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PSEUDO-INTERIORS OF HYPERSPACES *

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0. Introduction

In [14], R. M. Schori and J. E. West proved that the hyperspace of the unit interval I (the space of all non-empty closed subsets of I) is homeomorphic to the Hilbert cube $Q = [-1, 1]^{\infty}$. The hyperspace of a compact metric space X is denoted by 2^X; the set of all non-empty subcontinua of X by $C(X)$. Both 2^X and $C(X)$ are metrized by the Hausdorff metric. In [8], D. W. Curtis and R. M. Schori proved that 2^X is homeomorphic to Q iff X is a non-degenerate Peano continuum and that $C(X)$ is homeomorphic to Q iff X is a non-degenerate Peano continuum with no free arcs.

In [2] and [3], R. D. Anderson introduced the notion of Z-set and that of capset or pseudo-boundary respectively. These concepts will be defined in Section 1. However, the Z-sets coincide with those closed subsets of Q which are equivalent under a space homeomorphism to a subset which projects onto a point in infinitely many coordinates; and the capsets of Q are the subsets which are equivalent under a space homeomorphism to $\{x = (x_i)_i |$ for some i, $|x_i| = 1\}$. Both capsets and Z-sets have been extensively studied in infinite-dimensional topology. A pseudo-interior for Q is the complement of a capset and is known to be homeomorphic to separable Hilbert space l_2.

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In this paper the following pseudo-interiors are identified:

1° the collection of (connected) Z-sets in Q for 2^Q and $C(Q)$; and similarly for compact Q-manifolds (Theorems 2.2 and 2.4).

2° the collection of zero-dimensional closed subsets and the collection of Cantor sets for 2^I (Theorem 3.4).

As a corollary to 1° we obtain that the collection of compact subsets of l_2 is homeomorphic to l_2 (Corollary 2.3).

1. Preliminaries

For an arbitrary metric, not necessarily compact space X, we denote the collection of all non-empty compact subsets of X by 2^X and the collection of all non-empty compact subcontinua by $C(X)$. The *Hausdorff distance* $d_H(Y, Z)$ between two compact subsets Y and Z of X is defined as $\inf\{\varepsilon | Y \subset U_\varepsilon(Z) \text{ and } Z \subset U_\varepsilon(Y)\}$, where $U_\varepsilon(Y)$ denotes the ε-neighborhood of Y. If d and d' induce the same topology on X then d_H and d'_H induce the same topology on 2^X and $C(X)$. Every map $f: X \to X$ induces a map $2^f: 2^X \to 2^X$ by $2^f(K) = f(K)$.

As indicated in the introduction, Q is represented as $J^\infty = [-1, 1]^\infty$ with metric $d(x, y) = d((x_i)_i, (y_i)_i) = \sum_i 2^{-i} \cdot |x_i - y_i|$. We define two kinds of projection maps: for $x = (x_i)_i$, $\pi_i(x) = x_i \in J$ and $\bar{\pi}_i(x) = (x_1, \ldots, x_i) \in J^i$. The *pseudo-boundary* $B(Q)$ is the set $\{x = (x_i)_i | \text{ for some } i, |x_i| = 1\}$, which is a dense σ-compact subset of Q; the *pseudo-interior* s is its complement $(-1, 1)^\infty$ and is a dense G_δ and is known to be homeomorphic to separable Hilbert space l_2 ([1]). A closed subset K of Q is i-deficient if $\pi_i(K)$ is a point; K is *infinitely deficient* if K is i-deficient for infinitely many i. A closed subset K of Q is a Z-*set* (in Q) if for every ε there exists a map $f: Q \to Q \backslash K$ with $d(f, \mathrm{id}_Q) < \varepsilon$. This is not the standard definition, which reads: a closed subset K of Q is a Z-set iff for every non-empty homotopically trivial open set O, the set $O \backslash K$ is non-empty and homotopically trivial.[2] The first definition implies the latter: let O be non-empty and homotopically trivial and let K satisfy the first definition of Z-set. We show that $O \backslash K$ has trivial homotopy groups. Suppose that a map $\varphi: S^{n-1} = \partial I^n \to O \backslash K$ is given. Since O is homotopically trivial, there exists an extension $\bar{\varphi}: I^n \to 0$. For any ε there exists an ε^csmall map $f_\varepsilon: Q \to Q \backslash K$. For sufficiently small ε, $f_\varepsilon(\bar{\varphi}(I^n)) \subset O \backslash K$. Since $f_\varepsilon \circ \bar{\varphi} | \partial I^n$ can be chosen arbitrarily close to φ, the convexity of Q shows that there

[2] By 'X is homotopically trivial' we mean that X is arcwise connected and all homotopy groups of X in positive dimensions vanish. For ANR's this coincides with contractibility.

is also an extension $\bar{\varphi}': I^n \to O\backslash K$ of φ itself. The converse implication will be discussed somewhat later.

Obviously closed subsets of Z-sets are Z-sets. Also closed sets of infinite deficiency and compact subsets of s are Z-sets, for in these two cases we can find an integer i and a constant c such that $\pi_i^{-1}(c)$ does not intersect the set and such that the map $(x_i)_i \mapsto (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots)$ is within ε-distance of the identity. For the second definition of Z-set it is known that any homeomorphism $f: K \to K'$ between two Z-sets K and K' can be extended to an autohomeomorphism $\bar{f}: Q \to Q$. Moreover, if $d(f, \mathrm{id}_K) < \varepsilon$ then we can require that $d(\bar{f}, \mathrm{id}_Q) < \varepsilon$. This is the *Homeomorphism Extension Theorem* ([2], [5]). In particular, Q is homogeneous. On the other hand, we have the *Mapping Replacement Theorem* which says that for any $\varepsilon > 0$ and any map $h: K \to Q$, where K is compact, there exists an embedding $h': K \to Q$ such that $d(h, h') < \varepsilon$ and $h'(K)$ is infinitely deficient. A proof is based on the well-known fact that any compact metric space can be embedded in Q. So let $\varphi: K \to Q$ be any embedding. Define for $x \in K$,

$$h^{(N)}(x) = (\pi_1 h(x), \ldots, \pi_N h(x), \pi_1 \varphi(x), 0, \pi_2 \varphi(x), 0, \pi_3 \varphi(x), \ldots).$$

Then $h^{(N)}$ is an embedding because φ is an embedding, and $h^{(N)}$ is close to h for large N because $h(x)$ and $h^{(N)}(x)$ coincide in the lower-numbered coordinates. Together with the Homeomorphism Extension Theorem this implies that for any Z-set K in Q there is an arbitrarily small autohomeomorphism g of Q which maps K onto a set of infinite deficiency: let g be an extension of $h^{(N)}$, where $h = \mathrm{id}_K$ and N is sufficiently large; then g is such an autohomeomorphism. Since $g(K)$ is infinitely deficient, it satisfies the first definition of Z-set. This proves the equivalence of both definitions, and also shows that Z-sets are exactly those closed subsets of Q which can be mapped onto a set of infinite deficiency by an autohomeomorphism h of Q (where $d(h, \mathrm{id}_Q)$ can be made arbitrarily small).

With help of the notion of Z-set we can topologically characterize the pseudo-boundary as a subset of Q: a subset M of Q is a *capset* for Q ([3]; for the related concept of $(G, \mathscr{K})$-skeletoid, see [6]) if M can be written as $U_i M_i$, where each M_i is a Z-set in Q, $M_i \subset M_{i+1}$ and such that the following absorption property holds: for each ε and j and every Z-set K there is an $i > j$ and an embedding $h: K \to M_i$ such that $h|_{K \cap M_j} = \mathrm{id}_{K \cap M_j}$ and $d(h, \mathrm{id}_K) < \varepsilon$. This characterizes $B(Q)$. It is shown in [3] that, for M a capset and N a countable union of Z-sets, $M \cup N$ is a capset, and also that for M a capset and K a Z-set, $M\backslash K$ is a capset. In the next two lemmas we give two alternative characterizations of capsets which are more convenient for our purposes. The first is used for the case 2^I and for a proof of the second lemma, and the second lemma is used for the case 2^Q.

LEMMA (1.1): *Suppose M is a σ-compact subset of Q such that*
1° *For every $\varepsilon > 0$, there exists a map $h: Q \to Q \backslash M$ such that $d(h, \mathrm{id}_Q) < \varepsilon$*
2° *M contains a family of compact subsets $M_1 \subset M_2 \subset \ldots$ such that each M_i is a copy of Q and M_i is a Z-set in M_{i+1}, and such that for each ε there exists an integer i and a map $h: Q \to M_i$ with $d(h, \mathrm{id}_Q) < \varepsilon$.*
Then M is a capset for Q.

PROOF: From 1° it follows that every compact subset of M is a Z-set in Q, so M is a countable union of Z-sets. As remarked above, every countable union of Z-sets containing a capset is itself a capset, so it is sufficient to show that $\bigcup_i M_i$ is a capset. Let ε, j and a Z-set K be given. By 2°, there exist $i > j$ and $h: Q \to M_i$ such that $d(h, \mathrm{id}_Q) < \varepsilon/4$. By the Mapping Replacement Theorem there exists an embedding $g: Q \to M_i$ which maps Q onto a Z-set in M_i such that $d(h, g) < \varepsilon/4$. Then $d(g, \mathrm{id}_Q) < \varepsilon/2$. By the Homeomorphism Extension Theorem for M_i, there exists a homeomorphism $f: M_i \to M_i$ which extends $g^{-1}|g(K \cap M_j)$ and such that $d(f, \mathrm{id}) < \varepsilon/2$. Then $f \circ g: K \to M_i$ is an embedding of K into M_i which is the identity on $K \cap M_j$ and such that $d(f \circ g, \mathrm{id}) < \varepsilon$. Thus $\bigcup_i M_i$ is a capset, and therefore M is a capset as well.

LEMMA (1.2): *Suppose M is a σ-compact subset of Q such that*
1° *For every ε there exists a map $h: Q \to Q \backslash M$ such that $d(h, \mathrm{id}) < \varepsilon$.*
2° *There exists an isotopy $F = (F_t)_t: Q \times [1, \infty] \to Q$ such that $F_\infty = \mathrm{id}_Q$ and $F|Q \times [1, \infty)$ is a $1-1$ map into M.*
Then M is a capset for Q.

PROOF: Define $M_i = F([-1+1/i, 1-1/i]^\infty \times [1, i])$ and $h_i: Q \to M_i$ by $h_i(x) = F_i((1-1/i) \cdot x)$. Since $\lim_{i \to \infty} d(\mathrm{id}_Q, h_i) = 0$ and M_i is a Z-set in M_{i+1}, the conditions of Lemma 1.1 are satisfied.

A separable metric space M is a *Hilbert cube manifold* or *Q-manifold* if M is locally homeomorphic to Q. In [7], Chapman proved that every Q-manifold M is triangulable, i.e., $M \cong |P| \times Q$, where P is a countable locally finite complex. If M is compact, then P can be chosen finite and even such that $|P|$ is a combinatorial manifold with boundary. We denote points of $|P| \times Q$ by (p, x) or $(p, (x_i)_i)$ and define the projection maps $\pi_i(p, x) = x_i$ and $\pi_P(p, x) = p$. For a given triangulation $M = |P| \times Q$, a closed subset $K \subset M$ is called *i-deficient* if $\pi_i(K)$ is a point, and *infinitely deficient* if K is i-deficient for infinitely many i. A closed subset K of a compact manifold M is a Z-set if for every ε there is a map $f: M \to M \backslash K$ such that $d(f, \mathrm{id}_M) < \varepsilon$. As in the case of the Hilbert cube, this definition has become a standard one and is observed to be equivalent to the classical one (i.e., K is a Z-set iff for every non-empty homotopically trivial open set O, $O \backslash K$ is non-empty and homotopically trivial) by several authors. Only a restricted version of the Homeomorphism Extension Theorem holds, since homotopy conditions have to be met.

2. Pseudo-interiors for 2^Q and related results

First we show (Theorem 2.2) that the collection of (connected) Z-sets in Q forms a pseudo-interior for $2^Q(C(Q))$ by verifying the conditions of Lemma 1.2. Thus we rely heavily on the facts that $2^Q \cong Q$ and $C(Q) \cong Q$ ([8]). As a corollary, we show that $2^{l_2} \cong l_2$ (Corollary 2.3). Next these results are generalized to the manifold case (Theorem 2.4 and Corollary 2.5).

Notation:
by X^Y we mean the space of all continuous mappings from Y into X endowed with the compact-open topology.

LEMMA (2.1):
(a) *The collection of Z-sets in Q is a G_δ in 2^Q.*
(b) *The collection of connected Z-sets in Q is a G_δ in $C(Q)$.*

PROOF OF (a): Let

$$\mathscr{Z}_i = \{K \in 2^Q | \exists g \in Q^Q : g(Q) \cap K = \emptyset \text{ and } d(g, \mathrm{id}_Q) < 1/i\}.$$

Obviously $\mathscr{Z}_i$ is an open subset of 2^Q and $\mathscr{Z} = \bigcap_i \mathscr{Z}_i$ is exactly the collection of Z-sets in Q.

PROOF OF (b): This is a direct consequence of (a).

REMARK: Lemma 2.1 has a finite-dimensional analogue. In [9], Geoghegan and Summerhill give generalizations to Euclidean n-space E^n for many infinite-dimensional notions and results. In [9], Section 3, they define what they call Z_m-sets and strong Z_m-sets in E^n for $0 \leqq m \leqq n-2$. For $(n, m) \neq (3, 0), (4, 1)$ or $(4, 2)$, the Z_m-sets and strong Z_m-sets coincide. A third possible definition is: 'K is a Z_m^*-set if for all $i \leqq m+1$, the maps from I^i into $E^n \backslash K$ lie dense in $(E^n)^{I^i}$'. This definition is easily seen to imply the definition of Z_m-set given in [9] and to be implied by the definition of strong Z_m-set. The collection of Z_m^*-sets can be written as a countable intersection of open sets: let, for all $i \leqq m+1$, $\{f_k^i\}_k$ be a countable dense subset of $(E^n)^{I^i}$. Let

$$\mathscr{Z}_{i,k} = \{K \in 2^{E^n} | \exists g \in (E^n)^{I^i}: g(I^i) \cap K = \emptyset \text{ and } d(g, f_k^i) < 1/k\}.$$

Then

$$\bigcap_{\substack{i \leqq m+1 \\ k=1,2,\ldots}} \mathscr{Z}_{i,k}$$

is exactly the collection of Z_m^*-sets. Moreover, this set is dense in 2^{E^n} since the collection of finite subsets of E^n is a subcollection of it. If $m \leqq n-3$, its intersection with $C(E^n)$ is also dense in $C(E^n)$ since the collection of compact connected one-dimensional rectilinear polyhedra is a subcollection and is dense in $C(E^n)$.

THEOREM (2.2):
(a) *The collection* $\mathscr{Z}$ *of Z-sets in Q is a pseudo-interior for* 2^Q.
(b) *The collection* $\mathscr{Z}_C$ *of connected Z-sets in Q is a pseudo-interior for* $C(Q)$.

PROOF: Note that Lemma 1.2 is stated in terms of the pseudo-boundary and Theorem 2.2 in terms of the pseudo-interior. The maps h and $(F_t)_t$ which are asked for in the lemma will map connected sets onto connected sets, so that they prove (a) and (b) simultaneously.

As remarked in Section 1, every compact subset of s is a Z-set in Q. Therefore the map $h: Q \to s$, defined by

$$h(x) = (1-\varepsilon)\cdot x = ((1-\varepsilon)\cdot x_1, (1-\varepsilon)\cdot x_2, \cdots)$$

induces a map $2^h: 2^Q \to \mathscr{Z}$ as asked for in 1° of Lemma 1.2.

We shall construct F_t so that for $K \in 2^Q$ and $t < \infty$ the set $F_t(K)$ will be the union of two intersecting sets, one of which carries all information about K and the other of which is not a Z-set. First we consider the case that t is an integer. We define a sequence of maps $(\varphi_i)_i: Q \to Q$ by

$$\varphi_i(x) = \left(1 - \frac{1}{i}\right) \cdot (x_1, \ldots, x_{2i}, 0, x_{2i+1}, 0, x_{2i+2}, \ldots).$$

Obviously $\varphi_i(Q)$ is contained in s and projects onto 0 in all odd coordinates $\geqq 2i+1$. We define another auxiliary operator $T_{j,c}: 2^Q \to 2^Q$, where $j \geqq 1$ and $c \in [0, 2]$:

$$T_{j,c}(K) = \{(x_1, \ldots, x_{j-1}, x_j+y, x_{j+1}, \ldots) |\ |y| \leqq c \text{ and } |x_j+y| \leqq 1 \text{ and } (x_i)_i \in K\}.$$

As c varies from 0 to 2, $T_{j,c}(K)$ is transformed continuously from K into a set which occupies the whole interval in the jth direction. We have:

$$T_{j,0}(K) = K \text{ and } T_{j,2}(\bar{\pi}_j^{-1}(\bar{\pi}_j(K))) = \bar{\pi}_{j-1}^{-1}(\bar{\pi}_{j-1}(K)).$$

If $\pi_j(K) = \{0\}$ then $c = 2$ can be replaced by $c = 1$ in the above formula. Now we set:

$$F_i(K) = T_{2i+3,\frac{1}{2}}(\varphi_i(K)) \cup \bar{\pi}_{2i+3}^{-1}(\bar{\pi}_{2i+3}(\varphi_i(K))).$$

For every K this is a non-Z-set since the second term contains a subset of the form $\bar{\pi}_j^{-1}(x_1, \ldots, x_j)$ with $-1 < x_i < 1$ for $i = 1, \ldots, j$. Furthermore, $\pi_{2i+3}^{-1}(\frac{1}{2}) \cap F_i(K) = \pi_{2i+3}^{-1}(\frac{1}{2}) \cap T_{2i+3,\frac{1}{2}}(\varphi_i(K))$ is a translation of $\varphi_i(K)$ in the direction of the $2i+3^{rd}$ coordinate and therefore the first term contains all information about K and F_i is one-to-one.

Before we describe φ_t for arbitrary t, we restrict ourselves to $k = i+(n-1/n)$ where $i \geqq 1$ and $n \geqq 1$:

$$\varphi_i(x) = \left(1-\frac{1}{i}\right)\cdot(x_1, \ldots, x_{2i}, 0, x_{2i+1}, 0, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, \ldots)$$

$$\varphi_{i+\frac{1}{2}}(x) = \left(1-\frac{1}{i+\frac{1}{2}}\right)\cdot(x_1, \ldots, x_{2i}, x_{2i+1}, 0, 0, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, \ldots)$$

$$\varphi_{i+\frac{2}{3}}(x) = \left(1-\frac{1}{i+\frac{2}{3}}\right)\cdot(x_1, \ldots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, 0, 0, x_{2i+3}, 0, x_{2i+4}, 0, \ldots)$$

$$\varphi_{i+\frac{3}{4}}(x) = \left(1-\frac{1}{i+\frac{3}{4}}\right)\cdot(x_1, \ldots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, x_{2i+3}, 0, 0, 0, x_{2i+4}, 0, \ldots)$$

$$\varphi_{i+\frac{4}{5}}(x) = \left(1-\frac{1}{i+\frac{4}{5}}\right)\cdot(x_1, \ldots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, 0, 0, \ldots)$$

$$\varphi_{i+1}(x) = \left(1-\frac{1}{i+1}\right)\cdot(x_1, \ldots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, x_{2i+5}, \ldots).$$

For

$$t \in \left(i + \frac{n-1}{n},\ i + \frac{n}{n+1} \right)$$

φ_t is defined by linear interpolation between

$$\varphi_i + \frac{n-1}{n} \quad \text{and} \quad \varphi_i + \frac{n}{n+1}.$$

This way $\varphi_t(Q)$ projects onto 0 in all odd coordinates $\geqq 2i+3$ if $t \leqq i+1$.

For $i \geqq 1$ and $s \in [0, 1]$ we define

$$F_{i+s}(K) = T_{2i+3, \frac{1}{2}(1-s)} \circ T_{2i+5, \frac{1}{2}s}(\varphi_{i+s}(K)) \cup T_{2i+4, 2-2s} \circ T_{2i+5, 1-s}(\bar{\pi}_{2i+5}^{-1}(\bar{\pi}_{2i+5}(\varphi_{i+s}(K)))).$$

Note that this is consistent with the previous definition of $F_i(K)$. We check:

1° $(F_t)_t$ is continuous. For finite t this follows from the continuity of the operators $T_{j,c}$, 2^{φ_t} and $\bar{\pi}_j^{-1} \circ \bar{\pi}_j$; for $t \to \infty$ it is easily seen that $F_t(K) \to K$.

2° For every K, $F_t(K)$ is a non-Z-set if t is finite, for it contains a subset of the form $\bar{\pi}_j^{-1}(x_1, \ldots, x_j)$ with $-1 < x_i < 1$ for $i = 1, \ldots, j$.

3° $F = (F_t)_t$ is one-to-one on $2^Q \times [0, \infty)$: for the determination of t from $F_t(K)$ note that $t \in (i, i+1]$ iff $\pi_j(F_t(K)) = [-1, 1]$ for all $j > 2i+5$ and for no odd $j \leqq 2i+5$. Once it is determined that $t \in (i, i+1]$ then on that interval t is in one-to-one correspondence with

$$\pi_{2i+3} \circ F_t(K) = [-(1-(t-i))/2, (1-(t-i))/2]$$

(recalling that $\pi_{2i+3} \circ \varphi_t(x) = 0$ for $x \in Q$ and $t \leqq i+1$). Finally, for $t = i+s$ and $s \in (0, 1]$,

$$F_t(K) \cap \pi_{2i+3}^{-1}((1-s)/2) \cap \pi_{2i+5}^{-1}(s/2)$$

is a copy of K in a canonical way. Note that this set does not intersect the second term

$$T_{2i+4, 2-2s} \circ T_{2i+5, 1-s}(\bar{\pi}_{2i+5}^{-1} \bar{\pi}_{2i+5}(\varphi_{i+5}(K)))$$

since the latter set projects onto 0 in the $2i+3$rd coordinate, and if $s = 1$ also in the $2i+5$th coordinate.

4° If K is connected, then $F_t(K)$ is connected since $F_t(K)$ is the union of two connected sets which intersect in $\varphi_t(K)$.

The following corollary answers a question posed by R. M. Schori:

COROLLARY (2.3): *Both the collection of compact subsets of l_2 and the collection of compact connected subsets of l_2 are homeomorphic to l_2.*

PROOF: According to [1], l_2 is homeomorphic to $s = (-1, 1)^\infty$. Thus it is sufficient to show that the collection $\mathscr{L}$ (or $\mathscr{L}_C$) of closed (connected) subsets of Q which are contained in s forms a pseudo-interior for 2^Q (or $C(Q)$). Since this collection is a subset of $\mathscr{Z}$ ($\mathscr{Z}_C$) we only have to verify condition 1° of Lemma 1.2, and to show that $\mathscr{Z}$ and $\mathscr{Z}_C$ are G_δ's. But the map 2^h from the proof of Theorem 2.2 actually maps 2^Q and $C(Q)$ in $\mathscr{L}$ and $\mathscr{L}_C$ respectively, showing 1° of Lemma 1.2. Finally, we can write $\mathscr{L}$ ($\mathscr{L}_C$) as a G_δ by $\bigcap_i\{K \subset Q | K$ is closed (and connected) and $\pi_i(K) \subset (-1, 1)\}$. This completes the proof of the corollary.

THEOREM (2.4): *If M is a compact connected Q-manifold, then*

(a) *the collection $\mathscr{Z}^M$ of Z-sets in M is a pseudo-interior for 2^M.*

(b) *the collection $\mathscr{Z}^M_C$ of connected Z-sets in M is a pseudo-interior for $C(M)$.*

PROOF: As observed earlier, by [7] we may write $M = |P| \times Q$, where $|P|$ is a compact finite-dimensional manifold with boundary. Again we apply Lemma 1.2, where the M from the lemma is $2^M \backslash \mathscr{Z}^M$ or $C(M) \backslash \mathscr{Z}^M_C$ respectively. As before one can prove that $\mathscr{Z}^M$ and $\mathscr{Z}^M_C$ are G_δ-sets in 2^M and $C(M)$ respectively. Condition 1° of the lemma is proved by the map 2^h, where $h(p, x) = (p, (1-\varepsilon) \cdot x)$.

Let $H: |P| \times [1, \infty] \to |P|$ be an isotopy such that $H_\infty = \mathrm{id}$ and $H_t(|P|) \subset |P| \backslash |\partial P|$ for finite t (remember that we assume that $|P|$ is a compact manifold with boundary). Consider the map $F: 2^Q \times [1, \infty] \to 2^Q$ defined in the proof of Theorem 2.2. Define, for $p \in |P|$ and $K \subset Q$, $G_t(\{p\} \times K) = \{H_t(p)\} \times F_t(K)$. If L is a subset of $|P| \times Q$, then L can be written as a union $\bigcup_{p \in \pi_P(L)} \{p\} \times L_p$. Now define

$$G_t(L) = \bigcup_{p \in \pi_P(L)} G_t(\{p\} \times L_p).$$

Then $G = (G_t)_t$ satisfies 2° of Lemma 1.2. We need only show that $G_t(L)$ is a closed set.

From the definition of $F_t(K)$ one readily sees that

$$F_t(K) = \bigcup_{x \in K} F_t(\{x\}).$$

Therefore we can write

$$G_t(L) = \bigcup_{(p,x)\in L} \{H_t(p)\} \times F_t(\{x\}).$$

Let $(q_i, y_i)_i$ be a sequence in $G_t(L)$ converging to (q, y). We have to show that $(q, y) \in G_t(L)$. Let $q_i = H_t(p_i)$ and $y_i \in F_t(\{x_i\})$, where $(p_i, x_i) \in L$. There is a subsequence (p_{i_k}, x_{i_k}) converging to some point $(p, x) \in L$. Then $H_t(p) = \lim_k q_{i_k} = q$, and by continuity of F_t we have that $y \in F_t(\{x\})$. Therefore $(q, y) \in G_t(L)$.

Corollary (2.5): *For any connected l_2-manifold M, both the collection 2^M of compact subsets of M and the collection $C(M)$ of connected compact subsets of M are homeomorphic to l_2.*

Proof: According to [10] and [11] we can triangulate $M = |P| \times l_2$, where P is a locally finite simplicial complex. Of course, now we cannot assume that $|P|$ is a manifold with boundary.

Let K be a compact (connected) subset of M, then K has a closed neighborhood $|P'| \times l_2$, where P' is a finite (connected) subcomplex of P. The collection $\mathcal{O}^{P'}$ ($\mathcal{O}_C^{P'}$) of compact (connected) subsets of M which are contained in the topological interior of $|P'| \times l_2$ is an open neighborhood of K. Its closure in 2^M ($C(M)$), the set $\{K \subset M | K \text{ is compact (and connected) and } K \subset |P'| \times l_2\}$, is a pseudo-interior for $2^{|P'| \times Q}$ (for $C(|P'| \times Q)$) if we identify l_2 with $(-1, 1)^\infty \subset Q$. This is proved by an argument similar to that in the proof of Corollary 2.3. Therefore $\mathcal{O}^{P'}$ ($\mathcal{O}_C^{P'}$) is an open subset of a copy of l_2, showing that 2^M ($C(M)$) is an l_2-manifold.

Next we show that 2^M ($C(M)$) is homotopically trivial. By [12], this will prove that 2^M ($C(M)$) is homeomorphic to l_2. Let a map $f\colon \partial I^n \to 2^M$ (or $f\colon \partial I^n \to C(M)$) be given. Then $Y' = \bigcup_{y \in \partial I^n} f(y)$ is a compact union of compact sets, and therefore a compact subset of M. Choose a finite connected subcomplex P' of P and a compact convex subset D of l_2 such that $Y' \subset |P'| \times D$. Then

$$f(\partial I^n) \subset 2^{|P'| \times D} \quad (f(\partial I^n) \subset C(|P'| \times D)).$$

Moreover, $2^{|P'| \times D}$ and $C(|P'| \times D)$ are contractible: define, for

$$K \in 2^{|P'| \times D} \quad (K \in C(|P'| \times D))$$

and for $t \in [0, T]$ where T is sufficiently large $H(K, t)$ to be the closed t-neighborhood of K in some fixed convex metric for $|P'| \times D$. Then H

is a contraction of $2^{|P'|\times D}$ (or $C(|P'|\times D)$). Therefore f can be extended to $\bar{f}\colon I^n \to 2^{|P'|\times D} \subset 2^M$ (to $\bar{f}\colon I^n \to C(|P'|\times D) \subset C(M)$).

3. Pseudo-interiors for 2^I

In this section we show that both the collection of zero-dimensional subsets of I and the collection $\mathscr{C}$ of Cantor sets in I are pseudo-interiors for 2^I. It seems reasonable that similar statements are true for the hyperspace of more general spaces, but the author has been unable to prove a comparable statement even for the hyperspace of a finite graph.

In this section $I = [0, 1]$.

LEMMA (3.1):

(a) *The collection* $\mathcal{O}$ *of zero-dimensional closed subsets of a compact metric space* X *is a* G_δ *in* 2^X.

(b) *The collection* $\mathscr{C}$ *of Cantor sets in* X *is a* G_δ *in* 2^X.

PROOF OF (a): The collection $\mathcal{O}_n = \{A \subset X | A$ is closed and all components of A have diameter less than $1/n\}$ is an open subset of 2^X. For let $(A_i)_i \to A$, where $A_i \notin \mathcal{O}_n$ for all i. We show that $A \notin \mathcal{O}$. For every i there is a component K_i of A_i with diameter at least $1/n$. The sequence $(K_i)_i$ has a subsequence $(K_{i_k})_k$ which converges to a set K which is closed, connected and has diameter at least $1/n$ and is a subset of A. Therefore $A \notin \mathcal{O}_n$.

PROOF OF (b): We write $\mathscr{C}_n = \{A \subset X | A$ is closed and for all $x \in A$, there is a $y \neq x$ in A such that $d(x, y) < 1/n\}$. Since Cantor sets are exactly the compact metric spaces which are zero-dimensional and have no isolated points, it follows that $\mathscr{C} = \mathcal{O} \cap \bigcap_n \mathscr{C}_n$. We show that $\mathscr{C}_n$ is an open subset of 2^X: let $(A_i)_i \to A$, where $A_i \notin \mathscr{C}_n$ for all i. Let $U_\varepsilon(x)$ denote the ε-neighborhood of any point x. There is a sequence $(p_i)_i$ such that $U_{1/n}(p_i) \cap A_i = \{p_i\}$. This sequence has a limit point p and it is easily seen that $U_{1/n}(p) \cap A = \{p\}$. Therefore $A \notin \mathscr{C}_n$.

MAIN LEMMA (3.2): *There exist arbitrarily small maps* $h\colon 2^I \to \mathscr{C}$.

PROOF: The map h will be defined as a composition

$$2^I \xrightarrow{f} \mathrm{FSI} \xrightarrow{\bar{h}_N} \mathrm{FSC} \xrightarrow{g} \mathscr{C}, \qquad h = g \circ \bar{h}_N \circ f$$

where FSI (Finite Sequences of Intervals) is a collection of finite sequences of intervals, to be defined later, and FSC (Finite Sequences of Cantor sets) is a collection of finite sequences of topological Cantor sets, which will also be defined later on. The map f will be discontinuous, but g, $\bar{h}_N$ and $g \circ \bar{h}_N \circ f$ are continuous. In the subsequent discussion we assume a fixed $\varepsilon < \frac{1}{2}$ and N is the largest integer such that $N \cdot \varepsilon \leqq 1$. The map $h = g \circ \bar{h}_N \circ f$ will have distance less than 3ε to the identity.

Step 1. *The set* FSI.

Let FSI_n be the set of all sequences of n terms $\langle [a_1, b_1], \ldots, [a_n, b_n] \rangle$ such that

i) $0 \leqq a_1$ and $b_n \leqq 1$
ii) $a_{i+1} \geqq b_i$, i.e. the intervals do not overlap
iii) $b_i - a_i \geqq 2n \cdot \varepsilon^2$ if $1 < i < n$
iv) $b_i - a_i \geqq n \cdot \varepsilon^2$ if $i = 1, n$.

The metric on FSI_n is

$$\rho_n(\langle [a_1, b_1], \ldots, [a_n, b_n] \rangle, \langle [a'_1, b'_1], \ldots, [a'_n, b'_n] \rangle) = \max_i \max (|a'_i - a_i|, |b'_i - b_i|).$$

Define $\mathrm{FSI} = \bigcup_{n=1}^{N} \mathrm{FSI}_n$, where N is defined as above. Note that for $n > N$, $\mathrm{FSI}_n = \emptyset$ since for any element $\mathscr{A}$ of FSI_n, the sum of the lengths of the intervals of $\mathscr{A}$ is at least $(n-1) \cdot 2n \cdot \varepsilon^2 > (n-1) \cdot 2\varepsilon > 2 - 2\varepsilon > 1$ since $\varepsilon < \frac{1}{2}$, whereas $\mathscr{A}$ is a collection of non-overlapping subintervals of $[0, 1]$. We choose the following metric on FSI: $\rho(\mathscr{A}, \mathscr{B}) = \rho_n(\mathscr{A}, \mathscr{B})$ if $\{\mathscr{A}, \mathscr{B}\} \subset \mathrm{FSI}_n$, i.e., if both $\mathscr{A}$ and $\mathscr{B}$ consist of n intervals, and $\rho(\mathscr{A}, \mathscr{B}) = 1$ if for no n, $\{\mathscr{A}, \mathscr{B}\} \subset \mathrm{FSI}_n$, i.e., if $\mathscr{A}$ and $\mathscr{B}$ have a different number of terms.

Step 2. *The function* $f: 2^I \to \mathrm{FSI}$

Let $A \in 2^I$, then $U_\varepsilon(A)$, the open ε-neighborhood of A, is a finite union of disjoint subintervals of I, open relative to I. Let

$$f(A) = \langle [a_1, b_1], \ldots, [a_n, b_n] \rangle,$$

where the intervals $[a_i, b_i]$ are the closures of the components of $U_\varepsilon(A)$, arranged in increasing order, e.g., if $U_\varepsilon(A) = (a_1, b_1) \cup (b_1, b_2)$ then $f(A) = \langle [a_1, b_1], [b_1, b_2] \rangle$ and *not* $\langle [a_1, b_2] \rangle$. This assignment is not continuous: Let $A_\delta = \{0, 2\varepsilon+\delta\}$. If $\delta \geqq 0$, then

$$f(A_\delta) = \langle [0, \varepsilon], [\varepsilon+\delta, 3\varepsilon+\delta] \rangle$$

but if $\delta < 0$ then $f(A_\delta) = \langle [0, 3\varepsilon+\delta] \rangle$. But apart from this phenomenon f is continuous in the following sense: Let $\delta < \varepsilon$ and suppose for some $A, B \in 2^I$, $d_H(A, B) < \delta$, where d_H denotes the Hausdorff distance (see Section 1). Then each gap of $U_\varepsilon(A \cup B)$ (including a gap consisting of one point) corresponds to, i.e., is contained in a gap of $U_\varepsilon(A)$ since for $\delta < \varepsilon$ it cannot lie left or right from $U_\varepsilon(A)$. Conversely, each gap in $U_\varepsilon(A)$ *which has length* $\geqq 2\delta$ corresponds to, i.e., contains a gap of $U_\varepsilon(A \cup B)$. Let $f_B(A)$ be a function from 2^I to FSI which is obtained from $f(A)$ by replacing each gap in $U_\varepsilon(A)$ which has no counterpart in $U_\varepsilon(A \cup B)$ by a degenerate gap (see Fig. 1); e.g., if $f(A) = \langle [a_1, b_1], [a_2, b_2] \rangle$ with

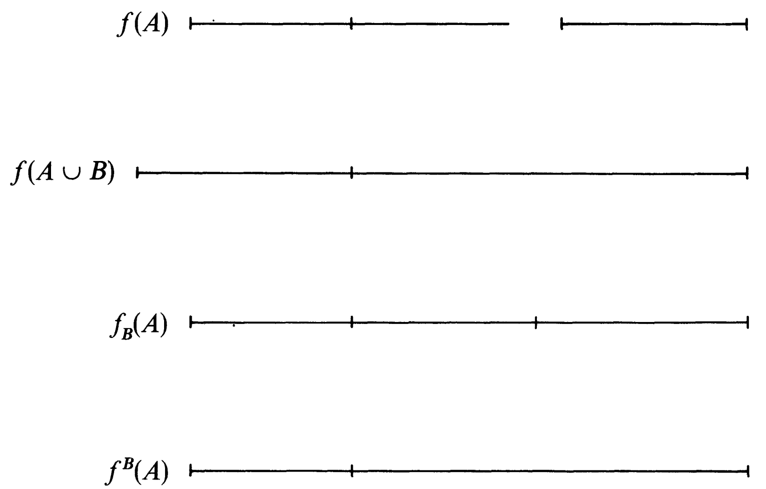


Figure 1

$a_2 - b_1 < 2\delta$ and if $U_\varepsilon(A \cup B) = (a'_1, b'_2)$ with $0 \leqq b'_2 - b_2 < \delta$ and $0 \leqq a_1 - a'_1 < \delta$, then let

$$f_B(A) = \left\langle \left[a_1, \frac{b_1+a_2}{2}\right], \left[\frac{b_1+a_2}{2}, b_2\right] \right\rangle.$$

Let $f^B(A)$ eliminate the degenerate gaps thus obtained (but not the other degenerate gaps); e.g., in the above example $f^B(A) = \langle [a_1, b_2] \rangle$. Then for $d_H(A, B) < \delta$ we have $d(f^B(A), f^A(B)) < \delta$ and also

$$d(f(A), f_B(A)) < \delta \quad \text{and} \quad d(f(B), f_A(B)) < \delta.$$

These notations will be used in the proof of the continuity of $g \circ \bar{h}_N \circ f$.

Step 3. *The set* FSC.

Let C be a topological Cantor set such that $C \subset I$ and $\{0, 1\} \subset C$ and $d_H(C, I) < \varepsilon$. Let $C(a, b)$ be the image of C under the 'linear' map which maps 0 onto a and 1 onto b. For $[a, b] \subset [0, 1]$ we also have $d_H(C(a, b), [a, b]) < \varepsilon$. We define FSC_n to be the collection of all sequences of n terms $\langle C(a_1, b_1), \ldots, C(a_n, b_n) \rangle$ such that

(i) $0 \leqq a_1 \leqq \ldots \leqq a_n \leqq 1$

(ii) $0 \leqq b_1 \leqq \ldots \leqq b_n \leqq 1$

(iii) $a_i < b_i$ for $i \leqq i \leqq n$.

Thus the sets $C(a_i, b_i)$ may overlap. Define $\text{FSC} = \bigcup_{n=1}^{N} \text{FSC}_n$. The metric of FSC is somewhat analogous to that on FSI: If

$$\mathscr{A} = \langle C(a_1, b_1), \ldots, C(a_n, b_n) \rangle \quad \text{and} \quad \mathscr{B} = \langle C(a'_1, b'_1), \ldots, C(a'_n, b'_n) \rangle$$

then

$$\rho(\mathscr{A}, \mathscr{B}) = \max_i d_H(C(a_i, b_i), C(a'_i, b'_i))$$

and if for no n, $\{\mathscr{A}, \mathscr{B}\} \subset \text{FSC}_n$ then $\rho(\mathscr{A}, \mathscr{B}) = 1$.

Step 4. *The map* $g\colon \text{FSC} \to \mathscr{C}$

We simply let $g(\mathscr{A})$ be the union of the terms of $\mathscr{A}$. Obviously g is continuous. Notice that by the characterization of Cantor sets given in the proof of Lemma 3.1, $g(\mathscr{A})$ is indeed a Cantor set.

Step 5. *Construction of* $\bar{h}_N$

From the remark at Step 2 it is easily seen that the function

$$\varphi\colon \langle [a_1, b_1], \ldots, [a_n, b_n] \rangle \mapsto \langle C(a_1, b_1), \ldots, C(a_n, b_n) \rangle$$

does not yield a continuous composition $g \circ \varphi \circ f$. Instead, we construct by induction a map $h_n\colon \text{FSI}_n \to \text{FSC}_n$ and set $\bar{h}_n = \bigcup_{i=0}^{n} h_i$ (i.e., $\bar{h}_n$ is the function which assigns $h_i(\mathscr{A})$ to $\mathscr{A}$ if $\mathscr{A} \in \text{FSI}_i$ and $i \leqq n$). The following induction hypotheses should be satisfied:

(i) If $\mathscr{A} = \langle [a_1, b_1], \ldots, [a_n, b_n] \rangle$, then $h_n(\mathscr{A}) = \langle C(a'_1, b'_1), \ldots, C(a'_n, b'_n) \rangle$, where $a'_1 = a_1$ and $b'_n = b_n$.

(ii) Additivity at 'large' gaps. If $\mathscr{A}$ can be broken up into $\mathscr{B}$ and $\mathscr{C}$ where $\mathscr{B} = \langle [a_1, b_1], \ldots, [a_i, b_i] \rangle$ and $\mathscr{C} = \langle [a_{i+1}, b_{i+1}], \ldots, [a_n, b_n] \rangle$ and $a_{i+1} - b_i \geqq 2\varepsilon^2$ then $h_n(\mathscr{A}) = \langle C(a_1, b'_1), \ldots, C(a'_n, b_n) \rangle$, where

$$\bar{h}_{n-1}(\mathscr{B}) = \langle C(a_1, b'_1), \ldots, C(a'_i, b'_i) \rangle$$

and

$$\bar{h}_{n-1}(\mathscr{C}) = \langle C(a'_{i+1}, b'_{i+1}), \ldots, C(a'_n, b_n) \rangle.$$

In particular, by (i) $b'_i = b_i$ and $a'_{i+1} = a_{i+1}$.

(iii) If

$$\mathscr{A} = \langle [a_1, b_1], \ldots, [a_i, b_i], [b_i, b_{i+1}], \ldots, [a_n, b_n] \rangle,$$

that is, if $a_{i+1} = b_i$, and if

$$\mathscr{B} = \langle [a_1, b_1], \ldots, [a_i, b_{i+1}], \ldots, [a_n, b_n] \rangle,$$

and if, moreover,

$$h_{n-1}(\mathscr{B}) = \langle C(a_1, b'_1), \ldots, C(a'_i, b'_{i+1}), \ldots, C(a'_n, b_n) \rangle,$$

then

$$h_n(\mathscr{A}) = \langle C(a_1, b'_1), \ldots, C(a'_i, b'_{i+1}), C(a'_i, b'_{i+1}), \\ C(a'_{i+2}, b'_{i+2}), \ldots, C(a'_n, b_n) \rangle,$$

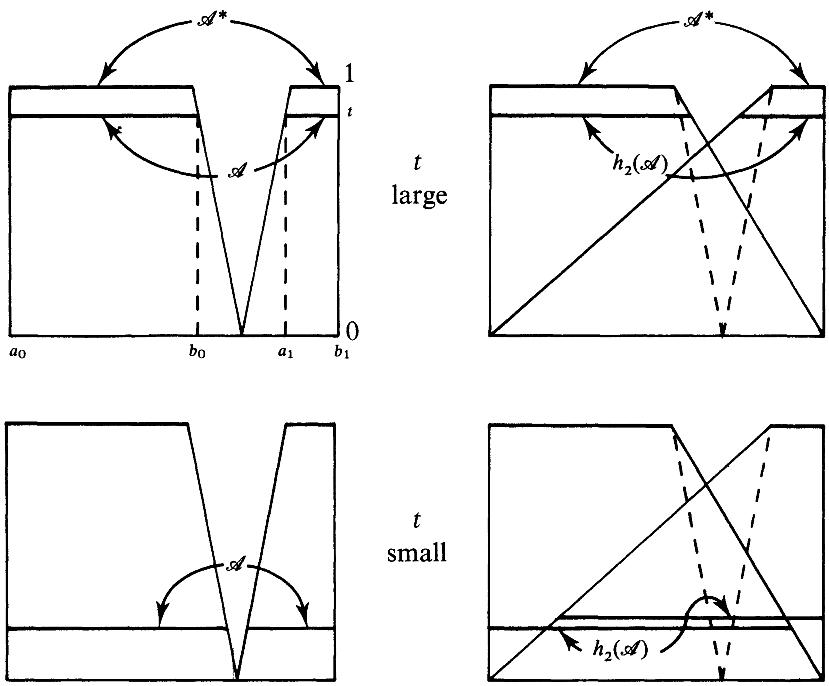
i.e., $a'_i = a'_{i+1}$ and $b'_i = b'_{i+1}$ and $gh_n(\mathscr{A}) = gh_{n-1}(\mathscr{B})$.

These induction hypotheses, and especially (iii), will be seen to insure continuity of $g \circ \bar{h}_N \circ f$. We give now the inductive construction of $h_n: \mathrm{FSI}_n \to \mathrm{FSC}_n$.

$n = 1$: set $h_1(\langle [a_1, b_1] \rangle) = \langle C(a_1, b_1) \rangle$, in accordance with (i).

$n = 2$: let $\mathscr{A} = \langle [a_1, b_1], [a_2, b_2] \rangle$ with both $b_1 - a_1 \geqq \varepsilon^2$ and $b_2 - a_2 \geqq \varepsilon^2$ and with $a_2 - b_1 \geqq 0$. If $a_2 = b_1$ then according to (iii) we have $h_2(\mathscr{A}) = \langle C(a_1, b_2), C(a_1, b_2) \rangle$. If $a_2 - b_1 \geqq 2\varepsilon^2$ then according to (ii), we have $h_2(\mathscr{A}) = \langle C(a_1, b_1), C(a_2, b_2) \rangle$. If $a_2 - b_1 = t \cdot 2\varepsilon^2$ with $0 < t < 1$, then b'_1 and a'_2 are constructed as in Figure 2 (the pictures show what happens if t is large (upper pictures), and what happens if t is small, (lower pictures)).

In formulas: let $\mathscr{A}^* = \langle [a_1, b_1^*], [a_2^*, b_2] \rangle$ be the result of enlarging the gap (b_1, a_2) symmetrically from its midpoint by a factor $1/t$. Thus



Note that above and below we have different $\mathscr{A}$ but the same $\mathscr{A}^*$.

Figure 2

$a_2^* - b_1^* = 2\varepsilon^2$. We put

$$h_2(\mathscr{A}) = \langle C(a_1, t \cdot b_1^* + (1-t) \cdot b_2), C(t \cdot a_2^* + (1-t) \cdot a_1, b_2) \rangle.$$

Note that this is consistent with the case $a_2 = b_1$ and $a_2 - b_1 \geqq 2\varepsilon^2$ as treated above.

$n+1$: Suppose $\bar{h}_n$ is already defined. Let

$$\mathscr{A} = \langle [a_1, b_1], \ldots, [a_{n+1}, b_{n+1}] \rangle \in \mathrm{FSI}_{n+1}.$$

If for all i, $a_{i+1} - b_i = 0$, i.e., if all gaps are degenerate, then by repeated application of (iii) we find that for all i, $C(a_i', b_i') = C(a_1, b_{n+1})$. If $\max_i (a_{i+1} - b_i) \geqq 2\varepsilon^2$, then $h_{n+1}(\mathscr{A})$ is determined by (ii). If for several i, $a_{i+1} - b_i \geqq 2\varepsilon^2$ then it is easily seen, using (ii) for $\bar{h}_n$, that $h_{n+1}(\mathscr{A})$ is independent of the choice of the gap at which $\mathscr{A}$ is broken up into $\mathscr{B}$ and $\mathscr{C}$. So let us assume that the length of the largest gap

$$\max_i (a_{i+1} - b_i) = 2t \cdot \varepsilon^2$$

with $0 < t < 1$. Let $\mathscr{A}^*$ be the result of widening each gap symmetrically from its midpoint by a factor $1/t$, so that the largest gap of $\mathscr{A}^*$ has width $2\varepsilon^2$. Now break up $\mathscr{A}^*$ into $\mathscr{B}$ and $\mathscr{C}$, where the gap in between $\mathscr{B}$ and $\mathscr{C}$ has width $2\varepsilon^2$. The reader may check that $\mathscr{B}$ and $\mathscr{C}$ are elements of $\mathrm{FSI}_1 \cup \ldots \cup \mathrm{FSI}_n$, in particular that they consist of intervals of sufficient length, noting that, since $\mathscr{B}$ and $\mathscr{C}$ have less terms than $\mathscr{A}$, they are allowed to consist of smaller intervals. Therefore $\bar{h}_n(\mathscr{B})$ and $\bar{h}_n(\mathscr{C})$ are defined. Let $\bar{h}_n(\mathscr{B}) = \langle C(a_1, b_1^*), \ldots, C(a_i^*, b_i^*)\rangle$ and

$$\bar{h}_n(\mathscr{C}) = \langle C(a_{i+1}^*, b_{i+1}^*), \ldots, C(a_{n+1}^*, b_{n+1})\rangle.$$

The construction of $h_{n+1}(\mathscr{A})$ *from* $\bar{h}_n(\mathscr{B})$ and $\bar{h}_n(\mathscr{C})$ is shown in Figure 3.

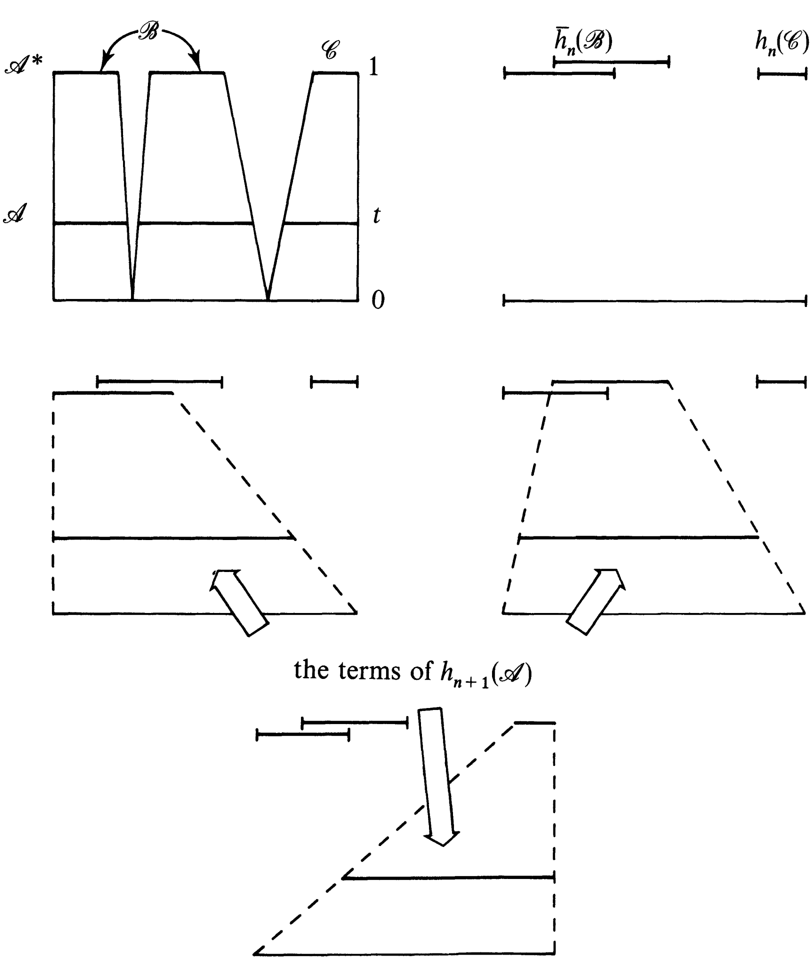


Figure 3

In formulas:

$$h_{n+1}(\mathscr{A}) = \langle C(a_1, t\cdot b_1^* + (1-t)\cdot b_{n+1}), C(t\cdot a_2^* + (1-t)\cdot a_1,\\ t\cdot b_2^* + (1-t)\cdot b_{n+1}), \ldots, C(t\cdot a_{n+1}^* + (1-t)\cdot a_1, b_{n+1})\rangle.$$

Thus each Cantor set is stretched somewhat toward $C(a_1, b_{n+1})$: only a little if t is close to 1 and almost all the way if t is close to 0.

It is an easy exercise to check the induction hypotheses and to prove that $d_H(A, g\circ \bar{h}_n \circ f(A)) < 3\varepsilon$. To show continuity, we refer to the functions f_B and f^B, defined at Step 2. From the remarks there and the continuity of g and $\bar{h}_N$ and the fact that $g\circ \bar{h}_N \circ f_B(A) = g\circ \bar{h}_N \circ f^B(A)$ for any two $A, B \in 2^I$ we easily see that $g\circ \bar{h}_N \circ f$ is continuous.

Let $I^* = \{\{t\}|t\in I\} \subset 2^I$. Then I^* is a Z-set in 2^I, since the map $f: 2^I \to 2^I$ defined by $f(K) = Cl(U_\varepsilon(K))$ is an ε'small map from 2^I into $2^I\backslash I^*$. Moreover, $I^* \cap \mathscr{C} = \emptyset$. Therefore the inclusion of I^* in Lemma 3.3 is harmless (see the remarks above Lemma 1.1).

LEMMA (3.3): *The set* $(2^I\backslash\mathscr{O}) \cup I^*$ *contains a family of copies of* Q *as asked for in Lemma* 1.1 (2°).

PROOF: If $K \subset I$, then let $[a_K, b_K]$ be the smallest closed interval containing K. Define $M_\varepsilon \subset 2^I$ by $M_\varepsilon = \{K \subset I | K$ is closed and $[a_K + (1-\varepsilon)\cdot(b_K - a_K), b_K] \subset K\}$. Let $\tilde{K}_\varepsilon$ be the image of K under a linear map which maps a_K onto a_K and b_K onto $a_K + (1-\varepsilon)\cdot(b_K - a_K)$. In formulas:

$$\tilde{K}_\varepsilon = \{a_K + (1-\varepsilon)\cdot(t - a_K)| t\in K\}.$$

Let

$$h_\varepsilon(K) = \tilde{K}_\varepsilon \cup [a_K + (1-\varepsilon)\cdot(b_K - a_K), b_K].$$

Then h_ε is a homeomorphism of 2^I onto M_ε with distance $\leqq \varepsilon$ to the identity. Since Lemma 3.2 and the remark on I^* show that every closed subset of $(2^I\backslash\mathscr{O}) \cup I^*$ is a Z-set, it follows that M_ε is a Z-set in 2^I. Because for $\delta < \varepsilon$, $h_\delta^{-1}(M_\varepsilon) = M_{(\varepsilon-\delta)/(1-\delta)}$ is a Z-set in 2^I by the same token, we see that M_ε is a Z-set in M_δ. Therefore the family $\{M_{1/i}\}_i$ satisfies 2° of Lemma 1.1, both for $M = (2^I\backslash\mathscr{O}) \cup I^*$ and for $M = 2^I\backslash\mathscr{C}$.

Combining Lemmas 3.2 and 3.3, we obtain the main theorem of this section:

THEOREM (3.4): *Both the collection of topological Cantor sets and the collection of zero-dimensional subsets in I are pseudo-interiors for 2^I.*

Finally, we mention the following conjecture (a definition of *fd* capset can be found in [3]):

CONJECTURE (R. M. Schori): *The collection of finite subsets of I is an fd capset for 2^I.*

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