beta} = |\mathrm{Lcl}_G H|$, and have to assume α infinite.

(i) $\alpha \leqq 2^\beta$. To see this, let S be a transversal to $N_G(H)$ in G. Then $|S| = \beta$. Let $s \in S$. Then as G is locally normal and $C_H(s) \leqq H \cap H^s$, we have $|H: H \cap H^s| < \infty$. Hence $H \cap H^s$ contains a normal subgroup $H(s)$ of H, of finite index in H, and containing H_G. Clearly $H_G = \bigcap_{s \in S} H(s)$, and so H/H_G can be embedded in the complete direct product of the β finite groups $H/H(s)$. Hence $|H/H_G| \leqq 2^\beta$.

(ii) $\beta \leqq \bar{\beta} \leqq 2^\alpha \leqq 2^{2^\beta}$. Trivially, $\beta \leqq \bar{\beta}$. Since G is locally normal and $|H/H_G| = \alpha$, we have $|H^G/H_G| = \alpha$. Since local conjugates of H lie between H_G and H^G, $\bar{\beta} \leqq 2^\alpha$. The last inequality is immediate from (i).

The remaining results in this section are technical facts needed for the proofs of Theorems 1′ and 2.

LEMMA 2.1 (Gorčakov [1]): *Let G be a locally normal group, and suppose that* $\{N_\lambda : \lambda \in \Lambda\}$ *is a set of normal subgroups of finite index of G such that* $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. *Suppose that* $|\Lambda|$ *is an infinite cardinal* α. *Then* $|G : Z(G)| \leqq \alpha$.

PROOF: The set of all intersections of finitely many N_λ may be indexed by a set of cardinal α, and so we may assume that the intersection of any two of the N_λ contains a third. For each $\lambda \in \Lambda$, let

$$H_\lambda = C_G(N_\lambda).$$

Then $|H_\lambda : H_\lambda \cap N_\lambda| < \infty$, as $|G : N_\lambda| < \infty$. Clearly $H_\lambda \cap N_\lambda$ is central in N_λ. We have $G = N_\lambda A$ for some finite set A. Now the centralizer of A in $H_\lambda \cap N_\lambda$ is clearly central in G, and has finite index in $H_\lambda \cap N_\lambda$. Thus, if $Z = Z(G)$, $|H_\lambda : H_\lambda \cap Z| < \infty$, and so $|H_\lambda Z/Z| < \infty$. Now every element g of G lies in a finite normal subgroup B of G, and we have $B \cap N_\lambda = 1$, and hence $[B, N_\lambda] = 1$, for some $\lambda \in \Lambda$. Thus $g \in H_\lambda$. It follows that G/Z is the union of the finite normal subgroups $H_\lambda Z/Z (\lambda \in \Lambda)$, and so $|G/Z| \leqq \alpha$.

LEMMA 2.2: *Let G be a locally normal group, let N be an infinite normal subgroup of G of cardinal* α, *and let* $C = C_G(N)$. *Then there exists a normal subgroup D of G such that* $C \leqq D$, $[D, G] \leqq C$, $[D, N] \leqq Z(G)$, *and* $|G : D| \leqq \alpha$.

PROOF: The set of all finite normal subgroups of G contained in N is a set of cardinal α. Clearly $C = \bigcap C_G(F)$, where F ranges over all such finite normal subgroups. Thus, if B/C is the centre of G/C, Lemma 2.1 gives

$$|G : B| \leqq \alpha. \tag{1}$$

On the other hand, if $Z = Z(G)$, then G/Z is well known to be residually finite. By Proposition 1, if $A/Z = C_{G/Z}(NZ/Z)$, we have $|G : A| \leqq \alpha$. Let $D = A \cap B$. From (1), $|G : D| \leqq \alpha$, and $[D, G] \leqq [B, G] \leqq C$, $[D, N] \leqq [A, N] \leqq Z(G)$.

The following result was used by Gorčakov [1]. We include a proof for completeness. Notice that it yields an alternative proof of Proposition 1.

LEMMA 2.3: *Let G be a residually finite locally normal group, and let N be a normal subgroup of G of infinite cardinal α. Then there exists a normal subgroup M of G such that $|G:M| \leqq \alpha$ and $M \cap N = 1$.*

PROOF: For each non-identity element of N, choose a normal subgroup of finite index of G which does not contain that element, and let L be the intersection of the subgroups so obtained. By Lemma 2.1, if S/L is the centre of G/L, $|G:S| \leqq \alpha$. Passing to G/L, we may assume that $|G:Z| \leqq \alpha$, where $Z = Z(G)$.

Let M be a subgroup of Z maximal subject to the condition $M \cap N \cap Z = 1$. Then $M \lhd G$, and if we can show that

$$|Z:M| \leqq \alpha \tag{2}$$

then $|G:M| \leqq \alpha$, and $N \cap M = 1$.

Write $U = Z/M$, and let V be the natural image of $N \cap Z$ in U. Then $|V| \leqq \alpha$, and every non-trivial subgroup of U intersects V non-trivially. Thus, for each prime p, the subgroup of elements of order dividing p in U lies in V, and so has cardinal at most α. The Sylow p-subgroup of U therefore also has cardinal at most α, and hence $|U| \leqq \alpha$, establishing (2).

Finally we recall the following facts about locally inner automorphisms:

LEMMA 2.4. (Stonehewer [5]): *Let G be a locally normal group, let $A \leqq G$ and $B \lhd G$. Then*

(i) *Every locally inner automorphism of A can be extended to a locally inner automorphism of G.*

(ii) *Every locally inner automorphism of G/B is naturally induced by a locally inner automorphism of G.*

An alternative proof can be given by viewing the group of locally inner automorphisms of a locally normal group as the completion of the group of inner automorphisms in its natural cofinite topology, as in [3]. However, since this seems to be a complication rather than a simplification at present, we shall not pursue it further.

3. Proof of Theorem 1′

Let $G \in \mathfrak{Y}$ and $H \leqq G$. The hypothesis is that $\alpha = |H/H_G|$ is infinite, and we have to show that $\beta = \alpha$ and $\bar{\beta} = 2^{\alpha}$.

We have $H = H_G S$ for some set S of cardinal α, and the definition of $\mathfrak{Y}$

gives $|G: C_G(S)| \leqq \alpha$. Since $C_G(S) \leqq N_G(H)$, $\beta = |G: N_G(H)| \leqq \alpha$. On the other hand, $G = N_G(H)T$ for some subset T of cardinal β. Since $G \in \mathfrak{Y}$, $|G: C_G(T)| \leqq \beta$, and since $H_G \geqq C_H(T)$, $\alpha = |H: H_G| \leqq \beta$. Thus, we have proved that

$$(1) \qquad \alpha = \beta$$

From Proposition 3, stated in the Introduction, we have $\bar{\beta} \leqq 2^\alpha$. To establish the reverse inequality, we consider families

$$(2) \qquad \{N_\lambda : \lambda \in \Lambda\}$$

of finite normal subgroups N_λ of G, satisfying the conditions

$$(3) \qquad [N_\lambda, N_\mu] = 1 (\lambda \neq \mu; \lambda, \mu \in \Lambda).$$

(4) *For each* $\lambda \in \Lambda$, *there exists an element* $x_\lambda \in N_\lambda$ *such that* $(H \cap N_\lambda)^{x_\lambda} \neq H \cap N_\lambda$.

It follows easily from these conditions that the N_λ are all distinct. The collection of all such families may be partially ordered by set-theoretic inclusion, and Zorn's Lemma allows us to choose a maximal family, say (2).

We shall show that $|\Lambda| \geqq \alpha$. If this is not so, then writing $N = \prod_{\lambda \in \Lambda} N_\lambda$, we have $|N| < \alpha$. Since $G \in \mathfrak{Y}$, $|G: C| < \alpha$, where $C = C_G(N)$. Now if $H \cap C \lhd C$, then $\beta(H \cap C) < \alpha$, and hence, by (1) applied to $H \cap C$, we obtain $|H \cap C: (H \cap C)_G| < \alpha$. Since $H_G \cap C \lhd G$, we conclude that $|H \cap C: H_G \cap C| < \alpha$, and since $|H: H \cap C| < \alpha$, that $|H: H_G \cap C| < \alpha$. Finally, we obtain $|H: H_G| < \alpha$, a contradiction. Therefore, $H \cap C$ is not normal in C. Let y be an element of C and h an element of $H \cap C$ such that $h^y \notin H \cap C$, and let M be a finite normal subgroup of G contained in C and containing $\langle h, y \rangle$. Then $(H \cap M)^y \neq H \cap M$, and M centralizes each N_λ since $M \leqq C = C_G(N)$. Thus, we may adjoin M to (1) to obtain a larger such family, a contradiction.

We have shown that $|\Lambda| \geqq \alpha$. It is now convenient, though not strictly necessary, to introduce a completion $\bar{G}$ of G in any cofinite topology of G (see [3]). Thus $\bar{G}$ is a compact topological group containing G as a normal dense subgroup, and the automorphisms induced on G by $\bar{G}$ are precisely its locally inner automorphisms. Let M be any subset of Λ, and for each finite subset Φ of Λ, let $\bar{G}(\Phi)$ denote the set of all elements $g \in \bar{G}$ such that

$$(N_\lambda \cap H)^g = (N_\lambda \cap H)^{x_\lambda} \qquad \text{if } \lambda \in M \cap \Phi,$$

$$(N_\lambda \cap H)^g = N_\lambda \cap H \qquad \text{if } \lambda \in \Phi \backslash M.$$

Then $\bar{G}(\Phi)$ contains the element $\prod_{\lambda \in M \cap \Phi} x_\lambda$ and so is not empty; furthermore it is an intersection of cosets of the closed subgroups $N_{\bar{G}}(N_\lambda \cap H)$ $(\lambda \in \Phi)$, and so is a closed subset of $\bar{G}$. The intersection of any finite number of sets $\bar{G}(\Phi)$ clearly contains another such set, and as $\bar{G}$ is compact, there exists an element $x = x_M$ in their intersection. Write $H_M = H^x$. Then $H_M \cap N_\lambda = H^x \cap N_\lambda = (H \cap N_\lambda)^x$ is equal to $(H \cap N_\lambda)^{x_\lambda}$ or $H \cap N_\lambda$ according as $\lambda \in M$ or $\lambda \notin M$. Therefore the subgroups H_M are all distinct. By Theorem 5.5 and Lemma 3.1 of [3], they are all locally conjugate to H in G. Since $|\Lambda| \geqq \alpha$, there are at least 2^α of them. Thus, $\bar{\beta} \geqq 2^\alpha$, and the proof of Theorem 1′ is complete.

4. An example

We notice first that the possibility $\alpha = \alpha(H) = \beta(H)$, $\bar{\beta}(H) = 2^{\beta(H)}$ certainly occurs for arbitrary infinite α. For example, take G to be the direct product of α copies of the symmetric group of degree 3 and H to be a Sylow 2-subgroup of G.

The example illustrating the other possibilities is essentially a version of the 'infinite extra-special p-group' first introduced by P. Hall in [2] for $p = 2$ and presented in a different way for arbitrary p by Tomkinson [8]. It will be convenient for us to give a third description of these groups.

Let A, B, C be three additive abelian groups, and suppose we are given a bilinear map of $A \times B$ into C. In anticipation of what is to come, we denote the map by $(a, b) \to [a, b]$. We define a binary operation on the set $A \times B \times C$ by

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' - [a', b]).$$

A routine verification shows that $A \times B \times C$ becomes a group with identity $(0, 0, 0)$. The set C_0 of all elements $(0, 0, c)$ with $c \in C$ is a central subgroup of G with abelian factor group. Thus G is nilpotent of class at most 2.

We now assume that the bilinear map is *non-degenerate*, in the sense that, for $a \in A$, $b \in B$,

(1) $$[a, B] = 0 \Leftrightarrow a = 0,$$

(2) $$[A, b] = 0 \Leftrightarrow b = 0.$$

Then it is easy to verify that C_0 is precisely the centre of G. We now identify A, B and C with subgroups of G via the maps $a \to (a, 0, 0)$, $b \to (0, b, 0)$, $c \to (0, 0, c)$ respectively. Then $[a, b]$ becomes the commutator of a and b, and $G = ABC$. We have $[G, A_G] \leqq A \cap C = 1$ since G/C is abelian, and so $A_G \leqq A \cap Z(G) = A \cap C = 1$. Arguing similarly with B, we have

$$A_G = B_G = 1. \tag{3}$$

The subgroup AC is abelian, and so $N_G(A) \geqq AC$. However

$$[B \cap N_G(A), A] \leqq BC \cap A = 1$$

and by (2), we obtain $B \cap N_G(A) = 1$. Since $G = ABC$, we obtain $N_G(A) = AC$. Arguing similarly with B gives

$$N_G(A) = AC, \qquad N_G(B) = BC. \tag{4}$$

In general G will not be an FC-group. Since $G = ABC$ and C is central, G will be an FC-group if and only if each element of $A \cup B$ has centralizer of finite index. In terms of the bilinear map, this says that

(5) *If $a \in A$, $b \in B$, then there exist subgroups X, Y of finite index in A and B respectively, such that $[X, b] = [a, Y] = 0$.*

This will certainly be satisfied if C is finite, since for fixed $a \in A$, the map $b \to [a, b]$ is a homomorphism of B into C, and similarly with the roles of A and B interchanged.

We now introduce particular choices for A, B, C. Let λ be an arbitrary infinite cardinal, let p be a prime, and let A be an elementary abelian p-group of rank λ. Let $C = \boldsymbol{Z}_p$ be an additive group of order p, and $B = A^* = \operatorname{Hom}(A, \boldsymbol{Z}_p)$. Then, if we write $[a, b]$ for the image of $a \in A$ under $b \in B$, (1), (2) and (5) hold, and so we obtain an FC-group G as above. Since G is a p-group it is locally normal, of nilpotency class 2 and with centre of order p.

Now $|B| = 2^\lambda$. Thus, from (3) and (4), we obtain

$$\alpha(A) = \lambda, \qquad \beta(A) = |B| = 2^\lambda. \tag{6}$$

$$\alpha(B) = 2^\lambda, \qquad \beta(B) = |A| = \lambda. \tag{7}$$

Thus all pairs (α, β) allowed by Proposition 3 and the Generalized Continuum Hypothesis can occur.

We next calculate $\bar{\beta}(A)$ and $\bar{\beta}(B)$. Now $AC \lhd G$ and $|AC| = \lambda$. Since AC contains every local conjugate of A, $\bar{\beta}(A) \leqq 2^{\lambda}$. But $\beta(A) = 2^{\lambda}$, by (6). Hence

$$\beta(A) = \bar{\beta}(A) = 2^{\lambda}. \tag{8}$$

It is in fact not difficult to see that every complement to C in AC is conjugate to A in G, so that the stronger result $\mathrm{Cl}_G A = \mathrm{Lcl}_G A$ holds (cf. [8] p. 210). However, this does not concern us here.

Now let $A^{**} = \mathrm{Hom}(A^*, \boldsymbol{Z}_p)$, and let L be the group obtained by the above construction from the triple $(A^{**}, A^*, \boldsymbol{Z}_p)$. Then A can be embedded in A^{**} in the usual way via the map χ (cf. [3] § 3), which assigns to an element $a \in A$ the element $\tau \to \tau(a)$ $(\tau \in A^*)$ of A^{**}, and the map $(a, b, c) \to (\chi(a), b, c)$ $(a \in A,\ b \in B = A^*,\ c \in C)$ clearly embeds G as a normal subgroup of L. Since $|A^{**}| = 2^{2^{\lambda}}$, the considerations used above show that $|\mathrm{Cl}_L B| = 2^{2^{\lambda}}$. We claim that the elements of L induce by conjugation locally inner automorphisms of G; from this it will follow that

$$\beta(B) = \lambda, \qquad \bar{\beta}(B) = 2^{2^{\lambda}} \tag{9}$$

so that all pairs $(\beta, \bar{\beta})$ allowed by Proposition 3 and the Generalized Continuum Hypothesis can occur.

Let D be the subgroup of L corresponding to A^{**}. Since the set of locally inner automorphisms of G is a group and $L = DG$, it will follow that L induces locally inner automorphisms on G if we can show that D does so. Let F_1 and F_2 be finite subgroups of A and B respectively, and $F = F_1 F_2 C$. Then F is a finite normal subgroup of G, and every finite set of elements of G lies in such an F. Let $d \in D$. Then d centralizes $F_1 C$, and commutation with d induces a homomorphism φ of F_2 into C. We have to show that φ can be induced by commutation with an element $a \in A$. Then a and d induce the same automorphism on F.

Now in the bilinear map notation, F_2 is a finite subgroup of A^*. Let E be its annihilator in A and let $E^{\perp}$ be the annihilator of E in A^*. Then $E^{\perp}$ is naturally isomorphic to $\mathrm{Hom}(A/E, \boldsymbol{Z}_p)$, and F_2 is a subgroup of $E^{\perp}$ which has zero annihilator in A/E. By the duality theory of finite abelian groups (or of finite dimensional vector spaces), $F_2 = E^{\perp}$, and every element of $\mathrm{Hom}(F_2, \boldsymbol{Z}_p)$ is naturally induced by an element of A/E, that is, by some $\chi(a)$ with $a \in A$. In particular, φ is so induced, and a is the required element.

We have thus shown that L induces locally inner automorphisms on G. In fact, it is not hard to see that L induces on G its full group of locally inner automorphisms, though we shall not require that fact here.

5. Proof of Theorem 2

Throughout this section, H denotes a subgroup of a locally normal group G, $\alpha = |H/H_G|$ is infinite, and $\bar{\beta} = |\mathrm{Lcl}_G H|$. We have to show that $\bar{\beta} = 2^\alpha$. We begin by stating the following immediate consequence of Lemma 2.4:

LEMMA 5.1: *Let B be a subgroup of a locally normal group A, $X \leqq A$, $Y \lhd A$. Then*

(i) $|\mathrm{Lcl}_X(B \cap X)| \leqq |\mathrm{Lcl}_A B|$.

(ii) $|\mathrm{Lcl}_Y(B \cap Y)| \leqq |\mathrm{Lcl}_A(B \cap Y)| \leqq |\mathrm{Lcl}_A B|$.

(iii) $|\mathrm{Lcl}_{A/Y} BY/Y| \leqq |\mathrm{Lcl}_A B|$.

The next lemma establishes a special case of Theorem 2, and in fact provides the key to the general situation.

LEMMA 5.2: *If $[H, G, G] = 1$, then $\bar{\beta} = 2^\alpha$.*

PROOF: Let L be the group of locally inner automorphisms of G and A the group of inner automorphisms of G. We may take the set of centralizers $C_L(F)$, where F ranges over the finite normal subgroups of G, as a basis of neighborhoods of 1 inducing on L the structure of a Hausdorff topological group. Then L is a cofinite group in the sense of [3], and by [3], Proposition 5.6, L is compact and A is dense in it. In other words, L is a completion of A in the topology which the latter inherits from L.

Now in proving Lemma 5.2, we may assume by Lemma 5.1 (iii), that $H_G = 1$. Writing $Z = Z(G)$, we then have $H \cap Z = 1$, and so

$$\langle H, Z\rangle = H \times Z. \tag{1}$$

Now every inner automorphism of G is trivial on HZ/Z and Z, and hence so is every locally inner automorphism. Thus $[H, L] \leqq Z$, $[Z, L] = 1$. Therefore, for $b \in L$, the map $\psi(b): h \to [h, b] = h^{-1}h^b$ is a homomorphism of H into Z, and ψ is a homomorphism of L into the additive group $H^* = \mathrm{Hom}(H, Z)$, consisting of all abstract group homomorphisms of H into Z.

Let U denote the image of ψ, and for each finite subgroup E of H, let

$$E^\perp = E_U^\perp = \{\theta \in U : \theta(e) = 1 \text{ for all } e \in E\}.$$

If F is a finite normal subgroup of G containing E, then F is L-invariant, $L/C_L(F)$ is finite, and $\psi(C_L(F) \leqq E^\perp$. Therefore $E^\perp$ has finite index in U.

The subgroups $E^{\perp}$ form a separating filter base of subgroups of finite index in U, and induce on it a cofinite topology in the sense of [3]. The fact that $\psi(C_L(F)) \leqq E^{\perp}$ as above shows that ψ is continuous with respect to the topologies of L and U. Since L is compact and U is Hausdorff, ψ is a closed map, and so maps the closure L of A onto the closure $\bar{T}$ of $T = \psi(A)$. Thus $U = \bar{T}$. We have

$$(2) \qquad \psi(A) = T, \qquad \psi(L) = \bar{T}.$$

Furthermore, $\bar{T}$ is compact, being the continuous image of the compact space L. Hence $\bar{T}$ is a completion of the cofinite group T (cf. [3]), in the topology induced on T by $\bar{T}$.

Let $b \in L$, $h \in H$. Then $h^b = h[h, b] = h \cdot (\psi(b))(h)$, and it follows easily that, for $b, b' \in L$, $H^b = H^{b'}$ if and only if $\psi(b) = \psi(b')$. Thus, from (2),

$$(3) \qquad |\mathrm{Lcl}_G H| = |\bar{T}|.$$

We now need to assemble enough information about T to be able to read off the cardinal of its completion $\bar{T}$ from [3] Corollary 3.6.

Now by assumption, $H_G = 1$. Therefore, if $1 \neq h \in H$, there exists an element $a \in A$ such that $h \notin H^a$. Since $h^a = h \cdot (\psi(a))(h)$, this means that $(\psi(a))(h) \neq 1$. Therefore

(4) *T separates the points of H,*

in the sense that if $1 \neq h \in H$, then there exists an element $\tau \in T$ such that $\tau(h) \neq 1$.

We also have:

(5) *If $\tau \in T$, then $\tau(H)$ is finite.*

For let α be an element of A such that $\psi(\alpha) = \tau$, and let a be an element of G inducing the inner automorphism α. Then, if $h \in H$, $(\psi(\alpha))(h) = [h, \alpha] = [h, a]$. But $[H, a]$ is certainly finite as G is locally normal.

We need to recall that

$$(6) \qquad |H| = \alpha.$$

Now since $\bar{T}$ is a completion of the cofinite group T, we obtain from [3] Corollary 3.6 that $|\bar{T}| = 2^{\sigma}$, where σ is the cardinal of the set of open subgroups of T. In view of (3), the proof of the lemma may be completed by showing that $\sigma = \alpha$. We now deduce this from (4), (5) and (6).

Since each open subgroup of T has finite index, the number of open subgroups of T is equal to the number of subgroups in any basis of meighborhoods of 0 in T. We therefore calculate the number of subgroups $E_T^{\perp} = \{\tau \in T : \tau(E) = 0\} = T \cap E_U^{\perp}$, as E ranges over the finite subgroups of H; these form a basis of neighborhoods of 0 in T.

For each $h \in H$, let $\chi(h)$ be the element $\tau \to \tau(h)\,(\tau \in T)$ of $T^* = \mathrm{Hom}(T, Z)$. Then χ is a homomorphism of H into the additive group T^*. If $1 \neq h \in H$, then by (4), there exists $\tau \in T$ such that $\tau(h) \neq 1$. Thus $(\chi(h))(\tau) \neq 1$, and $\chi(h) \neq 0$. Hence χ is injective, and allows us to identify H with a subgroup of T^*. Then

$$E_T^{\perp} = \bigcap_{\varphi \in E} \ker \varphi \tag{7}$$

for each finite $E \leqq H$.

For each finite $E \leqq H$, define

$$X(E) = \{\varphi \in H : \ker \varphi \geqq E_T^{\perp}\},$$

H being identified with a subgroup of T^* as above.

We claim that $X(E)$ is finite. In fact, $E_T^{\perp}$ is a subgroup of finite index of T, and so $T = E_T^{\perp} + D$, for some finite subgroup D of T. Applying (5) to the elements of D in turn, we find that there exists a finite subgroup D_1 of Z such that $\varphi(D) \leqq D_1$ for all $\varphi \in H \leqq T^*$. Since any element of $X(E)$ is clearly determined by its effect on D, restriction to D gives an injective map of $X(E)$ into the finite group $\mathrm{Hom}(D, D_1)$. Thus, $X(E)$ is indeed finite.

Now from (7), $E_T^{\perp}$ is the intersection of the kernels of the members of $X(E)$, and so it follows that, for finite subgroups E_1 and E_2 of H, $(E_1^{\perp})_T = (E_2^{\perp})_T$ if and only if $X(E_1) = X(E_2)$. Further, $E \leqq X(E)$, and so $\bigcup X(E) = H$, as E ranges over all finite subgroups of H. Since $|H| = \alpha$ and each $X(E)$ is finite, the number of sets $X(E)$ is α and hence the number of subgroups $E_T^{\perp}$ is α, since these are in bijective correspondence with the $X(E)$. The proof of Lemma 5.2 is now complete.

The next three lemmas provide further steps towards the general case of Theorem 2. The first is perhaps somewhat unexpected.

LEMMA 5.3: *If* $|G : H| < \alpha$, *then* $\bar{\beta} = 2^{\alpha}$.

PROOF: By Proposition 3, all we need to show is $\bar{\beta} \geqq 2^{\alpha}$. We may assume, by Lemma 5.1 (iii), that $H_G = 1$. Let Z be the centre of G. Then $H \cap Z = 1$, and so $|HZ/Z| = |H| = \alpha$. Let $U/Z = (HZ/Z)_{G/Z}$. If $|HZ/U| = \alpha$, then, as G/Z is residually finite, Theorem 1 and Lemma 5.1 (iii) give

$\bar{\beta} \geqq 2^\alpha$. We may therefore assume that $|HZ:U| < \alpha$. Then $|U/Z| = \alpha$, $|H \cap U| = \alpha$ and $(H \cap U)_G = 1$; also $|\mathrm{Lcl}_G H| \geqq |\mathrm{Lcl}_G(H \cap U)|$ by Lemma 5.1 (ii), and clearly $|G:H \cap U| < \alpha$. Hence, considering $H \cap U$ instead of H, we may assume that

$$(8) \qquad HZ = H \times Z \lhd G.$$

Let B be a normal subgroup of G of cardinal $\gamma < \alpha$ such that $G = HB$, and let $C = C_G(B)$. By Lemma 2.2, there exists a normal subgroup D of G such that $[D, G] \leqq C \leqq D$ and $|G:D| \leqq \gamma < \alpha$. Let $K = H \cap D$. Then $|K| = \alpha$ and $K_G = 1$. Furthermore, from (8) $[K, G] \leqq C \cap HZ = (H \cap C)Z$ as $Z \leqq C$. But $H \cap C = C_H(B) \leqq H_G = 1$. Hence $[K, G] \leqq Z$. Lemma 5.2 now gives $|\mathrm{Lcl}_G K| = 2^\alpha$, and Lemma 5.1 (ii) gives $\bar{\beta} = |\mathrm{Lcl}_G H| \geqq 2^\alpha$, as required.

LEMMA 5.4: *Let X be a normal subgroup of index $<\alpha$ of G, let $K = H \cap X$, and suppose $|K/K_X| < \alpha$. Then $\bar{\beta} = 2^\alpha$.*

PROOF: Let $L = K_X$. Consider the chain of subgroups $H \geqq H_G K \geqq H_G L \geqq H_G$. We have $|H:H_G K| \leqq |H:K| \leqq |G:X| < \alpha$, and $|H_G K:H_G L| \leqq |K:L| < \alpha$. It follows that $|L:L \cap H_G| = |H_G L:H_G| = \alpha$, and since $L_G \leqq H_G \cap L$, that $|L:L_G| = \gamma \geqq \alpha$.

There exists a normal subgroup B of G, of cardinal $< \alpha$, such that $G = XB$. Let $E = LB$. Then as $L \lhd X$, $L_G = \bigcap_{b \in B} L^b = L_E$. Thus $|L/L_E| = \gamma$, while $|E:L| < \gamma$.

By Lemma 5.3, $|\mathrm{Lcl}_E L| = 2^\gamma$. Hence, by Lemma 5.1 (i), $|\mathrm{Lcl}_G L| \geqq 2^\gamma \geqq 2^\alpha$. But if φ is a locally inner automorphism of G, then φ leaves X invariant, and so $L^\varphi = (H^\varphi \cap X)_X$. Therefore distinct local conjugates of L lie in distinct local conjugates of H, and $\bar{\beta} = |\mathrm{Lcl}_G H| \geqq 2^\alpha$. The reverse inclusion resulting from Proposition 3, the lemma is proved.

LEMMA 5.5: *Suppose that H is a p-group and that every p'-element of G normalizes H. Then $\bar{\beta} = 2^\alpha$.*

PROOF: Let Q be the subgroup of G generated by the p'-elements of G, and let P be any Sylow p-subgroup of G containing H. Then $Q \lhd G$ and $G = QP$. Thus, $H_G = H_P$, and since (by Proposition 3) all we have to establish is that $\bar{\beta} \geqq 2^\alpha$, Lemma 5.1 (i) allows us to replace G by P and assume that G is a p-group.

Let Z be the centre of G. Then, as in the first part of the proof of Lemma 5.3, we may assume that $H_G = 1$, so that $H \cap Z = 1$, and that $HZ \lhd G$.

Now in the case $\alpha = \aleph_0$, we have by Proposition 2 that $|G:N_G(H)|$ is infinite, and so we may appeal to Theorem B of Tomkinson [9] to conclude that $\bar{\beta} \geqq 2^\alpha$.

We next consider the case $\alpha > \aleph_0$. Let $1 = Z_0 \leqq Z_1 = Z \leqq Z_2 \leqq \dots$ be the upper central series of G. Since G is a locally normal p-group, $\bigcup_{i=0}^{\infty} Z_i = G$, the union being understood to run over integers $i \geqq 0$. There exists an integer $i \geqq 1$ such that $|Z_i \cap H| < \alpha$, $|Z_{i+1} \cap H| = \alpha$, since $|H| = \alpha$ and $H = \bigcup_j (H \cap Z_j)$. Since $(Z_{i+1} \cap H)_G = 1$, Lemma 5.1 (ii) allows us to assume that $H \leqq Z_{i+1}$.

Now G/Z is residually finite, and so Lemma 2.3 furnishes a normal subgroup X/Z of G/Z, of index $< \alpha$, such that $X \cap H \cap Z_i \leqq Z$. Let $K = H \cap X$. If $|K/K_X| < \alpha$, then Lemma 5.4 gives $\bar{\beta} = 2^\alpha$. But $[K, X] \leqq HZ \cap Z_i \cap X$ (as $HZ \lhd G$ and $H \leqq Z_{i+1}$) $= (H \cap Z_i)Z \cap X = (H \cap Z_i \cap X)Z \cap X = Z \cap X$. Hence $[K, X, X] = 1$. By Lemma 5.2, if $|K/K_X| = \alpha$, we obtain $|\mathrm{Lcl}_X K| = 2^\alpha$. Lemma 5.1 (ii) now yields $\bar{\beta} \geqq 2^\alpha$, as required.

Now we are in a position to complete the proof of Theorem 2 by a construction similar to that used in proving Theorem 1′.

Proof of Theorem 2: We again notice that $|G: N_G(H)|$ is infinite, by Proposition 2, and appeal to Theorem B of Tomkinson [9] to deal with the case $\alpha = \aleph_0$.

We now assume that $\alpha > \aleph_0$. As at the beginning of the proof of Lemma 5.3, writing $Z = Z(G)$, we may assume that $H_G = 1$, $H \cap Z = 1$, and $HZ \lhd G$. We have $(HZ)' = H'$, and so $H' \lhd G$. Hence $H' = 1$, and H is abelian. Since $\alpha > \aleph_0$, there exists a prime p such that the Sylow p-subgroup H_p of H has cardinal α. Since $H_p Z \lhd G$ and $H_p = H \cap H_p Z$, Lemma 5.1 (ii) allows us to replace H by H_p and assume that H is an abelian p-group. Then the normal closure L of H in G lies in the Sylow p-subgroup of HZ and so is an abelian p-group.

We now consider systems of pairs

$$\{(N_\lambda, x_\lambda): \lambda \in \Lambda\}, \tag{9}$$

where N_λ is a finite normal subgroup of G, x_λ is a p'-element of N_λ, and

$$(H \cap N_\lambda)^{x_\lambda} \neq H \cap N_\lambda \qquad (\lambda \in \Lambda), \tag{10}$$

$$[L \cap N_\mu, x_\lambda] = 1 \qquad (\lambda \neq \mu, \lambda, \mu \in \Lambda). \tag{11}$$

It is immediate that the elements x_λ occurring in such a system are all distinct. The set of all such systems may be partially ordered by set theoretic inclusion, and Zorn's Lemma yields the existence of a maximal such system, say the system (9).

We now claim that

$$|\Lambda| = \alpha. \tag{12}$$

Suppose then that this is not so, and let $N = \langle N_\lambda : \lambda \in \Lambda \rangle$, a normal subgroup of cardinal $< \alpha$ of G. Let $C = C_G(N)$. By Lemma 2.2, there exists a normal subgroup D of index $< \alpha$ in G, such that $[D, G] \leqq C \leqq D$ and $[D, N] \leqq Z$. Let $K = H \cap D$. If $|K/K_D| < \alpha$, then Lemma 5.4 yields $\bar{\beta} = 2^\alpha$. Therefore, we may suppose that $|K/K_D| = \alpha$.

Now K is a p-group. If every p'-element of D normalizes K, then Lemma 5.5 gives $|\mathrm{Lcl}_D K| = 2^\alpha$, Lemma 5.1 gives $|\mathrm{Lcl}_G H| \geqq 2^\alpha$, and Proposition 3 gives $\bar{\beta} = 2^\alpha$. Therefore we may assume that there exists a p'-element y of D which does not normalize K. Let M be a finite normal subgroup of G contained in D, containing y, such that $(K \cap M)^y \neq K \cap M$, that is, $(H \cap M)^y \neq H \cap M$. Let $\lambda \in \Lambda$. Since $[D, N] \leqq Z$, we have $[L \cap N_\lambda, y, y] = 1$, and since $L \cap N_\lambda$ is a normal p-subgroup of G and y is a p'-element, $[L \cap N_\lambda, y] = 1$. Furthermore,

$$[L \cap M, x_\lambda, x_\lambda] \leqq [D, G, N] \leqq [C, N] = 1,$$

and hence $[L \cap M, x_\lambda] = 1$ as before. Thus the pair (M, y) can be adjoined to the system (9), contradicting its maximality.

We have now established (10). The remainder of the proof is essentially a verbatim repetition of the last part of the proof of Theorem 1′, and we omit it. Theorem 2 is now established.

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