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EXTRASPECIAL SECTIONS OF PERIODIC FC-GROUPS

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1. Introduction

A group G is an FC-group if each element of G has only finitely many conjugates. We shall be concerned with the class $\mathfrak{X}$ of periodic FC-groups. $\mathfrak{X}$ is well known to coincide with the class of locally finite-normal groups and could be defined as the class of locally finite groups satisfying either:

(1.1) *if* $H \leqslant G$ *and* $|H| < \aleph_0$, *then* $|G: C_G(H)| < \aleph_0$,

or

(1.2) *if* $H \leqslant G$ *and* $|H| < \aleph_0$, *then* $|G: N_G(H)| < \aleph_0$,

Let $\mathfrak{F}$ denote the class of finite groups and Q, S, D the usual closure operations; it is clear that $\text{QSD}\mathfrak{F} \subseteq \mathfrak{X}$. In fact, $\text{QSD}\mathfrak{F}$ seems to be a large subclass of $\mathfrak{X}$ as P. Hall [3] showed that every countable $\mathfrak{X}$-group is a $\text{QSD}\mathfrak{F}$-group and Ju. M. Gorčakov [2] has shown that every residually finite $\mathfrak{X}$-group is a $\text{QSD}\mathfrak{F}$-group. P. Hall also gave an example, generalized in [6], of an $\mathfrak{X}$-group which is not a $\text{QSD}\mathfrak{F}$-group. Our aim here is to give further information about $\mathfrak{X}$-groups which are not in the class $\text{QSD}\mathfrak{F}$.

To present our results more clearly we introduce two further subclasses of $\mathfrak{X}$ using conditions similar to (1.1) and (1.2). We define $\mathfrak{Z}$ to be the class of locally finite groups G which satisfy:

(1.3) *if* $\mathfrak{m}$ *is an infinite cardinal and* $H \leqslant G$ *such that* $|H| < \mathfrak{m}$, *then* $|G: C_G(H)| < \mathfrak{m}$.

We let $\mathfrak{Y}$ denote the class of locally finite groups G satisfying:

(1.4) *if* $\mathfrak{m}$ *is an infinite cardinal and* $H \leqslant G$ *such that* $|H| < \mathfrak{m}$, *then* $|G : N_G(H)| < \mathfrak{m}$.

It was proved by P. Hall [3] and again by Gorčakov [2] that $\mathrm{QSD}\mathfrak{F} \subseteq \mathfrak{Z}$. It is clear that $\mathfrak{Z} \subseteq \mathfrak{Y}$ and the examples mentioned above are $\mathfrak{X}$-groups not in the class $\mathfrak{Y}$. (Hall used condition (1.3) to show that these groups were not in $\mathrm{QSD}\mathfrak{F}$ but it is just as easy to see that they are not $\mathfrak{Y}$-groups). Thus we have

$$\mathrm{QSD}\mathfrak{F} \subseteq \mathfrak{Z} \subseteq \mathfrak{Y} \subset \mathfrak{X}.$$

Our main result shows that the examples mentioned above are in some way typical of those $\mathfrak{X}$-groups not satisfying (1.4); we prove:

THEOREM A: *Let G be an $\mathfrak{X}$-group. Then G is a $\mathfrak{Y}$-group if and only if each extraspecial section of G is a $\mathfrak{Y}$-group.*

A section of a group G is a factor group H/K, where $K \lhd H \leqslant G$. An extraspecial p-group E is one in which $\zeta(E) = E'$ has order p and E/E' is elementary abelian. The extraspecial $\mathfrak{Y}$-groups are more easily identified by the following characterization:

THEOREM B: *Let E be an extraspecial p-group. Then E is a $\mathfrak{Y}$-group if and only if, for each infinite subgroup H of E and each maximal abelian subgroup A of H,* $|A| = |H|$.

Apart from the example in [6], a rather worse extraspecial group not in $\mathfrak{Y}$ has been constructed by A. Ehrenfeucht and V. Faber [1]. Their example is uncountable but *every* maximal abelian subgroup is countable.

The class $\mathfrak{Y}$ also has a connection with another problem concerning FC-groups. It was proved in [7] and again in [2] that if A is a subgroup of the $\mathrm{QSD}\mathfrak{F}$-group G, then $Lcl(A) = Cl(A)$ if and only if $Cl(A)$ is finite. (Here $Lcl(A)[Cl(A)]$ denotes the set of all subgroups of G which are locally conjugate [conjugate] to A.) However this result can be proved much more easily for the (possibly) larger class of $\mathfrak{Y}$-groups. In the proof of Theorem B of [7] it is shown that if $Cl(A)$ is infinite, then G has a countable normal subgroup N such that $Lcl(A \cap N)$ is uncountable. If $Lcl(A) = Cl(A)$, then $Lcl(A \cap N) = Cl(A \cap N)$ and so $A \cap N$ is a countable subgroup with $|G : N_G(A \cap N)|$ uncountable. Thus G is not a $\mathfrak{Y}$-group and we have:

THEOREM C: *Let A be a subgroup of the $\mathfrak{Y}$-group G. Then* $Lcl(A) = Cl(A)$ *if and only if* $Cl(A)$ *is finite.*

Theorem A indicates that in investigating $\mathfrak{X}$-groups which are not in the class $\mathrm{QSD}\mathfrak{F}$, we should begin by considering extraspecial groups. The main results that we prove may be summarized as:

THEOREM D: (i) *If E is an extraspecial $\mathfrak{Y}$-group of cardinality $\aleph_1$, then E is a $\mathfrak{Z}$-group.*

(ii) *There is an extraspecial $\mathfrak{Z}$-group of cardinality $\aleph_1$, which cannot be embedded in the central direct product of groups of order p^3.*

It seems unlikely that D(i) can be extended to show that $\mathfrak{Y}$ and $\mathfrak{Z}$ coincide for extraspecial groups of arbitrary cardinality but the construction of a counterexample would necessarily be very complicated. The example of D(ii) was intended to show that $\mathrm{QSD}\mathfrak{F} \neq \mathfrak{Z}$ but we have not been able to prove that this group is not a $\mathrm{QSD}\mathfrak{F}$-group. However it does show that a possible stronger conjecture is false. In [3], P. Hall actually proved that if G is a countable $\mathfrak{X}$-group, then $G \in \mathrm{QSD}(\mathfrak{F} \cap \mathfrak{V}(G))$, where $\mathfrak{V}(G)$ is the variety generated by G. The next result shows that the example G in D(ii) is not in the class $\mathrm{QSD}(\mathfrak{F} \cap \mathfrak{V}(G))$.

THEOREM E: (i) *Let E be an extraspecial $\mathrm{QSD}(\mathfrak{F} \cap \mathrm{Z}_p\mathfrak{A}_p)$-group where $\mathrm{Z}_p\mathfrak{A}_p$ denotes the variety of groups which are (central of exponent p)-by-(abelian of exponent p). Then E can be embedded in the central direct product of groups of order p^3.*

(ii) *There is an extraspecial subgroup of a central direct product of groups of order p^3 which is not itself a central direct product of groups of order p^3.*

We begin by proving the characterization of extraspecial $\mathfrak{Y}$-groups given in Theorem B as this will simplify later proofs. Our main result, Theorem A, is proved in Section 3. The remainder of the paper consists of the results concerning extraspecial p-groups. These results are all obtained in terms of symplectic spaces, using the well known relationship between extraspecial p-groups and non-degenerate symplectic spaces over $GF(p)$. The results we give do not depend on the underlying field of the symplectic spaces.

Although the example of D(ii) indicates that $\mathfrak{Z}$ is probably a larger class than $\mathrm{QSD}\mathfrak{F}$, we should emphasize that we have still not definitely constructed any examples of groups outside $\mathrm{QSD}\mathfrak{F}$ other than the known examples mentioned earlier. Apart from this obvious question it would also be interesting to know whether these investigations can be reduced entirely to studying extraspecial groups by showing that $\mathrm{QSD}\mathfrak{F}$-groups and $\mathfrak{Z}$-groups can be recognized by their extraspecial sections.

Our terminology concerning symplectic spaces is slightly different from that used in [5]. We call a subspace A *isotropic* if $(x, y) = 0$, for all $x, y \in A$; the term totally isotropic is used by Kapłansky. A *hyperbolic plane* is simply a 2-dimensional non-degenerate symplectic space H; that is, H has a basis $\{x, y\}$ such that $(x, y) = 1$. The expression $A \oplus B$ denotes the direct sum of subspaces A and B in the usual sense. If, in addition the subspaces A and B are orthogonal (i.e. $(a, b) = 0$, for all $a \in A$, $b \in B$) then we refer to the orthogonal sum of A and B. In what follows, it is usually clear when a direct sum is an orthogonal sum as one of the subspaces will be contained in the orthogonal complement of the other. (The orthogonal complement of a subspace U is defined to be $U^{\perp} = \{x \in V; (x, u) = 0 \text{ for all } u \in U\}$.) The only occasion on which we stress the orthogonality is when considering an orthogonal sum V of hyperbolic planes H_i. We denote this by

$$V = \bigoplus_{i \in I}^{\perp} H_i.$$

Kaplansky [5] refers to such a space as having a symplectic basis.

2. Extraspecial $\mathfrak{Y}$-Groups

LEMMA (2.1): *$\mathfrak{Y}$ and $\mathfrak{Z}$ are* QS-*closed classes.*

PROOF: Let $G \in \mathfrak{Y}$ and $U \leqslant G$. If H is an infinite subgroup of U, then $|G : N_G(H)| \leqslant |H|$ and so $|U : N_U(H)| \leqslant |G : N_G(H)| \leqslant |H|$ and $\mathfrak{Y}$ is S-closed.

Now let $N \lhd G$ and H/N be an infinite subgroup of G/N. Then there is a subgroup U of G such that $|U| = |H/N|$ and $H = UN$. $N_G(H) \geqslant N_G(U)$ and so

$$|G/N : N_{G/N}(H/N)| \leqslant |G : N_G(U)| \leqslant |U| = |H/N|$$

and so $G/N \in \mathfrak{Y}$.

To prove that $\mathfrak{Z}$ is Q-closed we follow a similar argument, noting that $C_G(H/N) \geqslant C_G(U)$.

PROOF OF THEOREM B: Let E be an infinite extraspecial $\mathfrak{Y}$-group and let A be a maximal abelian subgroup of E; then $C_E(A) = A$. If $|A| < |E|$ then $|E : C_E(A)| > |A|$. Since A has finite exponent it is a direct product of finite abelian groups and so may be expressed as $A = B \times Y$ where Y is a finite group containing $Z = \zeta(E)$. Then $|B| = |A|$ and

$$|E : C_E(A)| = |E : C_E(B)|$$

so that $|E : C_E(B)| > |B|$. But since $B \cap E' = 1$, we have $C_E(B) = N_E(B)$ and so $|E : N_E(B)| > |B|$, contrary to E being a $\mathfrak{Y}$-group. Therefore $|A| = |E|$.

Now let H be any infinite subgroup of E and A a maximal abelian subgroup of H. If $H \cap Z = 1$, then H is abelian and clearly $|A| = |H|$. So we may assume that $H \geqslant Z$. Let the elements of $\zeta(H)/Z$ be $\bar{x}_i$, $i \in I$. For each $i \in I$, there is an element $y_i \in E$ such that $[x_i, y_i] \neq 1$. Let $H_1 = \langle H, y_i; i \in I \rangle$; then $|H_1/H| \leqslant |\zeta(H)| \leqslant |A|$ and H_1 is extraspecial. Also, by (2.1), H_1 is a $\mathfrak{Y}$-group. Let A_1 be a maximal abelian subgroup of H_1 containing A. Then $|A_1| \leqslant |A| \cdot |H_1/H| \leqslant |A|^2 = |A|$ and, by the above, $|A_1| = |H_1|$. Therefore $|A| = |A_1| = |H_1| = |H|$, as required.

Conversely, let E be an infinite extraspecial group with an infinite subgroup U such that $|E : N_E(U)| > |U|$. Then $U \cap Z = 1$ and so $N_E(U) = C_E(U) \geqslant UZ$ and U is abelian. Since E/UZ is elementary abelian, there is a subgroup H of E such that $HC_E(U) = E$ and $H \cap C_E(U) = UZ$. Now let A be a maximal abelian subgroup of H containing UZ. Then $A \leqslant C_H(U) = UZ$ and so $|A| = |UZ| = |U|$. But $|H/A| = |H/UZ| = |E : C_E(U)| > |U|$ and we obtain $|A| < |H|$, as required.

3. Proof of Theorem A

It follows from (2.1) that if G is a $\mathfrak{Y}$-group then every section of G is a $\mathfrak{Y}$-group. So we assume that G is not a $\mathfrak{Y}$-group and show that there is an extraspecial section of G which is not in $\mathfrak{Y}$. Starting with the group G we shall repeatedly replace G with either a subgroup or factor group thus imposing more and more restrictions on G until we are left with an extraspecial group not in $\mathfrak{Y}$.

Let U be an infinite subgroup of G such that $|G : N_G(U)| > |U|$. Clearly we may assume that $\operatorname{core}_G(U) = 1$.

Let $Z = \zeta(G)$. $G/Z \in \mathrm{R}\mathfrak{F} \cap \mathfrak{X} \subseteq \mathrm{QSD}\mathfrak{F} \subseteq \mathfrak{Z}$ ([2], Theorem 2). Therefore $|G : C_G(UZ/Z)| \leqslant |UZ/Z| \leqslant |U|$ and so

$$|C_G(UZ/Z) : C_G(UZ/Z) \cap N_G(U)| > |U|.$$

Replacing G by $UC_G(UZ/Z)$ we may assume:

(1) $UZ \lhd G, |C : N_C(U)| > |U|$, *where* $C = C_G(UZ/Z)$.

If $c \in C$ then $U^c \leqslant UZ_c$, where $Z_c = [U, c]$ is a finite subgroup of Z. $Z_c \leqslant U^G \cap Z$ and $|U^G| = |U|$ so that $|U^G \cap Z| \leqslant |U|$. Thus there are at most $|U|$ finite subgroups of $U^G \cap Z$. Therefore there is a finite subgroup F of $U^G \cap Z$ such that UF contains more than $|U|$ conjugates of U. We choose F to have minimal order.

If $U^c \leqslant UF$, then $[U, c] \leqslant F$ and so $c \in C_G(UF/F)$. Now writing G for $UC_G(UF/F)$, C for $C_G(UF/F)$, we have

(2) $UF \lhd G,\ F \leqslant Z,\ C_G(UF/F) = C,\ |C : N_C(U)| > |U|,\ U \cap F = 1$.
($U \cap F = 1$ since $F \leqslant Z$ and core $U = 1$.)

There is a subgroup $D \leqslant F$ such that F/D is cyclic of prime order p, say. By the minimality of F, U has at most $|U|$ conjugates contained in UD but has more than $|U|$ contained in UF. It follows that UD has more than $|U|$ conjugates contained in UF.

Replacing G by G/D, U by UD/D etc., we have

(3) *F has order p*, $F \leqslant Z, U \cap F = 1, UF \lhd G, C = C_G(UF/F), G = UC$ *and* $|C : N_C(U)| > |U|$.

Let $u \in U$, $c \in C$; then $[c, u] \in F$ and so

$$[c, u^p] = [c, u]^p = 1,$$

i.e. C centralizes U^p. Also

$$[c, [u_1, u_2]] = [[c, u_1], [c, u_2]] = 1$$

and so C centralizes U'.

Therefore $U'U^p \lhd UC = G$.

Factoring out $U'U^p$, we may assume that U is elementary abelian and so $U \leqslant C_G(UF/F)$. We may also assume that $G = C_G(UF/F)$ and so obtain

(4) *F has order p, $F \leqslant Z$, UF is a normal elementary abelian p-group* $U \cap F = 1$, $G = C_G(UF/F)$ *and* $|G : N_G(U)| > |U|$.

Let $D = C_G(U)$, so that $D = C_G(UF) \lhd G$. Let $g_1, g_2 \in G$; then $[g_i, u] \in F$ and so

$$[g_i^p, u] = [g_i, u]^p = 1$$

and

$$[[g_1, g_2], u] = [[g_1, u], [g_2, u]] = 1.$$

Therefore G/D is an elementary abelian p-group and, since $D \leqslant N_G(U)$, we have $|G/D| > |U|$. We may now clearly replace G by a Sylow p-subgroup and so assume that

(5) *G is a p-group, $F \leqslant Z$, $|F| = p$, $U \cap Z = 1$, U and $G/C_G(U)$ are elementary abelian p-groups, $UF \lhd G$ and $|G : N_G(U)| > |U|$.*

Let $N \lhd G$ be maximal subject to $N \cap UF = 1$. Then if $U^x \leqslant UN$, we have $U^x \leqslant UN \cap UF = U(N \cap UF) = U$ and so $U^x = U$. Thus $N_G(UN) = N_G(U)$ and we can replace G by G/N. Thus, in addition to (5), we also have

(6) *F is the unique minimal normal subgroup of G and, in particular, Z is locally cyclic.*

We now construct an ascending chain of subgroups

$$UF = A_0 < A_1 < \ldots < A_\alpha < A_{\alpha+1} < \ldots \qquad (\alpha < \rho)$$

(where ρ is the least ordinal such that $|\rho| > |U|$) such that

(i) A_α/F is abelian and $|A_\alpha| = |U|$,
(ii) $A_\alpha \cap Z = F$,
(iii) $A_\alpha N_G(U) < A_{\alpha+1} N_G(U)$,
(iv) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$, for limit ordinals $\beta < \rho$.

Then, letting $A = \bigcup_{\alpha<\rho} A_\alpha$, we see that A/F is abelian, $A \cap Z = F$ and $|A : N_A(U)| > |U|$.

We construct the A_α inductively. The limit ordinal case is clear as (i) and (ii) follow immediately from (iv). So we may assume that $A_{\alpha-1}$ has been constructed. Let $C_{\alpha-1} = C_G(A_{\alpha-1}/F)$; then since $A_{\alpha-1}/F \cong A_{\alpha-1}Z/Z$ and $G/Z \in \mathfrak{Z}$, we have $|G/C_{\alpha-1}| \leqslant |A_{\alpha-1}| = |U|$. Therefore

$$|C_{\alpha-1}/C_{\alpha-1} \cap N_G(U)| > |A_{\alpha-1}|$$

and so $C_{\alpha-1} > C_{\alpha-1} \cap A_{\alpha-1} N_G(U)$.

Let $\langle \bar{z} \rangle$ be the unique subgroup of $A_{\alpha-1}Z/A_{\alpha-1}$ of order p, and let $a_1 \in C_{\alpha-1} - A_{\alpha-1} N_G(U)$. Then $\langle \bar{a}_1, \bar{z} \rangle$ is a finite abelian group and so

$|C_{\alpha-1} : C_{\alpha-1} \cap C_G(\langle \bar{a}_1, \bar{z} \rangle)|$ is finite. Therefore there is an element $a_2 \in C_{C_{\alpha-1}}(\langle \bar{a}_1, \bar{z} \rangle) - A_{\alpha-1} N_G(U)$. Thus $\langle \bar{a}_1, \bar{a}_2, \bar{z} \rangle$ is again a finite abelian group and since $\langle \bar{a}_1, \bar{a}_2 \rangle$ is not cyclic, $\langle \bar{a}_1, \bar{a}_2, \bar{z} \rangle$ is not cyclic. Therefore $\langle \bar{a}_1, \bar{a}_2, \bar{z} \rangle$ contains an element no power of which is equal to $\bar{z}$. That is, there is an element $a \in C_{\alpha-1} - A_{\alpha-1} N_G(U)$ such that $A_{\alpha-1} Z / A_{\alpha-1}$ has trivial intersection with $\langle \bar{a} \rangle$.

Let $A_\alpha = \langle A_{\alpha-1}, a \rangle$. Clearly A_α / F is abelian, $|A_\alpha| = |A_{\alpha-1}|$ and $A_\alpha N_G(U) > A_{\alpha-1} N_G(U)$. Also

$$A_\alpha \cap Z = \langle A_{\alpha-1}, a \rangle \cap A_{\alpha-1} Z \cap Z = A_{\alpha-1} \cap Z = F.$$

Replacing G by A, we have:

(7) *G is a p-group, G/F is abelian, $|F| = p$, U is elementary abelian and $|G : N_G(U)| > |U|$.*

If $x, y \in G$, then $[x, y] \in F$ and so $[x^p, y] = [x, y]^p = 1$ and so $x^p \in Z$ i.e. G/Z is elementary abelian. We can also repeat the argument preceding (6) so that we may assume (7) together with

(8) *G/Z is elementary abelian and Z is locally cyclic.*

Let X be maximal subject to $X \geqslant U$, $X \cap Z = F$ so that G/X is finite if Z is finite or is countable if Z is infinite. In either case $|G/X| \leqslant |U|$ and so $|X : N_X(U)| > |U|$. Also $X/F \cong XZ/Z$ is elementary abelian and so, replacing G by X, we have

(9) *G is a p-group such that $|G'| = p$ and G/G' is elementary abelian, $U \cap G' = 1$ and $|G : N_G(U)| > |U|$.*

It is now possible that $Z > G'$. If so, then there is a subgroup $Y \geqslant UG'$ such that $G/F = Y/F \times Z/F$ and clearly $\zeta(Y) = G'$. Replacing G by Y, we obtain the required result.

Although this proof relies heavily on the fact that G/Z is a $\mathfrak{Z}$-group (and not just a $\mathfrak{Y}$-group) it does not seem to be possible to adapt the methods to give the corresponding result for $\mathfrak{Z}$. It is possible to prove that an $\mathfrak{X}$-group which is not in $\mathfrak{Z}$ has a section G which is a p-group, contains a central subgroup F of order p and a normal subgroup $U \geqslant F$ such that $|C_G(U/F) : C_G(U)| > |U|$. The main difficulty in making further reductions is that when one considers centralizers it is not usually possible to factor out normal subgroups.

4. Extraspecial QSD$\mathfrak{F}$-groups

We begin by giving a reduction theorem for arbitrary extraspecial QSD$\mathfrak{F}$-groups.

LEMMA (4.1): *Let G be an extraspecial section of $\mathrm{Dr}_{i\in I} D_i$, where each D_i is finite. Then G is isomorphic to a section of $\mathrm{Dr}_{i\in I} E_i$, where $E_i \in \mathrm{QS}\{D_i\}$ and E_i is a monolithic p-group.*

[A group is *monolithic* if it has a unique minimal normal subgroup.]

PROOF: Let $G = H/K$ where $K \lhd H \leqslant D = \mathrm{Dr}_{i\in I} D_i$. Let T be a Sylow p-subgroup of H so that $TK = H$ and $G \cong T/(T \cap K)$. If S is a Sylow p-subgroup of D containing T, then $S = \mathrm{Dr}_{i\in I} S_i$, where S_i is a Sylow p-subgroup of D_i, and G is isomorphic to a section T/U of S.

We may assume that the index set I is well-ordered. For each $i \in I$ choose $N_i \lhd S_i$ maximal with respect to

$$N_i \cap TM_i \leqslant UM_i,$$

where $M_i = \langle N_j; j < i\rangle$.

Define $N = \mathrm{Dr}_{i\in I} N_i = \bigcup_{i\in I} M_i$. We show that $N \cap T \leqslant U$ so that G is isomorphic to the section NT/NU of $S/N \cong \mathrm{Dr}_{i\in I} S_i/N_i$. Suppose that $M_i \cap T \leqslant U$; then

$$\begin{aligned} M_{i+1} \cap T &= M_i N_i \cap T \\ &= M_i N_i \cap TM_i \cap T \\ &= M_i(N_i \cap TM_i) \cap T \\ &\leqslant M_i U \cap T \\ &= U(M_i \cap T) \\ &= U. \end{aligned}$$

Therefore, by induction, $M_i \cap T \leqslant U$ for all $i \in I$ and so

$$N \cap T = \bigcup_{i\in I} M_i \cap T \leqslant U.$$

$E_i = S_i/N_i$ is a p-group and $E_i \in \mathrm{QS}\{D_i\}$; it remains to show that E_i is monolithic. Let X_i/N_i be a normal subgroup of S_i/N_i. If $X_i \cap TN \leqslant UN$ then

$$X_i \cap TM_i \leqslant UN \cap TM_i = (UN \cap T)M_i = U(N \cap T)M_i = UM_i$$

and so, by the maximality of N_i, we get $X_i = N_i$. Writing H/K again for G as a section of $\mathrm{Dr}_{i\in I} E_i$ this means that for every non-trivial normal subgroup X_i of E_i, we have $X_i \cap H \not\leqslant K$. Let $L/K = \zeta(H/K)$ and $Z_i = \zeta(E_i)$. Then $(Z_i \cap H)K = L$ and so $Z_i \cap H = Z_i \cap L$ has order p. If Y_i is a minimal normal subgroup of E_i, then $(Y_i \cap H)K = L$ and so $Y_i \leqslant L \cap Z_i$ and we must have $Y_i = Z_i \cap L$. Thus $Z_i \cap L$ is the monolith of E_i.

To obtain Theorem $E(i)$ we first need to consider the symplectic space associated with an extraspecial group. We state this explicitly as

THEOREM (4.2): *Let G be an extraspecial p-group with $\zeta(G) = \langle z \rangle$.*

(i) *$\bar{G} = G/\zeta(G)$ becomes a non-degenerate symplectic space over $GF(p)$ if we define $(\bar{x}, \bar{y}) = a$, where $\bar{x} = x\zeta(G)$, $\bar{y} = y\zeta(G)$ and $[x, y] = z^a$.*

(ii) *G is a central direct product of groups of order p^3 if and only if the symplectic space $\bar{G}$ is an orthogonal sum of hyperbolic planes.*

It is well-known (e.g. [5], p. 45) that a non-degenerate symplectic space of countable dimension is an orthogonal sum of hyperbolic planes so we see immediately that every countable extraspecial p-group is a central direct product of groups of order p^3.

PROOF OF THEOREM E(i):

By Lemma (4.1) we may assume that $G = H/K$ where

$$K \lhd H \leqslant E = \operatorname*{Dr}_{i\in I} E_i$$

and each E_i is a monolithic p-group. Let $L/K = \zeta(H/K)$, $Z_i = \zeta(E_i)$ and N_i be the monolith of E_i. If $N = \mathrm{Dr}_{i\in I} N_i$, then $(N \cap H)K = L$. $H/(N \cap H) \cong NH/H$ is elementary abelian and so there is a subgroup M such that $ML = H$ and $M \cap L = N \cap H$. Thus $H/K \cong M/(H \cap N)$ and, replacing H/K by $M/(H \cap N)$, we may assume that $K \leqslant N$ and hence $H \cap N = L$.

N is elementary abelian and so there is a subgroup X such that $N = LX$, $K = L \cap X$ and so $H/K \cong HX/KX$. Replacing H/K by HX/KX we may now assume that $L = N$.

Let $Z = \zeta(E) = \mathrm{Dr}_{i\in I} Z_i$. Since $H \cap Z = N$, there is a subgroup $U \geqslant H$ such that $UZ = E$, $U \cap Z = N$. We show that the extraspecial group U/K is a central direct product of groups of order p^3. For each $i \in I$, there is a finite extraspecial subgroup V_i such that $Z_i V_i = E_i$,

$Z_i \cap V_i = N_i$. Let $V = \mathrm{Dr}_{i \in I} V_i$ so that $ZV = E$ and $Z \cap V = N$. Then the extraspecial group V/K is the central direct product of the groups V_i and so the associated symplectic space $\overline{V/K}$ is an orthogonal sum of hyperbolic planes. But the symplectic spaces $\overline{U/K}$ and $\overline{V/K}$ are clearly isomorphic under the mapping which takes $\bar{u} \in \overline{U/K}$ to its projection on $\overline{V/K}$ in the symplectic space $\overline{Z/K} \oplus \overline{V/K}$. Thus $\overline{U/K}$ is an orthogonal sum of hyperbolic planes and H/K is embedded in the central product U/K of groups of order p^3.

5. Symplectic spaces associated with $\mathfrak{Z}$-groups and $\mathfrak{Y}$-groups

We begin by defining $\mathfrak{Y}$-spaces and $\mathfrak{Z}$-spaces in such a way that an extraspecial p-group G is a $\mathfrak{Y}$-group ($\mathfrak{Z}$-group) if and only if its associated symplectic space $\bar{G}$ is a $\mathfrak{Y}$-space ($\mathfrak{Z}$-space). For $\mathfrak{Z}$ we can simply translate (1.3) into the language of symplectic spaces so that a non-degenerate symplectic space V over a field $\mathfrak{k}$ is a $\mathfrak{Z}$-space if it satisfies:

if $\mathfrak{m}$ *is an infinite cardinal and* $U \subseteq V$ *such that* $\dim U < \mathfrak{m}$, *then* $\dim(V/U^{\perp}) < \mathfrak{m}$.

For $\mathfrak{Y}$ we need to use the characterization of extraspecial $\mathfrak{Y}$-groups given in Theorem B. Thus a non-degenerate symplectic space V over a field $\mathfrak{k}$ is a $\mathfrak{Y}$-space if it satisfies:

for each infinite-dimensional subspace U of V and for each maximal isotropic subspace A of U, $\dim A = \dim U$.

We also require a rather stronger condition, calling V a $\mathfrak{W}$-space if it satisfies:

if $\mathfrak{m}$ *is an infinite cardinal and* $U \subseteq V$ *such that* $\dim U < \mathfrak{m}$, *then there is a subspace* $W \supseteq U$ *such that* $\dim W < \mathfrak{m}$ *and* $V = W \oplus W^{\perp}$.

It is clear that an orthogonal sum of hyperbolic planes is a $\mathfrak{W}$-space and that every non-degenerate subspace of a $\mathfrak{W}$-space is a $\mathfrak{Z}$-space. We shall show that every $\mathfrak{Z}$-space of dimension $\aleph_1$ can be embedded in a $\mathfrak{W}$-space. First we embed in a larger space so that a given subspace is contained in an orthogonal summand of the same dimension.

Lemma (5.1): *Let U be an infinite-dimensional subspace of the $\mathfrak{Z}$-space V. Then V can be embedded in a $\mathfrak{Z}$-space $\bar{V} = \bar{V}(V, U)$ such that*

$\dim \bar{V} = \dim V$ *and* $\bar{V}$ *contains a subspace* $\bar{U} = \bar{U}(U) \supseteq U$ *such that* $\dim \bar{U} = \dim U$ *and* $\bar{V} = \bar{U} \oplus \bar{U}^*$.

Furthermore, if $W \subseteq U$ *and* $V = W \oplus W^{\perp}$, *then* $\bar{V} = W \oplus W^*$. [Here X^* denotes the orthogonal complement of X in $\bar{V}$ and $X^{\perp}$ the orthogonal complement in V.]

Proof: By adjoining elements to U, if necessary, we may assume that $U \cap U^{\perp} = 0$. Let $V = U \oplus U^{\perp} \oplus X$; since V is a $\mathfrak{Z}$-space, $\dim X \leqslant \dim U$ and we can choose a basis $\{x_i; i \in I\}$ of X so that $|I| \leqslant \dim U$.

Let $\bar{V}$ be spanned by V and basis elements y_i, $i \in I$. Define an alternate product on $\bar{V}$ by

$$
\begin{aligned}
&(u, y_i) = (u, x_i) && \text{for all } u \in U,\\
&(w, y_i) = 0 && \text{for all } w \in U^{\perp},\\
&(x_j, y_i) = (y_j, y_i) = 0 && \text{for all } i, j \in I.
\end{aligned}
$$

Let $\bar{U}$ be the subspace of $\bar{V}$ spanned by U and the y_i, $i \in I$; then $\dim \bar{U} = \dim U$.

$$\bar{U}^* = \langle U^{\perp}, x_i - y_i; i \in I\rangle \qquad \text{and so } \bar{U} \oplus \bar{U}^* = \bar{V}.$$

If $W \subseteq U \subseteq V$ such that $V = W \oplus W^{\perp}$, then

$$W^* \supseteq \langle W^{\perp}, x_i - y_i; i \in I\rangle.$$

Since

$$x_i \in W \oplus W^{\perp} \subseteq W \oplus W^*,$$

we have $y_i \in W \oplus W^*$ and so $W \oplus W^* = \bar{V}$.

It remains to show that $\bar{V}$ is a $\mathfrak{Z}$-space. Writing Y for $\langle y_i; i \in I\rangle$, we have $\bar{V} = U \oplus U^{\perp} \oplus X \oplus Y$.

Let $S \subseteq \bar{V}$ and $\dim S = \mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal. Let A, B and C denote the projections of S on U, $U^{\perp} \oplus X$ and Y, respectively. Since V is a $\mathfrak{Z}$-space, we have $\dim (V/A^{\perp}) \leqslant \mathfrak{m}$ and so

$$\dim (X/(X \cap A^{\perp})) \leqslant \mathfrak{m}.$$

Since $x_i - y_i \in A^*$, for all $i \in I$, it follows that

$$\dim (Y/(Y \cap A^*)) \leqslant \mathfrak{m}.$$

Therefore

$$\dim(\bar{V}/A^*) \leqslant \dim(V/A^\perp) + \dim(Y/(Y \cap A^*)) \leqslant \mathfrak{m}.$$

$B^* \supseteq Y$ and so $\dim(V/B^*) = \dim(V/B^\perp) \leqslant \mathfrak{m}$. $C^* \supseteq U^\perp \oplus X \oplus Y$ and so $\dim(\bar{V}/C^*) = \dim(V/(V \cap C^*))$. If $\phi : Y \to X$ is the mapping which takes y_i to x_i, for each $i \in I$, then $V \cap C^* = (C\phi)^\perp$. Since $C\phi$ is a subspace of the $\mathfrak{Z}$-space V, we have $\dim(V/(C\phi)^\perp) \leqslant \mathfrak{m}$ and hence $\dim(\bar{V}/C^*) \leqslant \mathfrak{m}$. $S^* \supseteq A^* \cap B^* \cap C^*$ and so we obtain $\dim(\bar{V}/S^*) \leqslant \mathfrak{m}$.

THEOREM (5.2): *A $\mathfrak{Z}$-space V such that* $\dim V$ *is a regular cardinal can be embedded in a space $\bar{V}$ satisfying the condition*:

if $U \subseteq \bar{V}$ and $\dim U < \dim \bar{V}$ *then there is a subspace $W \supseteq U$ such that $\bar{V} = W \oplus W^\perp$ and* $\dim W < \dim \bar{V}$.

PROOF: Let ρ be the least ordinal with cardinality $\dim V$; then V has a basis $\{x_i; i < \rho\}$. Let $V_\alpha = \langle x_i; i < \alpha\rangle$ so that $V = \bigcup_{\alpha<\rho} V_\alpha$.

We construct spaces $V(\alpha) \supseteq V$ and subspaces $\bar{V}_\alpha$ of $V(\alpha)$ such that

(1) $$\dim V(\alpha) = \dim V, \qquad \dim \bar{V}_\alpha = \dim V_\alpha,$$

(2) $$\text{if} \quad \beta \leqslant \alpha, \quad V \subseteq V(\beta) \subseteq V(\alpha),$$

$$V_\beta \subseteq \bar{V}_\beta \subseteq \bar{V}_\alpha \quad \text{and} \quad V(\alpha) = \bar{V}_\beta \oplus \bar{V}_\beta^\perp = \bar{V}_\alpha + V.$$

Then we may define $\bar{V} = \bigcup_{\alpha<\rho} V(\alpha) = \bigcup_{\alpha<\rho} \bar{V}_\alpha$. Since $\dim \bar{V}$ is a regular cardinal, any subspace of $\bar{V}$ with dimension less than $\dim \bar{V}$ is contained in some $\bar{V}_\alpha$ $(\alpha < \rho)$ and $\bar{V} = \bar{V}_\alpha \oplus \bar{V}_\alpha^\perp$ and the result follows.

The spaces $V(\alpha)$ and $\bar{V}_\alpha$ are constructed inductively. We may suppose that $V(\beta)$ and $\bar{V}_\beta$ have been constructed for each $\beta < \alpha$.

Case (i): $\alpha = \gamma + 1$.

We have $V(\gamma) = V + \bar{V}_\gamma \oplus \bar{V}_\gamma = \bar{V}_\gamma^\perp$. Let $U = V_\alpha + \bar{V}_\gamma \subseteq V(\gamma)$ and define

$$V(\alpha) = \bar{V}(V(\gamma), U)$$

as in Lemma (5.1) and

$$\bar{V}_\alpha = \bar{U}(U) \supseteq V_\alpha + \bar{V}_\gamma.$$

By the Lemma, whenever $\beta < \alpha$, $\bar{V}_\beta$ is an orthogonal summand of $V(\alpha)$.

$V(\alpha) = V(\gamma)+\bar{V}_\alpha$, by construction, and the induction hypothesis gives $V(\alpha) = V+\bar{V}_\gamma+\bar{V}_\alpha = V+\bar{V}_\alpha$. Thus $V(\alpha)$ and $\bar{V}_\alpha$ satisfy the conditions (1) and (2).

Case (ii): α a limit ordinal.

Let $V_0(\alpha) = \bigcup_{\beta<\alpha} V(\beta)$ and let $U = V_\alpha + \bigcup_{\beta<\alpha} \bar{V}_\beta \subseteq V_0(\alpha)$. Define

$$V(\alpha) = \bar{V}(V_0(\alpha), U) \quad \text{and} \quad \bar{V}_\alpha = \bar{U}(U) \supseteq V_\alpha + \bigcup_{\beta<\alpha} \bar{V}_\beta.$$

For each $\beta < \alpha$ and for each γ with $\beta \leqslant \gamma < \alpha$,

$$V(\gamma) = \bar{V}_\beta \oplus (V(\gamma) \cap \bar{V}_\beta^\perp)$$

and so $\bar{V}_\beta$ is an orthogonal summand of $V_0(\alpha)$. By Lemma (5.1), $\bar{V}_\beta$ is an orthogonal summand of $V(\alpha)$. By construction,

$$V(\alpha) = V_0(\alpha)+\bar{V}_\alpha = \bigcup_{\beta<\alpha} (V+\bar{V}_\beta)+\bar{V}_\alpha = V+\bar{V}_\alpha.$$

Thus $V(\alpha)$ and $\bar{V}_\alpha$ again satisfy conditions (1) and (2).

Since the space $\bar{V}$ in the theorem is clearly a $\mathfrak{W}$-space if its dimension is $\aleph_1$, this shows that every $\mathfrak{Z}$-space of dimension $\aleph_1$ can be embedded in a $\mathfrak{W}$-space. Combining this with Theorem D(i), which we prove now, gives

THEOREM (5.3): *A $\mathfrak{Y}$-space of dimension $\aleph_1$ can be embedded in a $\mathfrak{W}$-space.*

PROOF: It remains only to show that a $\mathfrak{Y}$-space of dimension $\aleph_1$ is a $\mathfrak{Z}$-space.

Suppose that V has a subspace U of dimension $\aleph_0$ such that $\dim(V/U^\perp) = \aleph_1$. Then $U = A+B$, the sum of two isotropic subspaces, and $U^\perp = A^\perp \cap B^\perp$.

Therefore V has an isotropic subspace A, say, such that $\dim A = \aleph_0$ and $\dim(V/A^\perp) = \aleph_1$. There is a subspace W of V such that $V = W \oplus A^\perp$. Let $X = A+W$; then $A^\perp \cap X = A$ and so A is a maximal isotropic subspace of X although $\dim X = \aleph_1$ and $\dim A = \aleph_0$.

6. The counterexamples

We begin by giving an example of a non-degenerate subspace V of an orthogonal sum of hyperbolic planes which is not a $\mathfrak{W}$-space. In particular V itself is not an orthogonal sum of hyperbolic planes.

Let

$$X = \bigoplus_{0 \leqslant i < \omega_1}^{\perp} \langle x_i, y_i \rangle,$$

where $\langle x_i, y_i \rangle$ is a hyperbolic plane such that $(x_i, y_i) = 1$ and ω_1 is the least uncountable ordinal.

Let V be the subspace of X spanned by the elements x_i and $y_i - y_j$ with $0 \leqslant i, j < \omega_1$. Writing z_i for $y_i - y_0$, we see that

$$V = \langle x_0 \rangle + \bigoplus_{1 \leqslant i < \omega_1}^{\perp} \langle x_i, z_i \rangle,$$

$(x_i, z_i) = 1$ and $(z_i, x_0) = 1$.

Let

$$U_0 = \langle x_0 \rangle + \bigoplus_{1 \leqslant n < \omega}^{\perp} \langle x_n, z_n \rangle$$

and suppose that $V = U \oplus U^{\perp}$ where $U \supseteq U_0$. We show that U has uncountable dimension.

For each $i < \omega_1$, $x_i \in U \oplus U^{\perp}$ and so there is an element $u \in U$ such that $u - x_i \in U^{\perp}$. Let $u = kx_0 + w$, where $k \in \mathfrak{k}$ and

$$w \in W = \bigoplus_{1 \leqslant i < \omega_1}^{\perp} \langle x_i, z_i \rangle.$$

For each $n < \omega$ with $n \neq i$, $(z_n, u - x_i) = 0$ and so $(z_n, w) = -k$, for infinitely many n. It follows that $k = 0$ and so $w - x_i \in U^{\perp}$. That is, for each $i < \omega_1$, there is an element $w_i \in W \cap U$ such that $w_i - x_i \in U^{\perp}$. Similarly, there is an element $v_i \in W \cap U$ such that $v_i - z_i \in U^{\perp}$.

Now $(w_i, v_j - z_j) = 0$ and $(v_j, w_i - x_i) = 0$. Therefore $(w_i, v_j) = (w_i, z_j)$ and $(v_j, w_i) = (v_j, x_i)$ and hence $(w_i, z_j) = (x_i, v_j)$.

If w_i and v_j are written as linear combinations of x's and z's then the above shows that the coefficient of x_j in w_i is equal to the coefficient of z_i in v_j. Since $(x_0, v_j - z_j) = 0$, we have $(x_0, v_j) = 1$, for each j, and so there is some z_i having a non-zero coefficient in v_j. Therefore, for each j, there is a w_i in which the coefficient of x_j is non-zero. It follows that there are uncountably many different w_i's and $\dim(W \cap U)$ is uncountable.

It should be noted that an orthogonal summand V of an orthogonal sum of hyperbolic planes is also an orthogonal sum of hyperbolic planes. The method used by I. Kaplansky [4] for modules shows that V is an orthogonal sum of spaces of countable dimension. The structure of symplectic spaces of countable dimension then gives the required result.

We saw in Section 5 that every $\mathfrak{Y}$-space of dimension $\aleph_1$ can be embedded in a $\mathfrak{W}$-space. Because $\mathfrak{W}$-spaces seem to be very close to orthogonal sums of hyperbolic planes one might hope to prove a stronger embedding theorem. However our next example is a $\mathfrak{W}$-space of dimension $\aleph_1$ which cannot be embedded in an orthogonal sum of hyperbolic planes.

Let V have a basis consisting of x_i, y_i and z_α, where i takes all ordinal values less than ω_1 and α takes all *limit* ordinal values less than ω_1. We define an alternate product on V by

$$(x_i, x_j) = (y_i, y_j) = (z_\alpha, z_\beta) = 0$$
$$(x_i, y_j) = \delta_{ij}$$
$$(y_i, z_\alpha) = 0$$
$$(x_i, z_\alpha) = \begin{cases} 1 & \text{if } i < \alpha, \\ 0 & \text{if } i \geqslant \alpha. \end{cases}$$

For each $\varepsilon < \omega_1$, let

$$V_\varepsilon = \langle x_i, y_i, z_\alpha;\ i < \varepsilon,\ \alpha \leqslant \varepsilon\rangle;$$

clearly $V = \bigcup_{\varepsilon<\omega_1} V_\varepsilon$. Each countable dimensional subspace of V is contained in some V_ε and so to show that V is a $\mathfrak{W}$-space it is sufficient to show that $V = V_\varepsilon \oplus V_\varepsilon^\perp$. Certainly $V_\varepsilon^\perp \supseteq \langle x_j, y_j; j \geqslant \varepsilon\rangle$ and if $\varepsilon = \alpha+n$, where α is a limit ordinal and $1 \leqslant n < \omega$, then

$$z_\alpha + y_\alpha + y_{\alpha+1} + \dots + y_{\alpha+n-1} - z_\beta \in V_\varepsilon^\perp, \qquad \text{for all } \beta > \varepsilon.$$

If ε is a limit ordinal then $z_\varepsilon - z_\beta \in V_\varepsilon^\perp$, for all $\beta > \varepsilon$. In both cases $z_\beta \in V_\varepsilon + V_\varepsilon^\perp$ and hence $V_\varepsilon \oplus V_\varepsilon^\perp = V$, as required.

Now suppose that

$$V \subseteq H = \mathop{\oplus^\perp}_{i<\omega_1} H_i,$$

where $H_i = \langle a_i, b_i\rangle$ and $(a_i, b_i) = 1$. Write

$$K_i = \mathop{\oplus^\perp}_{j<i} H_j.$$

Let i be any ordinal less than ω_1; then there is a least ordinal $j(i) < \omega_1$ such that $V_i \subseteq K_{j(i)}(\cap V)$. If $j < \omega_1$, then there is a least ordinal $i(j)$ such that $K_j \cap V \subseteq V_{i(j)}$. Given an ordinal $i < \omega_1$, we define

$$i_0 = i, \qquad j_0 = j(i_0),$$
$$i_n = i(j_{n-1}), \qquad j_n = j(i_n), \qquad \text{for all integers } n \geqslant 1.$$

Then

$$\bigcup_{n=1}^{\infty} K_{j_n} \cap V = \bigcup_{n=1}^{\infty} V_{i_n}.$$

Let $\alpha = lub\{j_n\}$ and $\beta = lub\{i_n\}$; then we have

$$(*) \qquad \bigcup_{i<\beta} V_i = K_\alpha \cap V.$$

We shall call a limit ordinal β for which there exists a limit ordinal $\alpha = \alpha(\beta)$ satisfying (*) a *β-ordinal*. We have shown that if i is any ordinal less than ω_1, then there is a β-ordinal β such that $i \leqslant \beta < \omega_1$. In particular, there are uncountably many β-ordinals.

If β is a β-ordinal and $\alpha = \alpha(\beta)$, then K_α contains an element k_α such that $k_\alpha - z_\beta \in K_\alpha^\perp$. Suppose, if possible, that there is no element $k \in H$ such that $k - z_\beta \in K_{\alpha(\beta)}^\perp$ for uncountably many β-ordinals β. Then, for each $\alpha < \omega_1$, there is a smallest β-ordinal $\beta(\alpha)$ such that, for each element $k \in K_\alpha$ and for each β-ordinal $\gamma \geqslant \beta(\alpha)$, $k - z_\gamma \notin K_{\alpha(\gamma)}^\perp$.

Choose some β-ordinal β_0 and define $\alpha_0 = \alpha(\beta)$ and, for each integer $n \geqslant 1$, $\beta_n = \beta(\alpha_{n-1})$, $\alpha_n = \alpha(\beta_n)$. Let $\beta = lub\{\beta_n\}$ and $\alpha = lub\{\alpha_n\}$; then

$$\bigcup_{i<\beta} V_i = \bigcup_{n=1}^{\infty} \Big(\bigcup_{i<\beta_n} V_i\Big) = \bigcup_{n=1}^{\infty} (K_{\alpha_n} \cap V) = K_\alpha \cap V,$$

so that β is a β-ordinal and $\alpha = \alpha(\beta)$. If $k \in K_\alpha$, then $k \in K_{\alpha_n}$ for some n and so $k - z_\gamma \notin K_{\alpha(\gamma)}^\perp$ for any β-ordinal $\gamma \geqslant \beta_{n+1}$. In particular there is no element $k \in K_\alpha$ such that $k - z_\beta \in K_\alpha^\perp$. This contradiction shows that there is some element $k \in H$ such that $k - z_\beta \in K_{\alpha(\beta)}^\perp$, for uncountably many β-ordinals β.

It follows that

$$(k, y_i) = (k, z_\alpha) = 0, \qquad \text{for all } i, \alpha < \omega_1,$$
$$(x_i, k) = 1, \qquad \text{for all } i < \omega_1.$$

We now consider the subspace $\bar{V} = V + \langle k \rangle$ of H. Letting $\bar{V}_\varepsilon = V_\varepsilon + \langle k \rangle$, we clearly have $\bar{V} = \bigcup_{\varepsilon < \omega_1} \bar{V}_\varepsilon$. This allows us to repeat the arguments above, defining a $\bar{\beta}$-ordinal to be a limit ordinal β for which there exists a limit ordinal $\alpha = \bar{\alpha}(\beta)$ such that

$$\bigcup_{i<\beta} \bar{V}_i = K_\alpha \cap \bar{V}.$$

We are then able to show that there is some element $h \in H$ such that $h - x_\beta \in K^\perp_{\bar{\alpha}(\beta)}$, for uncountably many $\bar{\beta}$-ordinals β.

It follows that

$$(h, x_i) = (h, y_i) = (h, z_\alpha) = 0, \qquad \text{for all } i, \alpha < \omega_1,$$
$$(h, k) = 1.$$

Suppose that $h \in K_\delta$ and let γ be a $\bar{\beta}$-ordinal such that $\alpha(\gamma) > \delta$, and $k - z_\gamma \in K^\perp_{\alpha(\gamma)}$. But $(h, k - z_\gamma) = 1$ and this is a contradiction to $h \in K_\delta \subseteq K_{\alpha(\gamma)}$. Thus V cannot be embedded in an orthogonal sum of hyperbolic planes.

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