COMPOSITIO MATHEMATICA

S. J. PATTERSON

The laplacian operator on a Riemann surface

Compositio Mathematica, tome 31, nº 1 (1975), p. 83-107

<http://www.numdam.org/item?id=CM_1975__31_1_83_0>

COMPOSITIO MATHEMATICA, Vol. 31, Fasc. 1, 1975, pag. 83–107
Noordhoff International Publishing
Printed in the Netherlands

THE LAPLACIAN OPERATOR ON A RIEMANN SURFACE

S. J. Patterson

1. Introduction

The object of this paper is to prove and apply some formulae concerning the function

$$P(z, \zeta) = \text{Im }(z)/|z-\zeta|^2$$
$$P(z, \infty) = \text{Im }(z)$$

where $z \in \boldsymbol{H}$ (the upper half-plane) and $\zeta \in \boldsymbol{R} = \partial \boldsymbol{H}$. These formulae are merely versions of well-known formulae in the theory of the hypergeometric series. However they have considerable relevance to the theory of the Laplace operator on $\boldsymbol{H}$. We shall consider two such applications.

Firstly we shall be able to give an explicit form for the Selberg trace formula for automorphic forms of fractional weight. In his paper [10] Selberg gives formulae for the so-called 'Selberg transform' which relate a point-pair invariant to the spectral decomposition of the corresponding operator. By means of certain differential operators (the Maass operators) this extends to integral weights (where differentials are of weight 1 not 2). The proper interpretation of this is in terms of representation theory, as in [5]. Our formulae extend these results to forms of arbitrary weight.

Secondly we can interpret certain series introduced by Elstrodt [1] as Eisenstein series. Such series were first studied by Maass and Selberg in connection with various discontinuous groups with non-compact quotient. In fact, to each 'infinite part' of the quotient there is a corresponding Eisenstein series. If the discontinuous group is now a Fuchsian group of the second kind the infinite part is very large and the discussion of the spectrum of the Laplace operator is complicated (see [1], [2], [3]).

However, for a certain restricted class of groups we can give a more or less complete discussion. This is done in section 5. The conclusions are quite striking and they indicate that such groups are not very different from groups of the first kind. One would then hope that there results could be extended to all finitely generated groups of the second kind. This involves the analytic continuation of certain series. This can be done by the method sketched by Selberg [11] but it is rather long and involved. We shall give a sketch of this theory in section 6.

The general philosophy of this paper hinges on two points. The first is Selberg's theory of point-pair invariants [10] (which constitutes an induction on the dimensions of the spaces involved) which is a powerful technique. The other point is putting the Poisson kernel $P(z, \zeta)$ at the centre of the discussion. This is the policy generally accepted in the corresponding theory of general Lie groups. This method I learnt through a study of the work of Elstrodt [1], [2]. He also uses systematically the technique of first studying any problem first for the trivial and elementary groups. This is essentially the method used in section 3 of this paper.

2. The formulae

In this section we shall prove several formulae. The first and most basic is, for $z, w \in \boldsymbol{H}$, $k \in \boldsymbol{R}$ that

$$(1) \quad \int_{\boldsymbol{R}} P(z, \zeta)^s (z, \zeta)^k P(w, \zeta)^{1-s} (w, \zeta)^{-k} d\zeta$$

$$= \pi \cdot e^{-2\pi i k}((z, w))^k \left(\frac{\Gamma(1-2s)}{\Gamma(1-s+k)\Gamma(1-s-k)} \sigma^{-s} F(s+k, s-k; 2s; \sigma^{-1}) \right.$$

$$\left. + \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} \sigma^{s-1} F(1-s+k, 1-s-k; 2-2s; \sigma^{-1}) \right)$$

where

$$(z, \zeta) = (\bar{z}-\zeta)/(z-\zeta)$$

$$((z, w)) = (w-\bar{z})/(z-\bar{w})$$

$$\sigma = |z-\bar{w}|^2/4 \operatorname{Im}(z) \operatorname{Im}(w).$$

Most of the other formulae that we shall need can be deduced from this so that we prove this first and state the other formulae later. Except for $((w, z))^k$ we shall understand x^k to mean $\exp(k \log x)$ where

$$0 \leqq \operatorname{Im}(\log x) < 2\pi.$$

With this convention we define $((z, w))^k$ to be

$$|w-\bar{z}|^{-2k}(i(w-\bar{z}))^{2k}.$$

To prove (1) we expand the integral and we see that it is

$$\int \left(\frac{\zeta - z}{\zeta - w}\right)^{-s-k} \left(\frac{\zeta - \bar{z}}{\zeta - \bar{w}}\right)^{k-s} ((\zeta - w)(\zeta - z))^{-1} d\zeta \operatorname{Im}(z)^s \operatorname{Im}(w)^{1-s}$$

where the conventions about the arguments are as follows. Let C be the circle through $z, w, \bar{z}, \bar{w}$. Let C_1 be the arc of C joining z and w and lying in $\boldsymbol{H}$. Then we cut $\boldsymbol{C}_\infty$ along C_1 and $\bar{C}_1$. We take that branch of $(\zeta - z/\zeta - w)^{-s-k}$ which is 1 at $\zeta = \infty$; likewise the branch of $(\zeta - \bar{z}/\zeta - \bar{w})^{k-s}$ which is 1 at $\zeta = \infty$. Henceforth we regard ζ as a complex variable.

Now let

$$t(\zeta) = ((\zeta - z)(w - \bar{z}))/((\zeta - \bar{z})(w - z)).$$

Then $t(z) = 0$, $t(w) = 1$, $t(\bar{z}) = \infty$, $t(\bar{w}) = \tau$ where, σ being as above,

$$\sigma^{-1} + \tau^{-1} = 1.$$

Also $t(C_1) = [0, 1]$, $t(\bar{C}_1) = [\tau, \infty]$. The path of integration becomes a loop γ around $[0, 1]$. On substituting the integral becomes

$$\sigma^{s-1}((z, w))^k e^{-2\pi i k} \sin \pi(s+k) \int_0^1 t^{-k-s}(1-t)^{k+s-1}(1-\tau^{-1}t)^{s-k-1} dt.$$

The integral is a hypergeometric function and on identifying it we see that the expression above is

$$\pi((z, w))^k e^{-2\pi i k} \sigma^{s-1} F(1-s-k, 1-s+k; 1; \tau^{-1}).$$

The expression (1) follows at once on using the well-known relations between the hypergeometric functions of argument z and $1-z$.

In (1) we now let $w \to \infty i$. We see that if Re $(s) > \frac{1}{2}$ that

$$\int_{\boldsymbol{R}} P(z, \zeta)^s (z, \zeta)^k d\zeta = \pi \cdot e^{\pi i k} 2^{2(1-s)} P(z, \infty)^{1-s} \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} \tag{2}$$

Now let

$$z = -1/(z_1 - \eta), \qquad \zeta = -1/(\zeta_1 - \eta).$$

Clearly we have

$$(z, \zeta) = (z_1, \zeta_1)/(z_1, \eta)$$

and

$$P(z, \zeta) = P(z_1, \zeta_1)|\zeta_1 - \eta|^2.$$

Thus from (2) we have

$$\int_{\boldsymbol{R}} P(z, \zeta)^s ((z, \zeta)/(z, \eta))^k |\zeta - \eta|^{2s-2} d\zeta = \pi e^{\pi i k} 2^{2(1-s)} P(z, \eta)^{1-s} \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)}$$

Define $\varepsilon(\zeta, \eta)$ by

$$(z, \eta)^k ((z, \zeta)/(z, \eta))^k = \varepsilon(\zeta, \eta)(z, \zeta)^k.$$

It is easy to check that this does not depend on z. We can also show that for ζ, η finite that

$$\varepsilon(\zeta, \eta) = \begin{cases} 1 & \text{if } \zeta \leqq \eta \\ e^{2\pi i k} & \text{if } \zeta > \eta. \end{cases}$$

We shall also set $\varepsilon(\zeta, \infty) = 1$. With this notation we obtain

$$\int_{\boldsymbol{R}} P(z, \zeta)^s (z, \zeta)^k \varepsilon(\zeta, \eta) |\zeta - \eta|^{-2(1-s)} d\zeta = \pi e^{\pi i k} 2^{2(1-s)} P(z, \eta)^{1-s} (z, \eta)^k \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} \tag{3}$$

(1), (2) and (3) make up the first set of formulae that we need. We have to look next at how the quantities behave under transformation by elements of Con ($\boldsymbol{H}$) (the group of conformal transformations of $\boldsymbol{H}$, isomorphic to PSL(2, $\boldsymbol{H}$)). For $\gamma \in \mathrm{Con}\,(\boldsymbol{H})$ set

$$j(\gamma, z) = \gamma'(z)/|\gamma'(z)|.$$

Then after an easy calculation we see that

$$\varepsilon(\gamma(\zeta), \gamma(\infty))j(\gamma, z)^k(\gamma(z), \gamma(\zeta))^k = e^{2\pi ik}(z, \zeta)^k. \tag{4}$$

Now, for $g_1, g_2 \in \mathrm{Con}\,(\boldsymbol{H})$ define $\sigma^k(g_1, g_2)$ by

$$j(g_1 g_2, z)^k = \sigma^k(g_1, g_2)j(g_1, g_2(z))^k j(g_2, z)^k. \tag{5}$$

We obtain from (4) and (5) that

$$e^{2\pi ik}\varepsilon(g_1 g_2(\zeta), g_1 g_2(\infty))\sigma^k(g_1, g_2) = \varepsilon(g_1 g_2(\zeta), g_1(\infty))\varepsilon(g_2(\zeta), g_2(\infty)). \tag{6}$$

From (4) and (1) we deduce that

$$j(\gamma, z)^k j(\gamma, w)^{-k}((\gamma(z), \gamma(w)))^k = ((z, w))^k. \tag{7}$$

We shall need one more formula which is easily deduced from (1). This is, for Re $(t) > \frac{1}{2}$

$$\begin{aligned}(8)\quad &(2\pi i)^{-1}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_{\boldsymbol{R}} P(z, \zeta)^s(z, \zeta)^k P(w, \zeta)^{1-s}(w, \zeta)^{-k}d\zeta \\ &\times \frac{(2s-1)\cdot \sin 2\pi s}{\sin \pi(s+k)\cdot \sin \pi(s-k)}\,\frac{ds}{(s(1-s)-t(1-t))} \\ &= 2e^{-2\pi ik}((z, w))^k\,\frac{\Gamma(t+k)\Gamma(t-k)}{\Gamma(2t)}\,\sigma^{-t}F(t+k, t-k; 2t; \sigma^{-1}) \\ &\quad -2e^{-2\pi ik}((z, w))^k \sum_{0 \leq m < |k|-\frac{1}{2}} \frac{\Gamma(2|k|-m)(-1)^m}{m!\Gamma(2|k|-2m)} \\ &\times \frac{2(|k|-m)-1}{(|k|-m)(1-|k|+m)-t(1-t)}\,\sigma^{-|k|+m}F(2|k|-m, -m; 2|k|-2m; \sigma^{-1})\end{aligned}$$

To prove this we recall that in [12] it is proved that for a fixed $\delta > 0$, and $\sigma > 1$ we have for large s so that $|\arg(s)| < \pi-\delta$

$$\sigma^{-s}\,\frac{\Gamma(s+k)\Gamma(s-k)}{\Gamma(2s)}F(s+k, s-k; 2s; \sigma^{-1}) \sim e^{-us}\sum_{n=0}^{\infty} c_n(\sigma, k)s^{-n-\frac{1}{2}} \tag{9}$$

where u is the positive solution of

$$\cosh u = 2\sigma - 1.$$

We can now use (1) to evaluate the inner integral in (8). Then (9) justifies moving the line of integration to the right and the right hand side of (8) consists simply of the contributions from the poles.

3. The Laplace operators on H

We shall assume the discussion of the operators Δ_k given by Elstrodt in [2]. Suppose that $\lambda = t(1-t)$ with $\mathrm{Re}\,(t) > \frac{1}{2}$,

$$t \notin \{|k| - m : 0 \leqq m < |k| - \tfrac{1}{2}\}.$$

Then the operator $(-\Delta_k - \lambda I)^{-1}$ has the kernel (σ being as in Section 2)

$$\frac{e^{-2\pi i k}}{4\pi}((z, w))^k \cdot \frac{\Gamma(t+k)\Gamma(t-k)}{\Gamma(2t)} \sigma^{-t} F(t+k, t-k; 2t; \sigma^{-1}).$$

By (8) this is equal to

$$\frac{1}{8\pi}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-i\infty}\int_{-\infty}^{\infty} P(z, \zeta)^s (z, \zeta)^k P(w, \zeta)^{1-s}(w, \zeta)^{-k} d\zeta$$
$$\times \frac{(2s-1)\sin 2\pi s}{\sin \pi(s+k)\sin \pi(s-k)} \frac{1}{s(1-s)-t(1-t)} ds$$
$$+ \frac{e^{-2\pi i k}}{4\pi}((z, w))^k \sum_{0 \leqq m < |k| - \frac{1}{2}} \frac{\Gamma(2|k|-m)(-1)^m}{m!\Gamma(2|k|-2m)}$$
$$\times \frac{2(|k|-m)-1}{(|k|-m)(1-|k|+m)-t(1-t)} \sigma^{-|k|+m} F(2|k|-m, -m; 2|k|-2m; \sigma^{-1}).$$

Let us now interpret this formula. We can check, as in [1], [2], that $P(z, \zeta)^s (z, \zeta)^k$ is a generalized eigenfunction of $-\Delta_k$ with eigenvalue $s(1-s)$. Also

$$F(2|k|-m, -m; 2|k|-2m; \sigma^{-1})\sigma^{-|k|+m}((z, w))^k \tag{10}$$

is, qua function of z, an eigenfunction with eigenvalue

$$(|k|-m)(1-|k|+m).$$

Hence the formula above represents the spectral decomposition of $(-\Delta_k - \lambda I)^{-1}$. It follows that the spectrum is contained in $[\frac{1}{4}, \infty[\cup E_k$ where $E_k = \{(|k|-m)(1-|k|+m) : 0 \leqq m < |k|-\frac{1}{2}\}$. For simplicity we shall refer to the part of the spectrum in E_k as the *discrete series* and the part in $[\frac{1}{4}, \infty[$ as the *principal series* in accordance with the representation theory viewpoint. The kernel (10) is, up to a constant factor, a reproducing kernel on the eigenspace with eigenvalue $(|k|-m)(1-|k|+m)$. The principal series is spanned by $P(z, \zeta)^s(z, \zeta)^k(\zeta \in \mathbf{R} \cup \{\infty\})$.

Now let the kernel $q(z, w)$ represent an operator commuting with $-\Delta_k$. Suppose that

$$\int |q(z, w)| \operatorname{Im}(w)^{\frac{1}{2}} d\sigma(w)$$

(σ is the hyperbolic measure; the various uses of the symbol σ should cause no confusion) converges. Then Selberg's theory ([10]) shows that, if we define h by

$$h(t) \cdot \operatorname{Im}(z)^t = \int q(z, w) \cdot \operatorname{Im}(w)^t \cdot d\sigma(w) \tag{11}$$

then the operator represented by $q(z, w)$ is 'diagonal' with respect to the spectral decomposition of $-\Delta_k$ and the 'entry' at the eigenvalue $t(1-t)$ is $h(t)$. From (11) one sees that h is bounded on the spectrum. Hence one sees that the operator is L^2. Thus we obtain the representation

$$\begin{aligned} q(z, w) = {} & \frac{1}{16\pi^2 i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_{-\infty}^{+\infty} P(z, \zeta)^s (z, \zeta)^k P(w, \zeta)^{1-s} (w, \zeta)^{-k} \\ & \times \frac{(2s-1) \cdot \sin 2\pi s}{\sin \pi(s+k) \cdot \sin \pi(s-k)} h(s) d\zeta ds \\ & + \frac{e^{-2\pi i k}}{4\pi} ((z, w))^k \sum_{0 \leqq m < |k|-\frac{1}{2}} \frac{\Gamma(2|k|-m)(-1)^m}{m! \Gamma(2|k|-2m-1)} h(|k|-m) \\ & \times \sigma^{-|k|+m} F(2|k|-m, -m; 2|k|-2m; \sigma^{-1}) \end{aligned} \tag{12}$$

where the integral over s has to be regarded as

$$\lim_{T \to \infty} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}$$

the limit being taken in the L^2 sense.

This representation is not very useful as it stands and it is better to change our viewpoint (with Selberg) and to take h as the basic function and then to study the corresponding function q which we denote by q_h. We suppose that h satisfies the following three conditions:

(i) for some $\varepsilon > 0$ can be extended to an analytic function in $|\mathrm{Re}\,(s-\frac{1}{2})| < \frac{1}{2}+\varepsilon$,

(ii) $h(s) = h(1-s)$, and,

(iii) for some $c > 0$, $|h(s)| \leqq c(1+|s|^2)^{-1-\varepsilon}$ in the strip defined in (i).

Note that these conditions say nothing about the value of $h(s)$ when $\mathrm{Re}\,(s) > 1+\varepsilon$. Under these conditions the double integral in (12) converges absolutely. Without loss of generality we can suppose that $1+\varepsilon/2 \notin \{|k|-m : m \in \boldsymbol{Z}\}$. Then we can use (1) to evaluate the integral over ζ. One sees first of all that there is a function $K_h(\sigma)$ so that

$$q_h(z, w) = e^{-2\pi ik}((z, w))^k K_h(\sigma). \tag{13}$$

This has to be true on general grounds in any case. Using the estimate (9) we can move the line of integration to $\mathrm{Re}\,(s) = 1+\varepsilon/2$. Then we find

$$\begin{aligned} K_h(\sigma) = {} & \frac{1}{8\pi^2 i} \int_{\mathrm{Re}\,(s) = 1+\varepsilon/2} \sigma^{-s} \frac{\Gamma(s+k)\Gamma(s-k)}{\Gamma(2s)} \\ & \times F(s+k, s-k; 2s; \sigma^{-1})(2s-1)h(s)ds \\ & + \frac{1}{4\pi} \sum_{0 \leqq m < |k|-1-\varepsilon/2} \frac{\Gamma(2|k|-m)(-1)^m}{m!\Gamma(2|k|-2m-1)} h(|k|-m)\sigma^{-|k|+m} \\ & \times F(2|k|-m, -m; 2|k|-2m; \sigma^{-1}). \end{aligned} \tag{14}$$

From this we deduce that there is a constant A so that, if $|k| \leqq 1$,

$$|K_h(\sigma)| \leqq A\sigma^{-1-\varepsilon/2} \tag{15}$$

To prove this we observe that the double integral in (12) converges absolutely and hence we need only prove it for large σ. The conclusion will then follow directly if we can show that

$$\sigma^{-s} \frac{\Gamma(s+k)\Gamma(s-k)}{\Gamma(2s)} F(s+k, s-k; 2s; \sigma^{-1})$$

with $\sigma \geqq 2$ is bounded uniformly in σ on $\mathrm{Re}\,(s) = 1+\varepsilon/2$. However this function has the integral presentation

$$\sigma^{-s}\int_0^1 u^{s-k-1}(1-u)^{s+k-1}(1-u/\sigma)^{-s-k}du$$

from which our conclusion follows immediately.

The restriction that $|k| \leqq 1$ is unfortunate but not important since through an application of the theory of Maass operators we can recover most of the results we need for arbitrary k. If we replace the strip in condition (i) above by $|\mathrm{Re}\,(s-\frac{1}{2})| < |k|+\frac{1}{2}+\varepsilon$ then (15) follows.

It follows from (15) that (11) converges. The correspondence between (11) and (12) (i.e. between h and K_h) is called the *Selberg transform*. The discussion above gives conditions for its validity. For the sake of precision we give a set of conditions which the reader will have no difficulty in verifying from the discussion above.

(i′) For some $\varepsilon > 0$ h can be extended to an analytic function in $|\mathrm{Re}\,(s-\frac{1}{2})| < |k|+\frac{1}{2}+\varepsilon$ except for the points $|k|-m \notin [0,1]$
$(0 \leqq m < |k|, m \in \boldsymbol{Z})$

at which h has a removable singularity.

(ii′) $h(s) = h(1-s)$.

(iii′) For some $c > 0$, $|h(s)| < c(1+|s|^2)^{-1-\varepsilon}$ in the strip defined in (i′).

Under these conditions (15) holds for a suitable $\varepsilon' > 0$. It is worth remarking that these conditions may be relaxed but for most applications there is no point in doing so.

Finally we need the special case when $\sigma = 1$. To calculate this we set $z = w$ in (12). Then

$$K_h(1) = \frac{1}{16\pi^2 i}\int_{-\infty}^{\infty} P(z,\zeta)d\zeta \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(2s-1)\cdot \sin 2\pi s}{\sin \pi(s+k)\cdot \sin \pi(s-k)}\, h(s)ds$$

$$+\frac{1}{4\pi}\sum_{0\leq m<|k|-\frac{1}{2}} \frac{\Gamma(2k-m)(-1)^m}{m!\,\Gamma(2k-2m-1)}\, h(|k|-m)F(2|k|-m,\ -m;\ 2|k|-2m;\ 1).$$

This simplifies and we find

$$\text{(16)}\quad K_h(1) = (4\pi)^{-1}\int_{-\infty}^{\infty} \frac{r\cdot \sinh 2\pi r}{\cosh 2\pi r+\cos 2\pi k}\, h(\tfrac{1}{2}+ir)dr$$
$$+(4\pi)^{-1}\sum_{0\leq m<|k|-\frac{1}{2}} (2|k|-2m-1)h(|k|-m).$$

4. The Selberg trace formula

Let G be a Fuchsian group of the first kind. For simplicity of exposition we shall assume that there are no parabolic elements in G. In [10] Selberg showed how to develop his theory for forms of arbitary weight; the theory is treated in more detail in [9].

Let V be a Hermitian vector space and let χ be a multiplier of weight k taking values in $U(V)$ (the group of unitary operators on V); a version of the theory suitable for our needs is described in [7; Section 6]. Note that χ is also a multiplier of weight $k+m$ for any integer m and that $\bar{\chi}$ is a multiplier of weight $m-k$ for any integer m.

Let h be a function satisfying conditions (i′), (ii′), (iii′) of Section 3; let K_h, q_h also be as in Section 3. Define

$$Q_h(z, w) = \sum_{g \in G} \chi(g) j(g, w)^{-k} q_h(z, g(w)). \tag{17}$$

By (15) this converges. For $g \in G$ we have

$$Q_h(z, g(w)) = j(g, w)^k Q_h(z, w) \chi(g)^{-1} \tag{18}$$

$$Q_h(g(z), w) = \chi(g) Q_h(z, w) j(g, z)^{-k} \tag{19}$$

These follow directly from the series definition and (7). Let us also record, in the notation of (5), that for $g_1, g_2 \in G$

$$\chi(g_1 g_2) = \sigma^k(g_1, g_2)\chi(g_1)\chi(g_2) \tag{20}$$

Let $||x||$ denote the norm of an element x of V. We define $L(G, V)$ to be the Hilbert space of functions $f: \boldsymbol{H} \to V$ satisfying, for $g \in G$,

$$j(g, z)^k f(g(z)) = \chi(g) f(z) \tag{21}$$

and

$$\int_{G \backslash \boldsymbol{H}} ||f||^2 d\sigma < \infty.$$

The integral (12) converges uniformly on compact subsets of $\boldsymbol{H} \times \boldsymbol{H}$; hence $q_h(z, w)$ is continuous. Again, by (15) the series in (17) converges uniformly on compact subsets of $\boldsymbol{H} \times \boldsymbol{H}$; hence $Q_h(z, w)$ is continuous. Under our assumptions $G \backslash \boldsymbol{H}$ is compact and so, by (18), (19), $Q_h(z, w)$

is bounded. Hence the map $\tau : L(G, V) \to L(G, V)$ given by

$$f \mapsto \int_{G\backslash \boldsymbol{H}} Q_h(\cdot, w) f(w) d\sigma(w)$$

is well-defined (by (18), (19)) and is of trace-class by the remarks above. Also τ commutes with $-\Delta_k$. It is not difficult to check that if h is real on $\mathrm{Re}\,(s) = \frac{1}{2}$ (in which case, in view of (ii) it is real on $\mathrm{Im}\,(s) = 0$) and h is real at $s = |k| - m$ ($m \in \boldsymbol{Z}$; we have to treat this case differently) then τ is self-adjoint. Now, $-\Delta_k$ is an essentially self-adjoint operator on a dense subspace of $L(G, V)$; its spectrum lies in $[0, \infty[\cup E_k$ (see [2]). The dimension of the eigenspace with eigenvalue $(|k| - m)(1 + m - |k|)$ ($m \in \boldsymbol{Z}$, $0 \leqq m \leqq |k|$) is equal to the dimension of the space of analytic automorphic forms of weight $|k| - m$ and multiplier χ if $k \geqq 0$, and if $k \leqq 0$ to the dimension of the space of analytic automorphic forms of weight $|k| - m$ and multiplier $\bar{\chi}$ (see [7]).

If we choose h so that τ is self-adjoint then we can find an orthonormal base $\{\varphi_j\}$ in $L(G, V)$ with respect to which τ is diagonal. As the τ for different h commute (see [10]) we can do this for all such τ simultaneously. It then follows that $-\Delta_k$ and the τ which are not self-adjoint are also diagonal with respect to this base. Now definte λ_j by

$$-\Delta_k \varphi_j = \lambda_j \cdot \varphi_j.$$

Also define s_j by

$$s_j(1 - s_j) = \lambda_j$$

(for the moment it does not matter which solution we take.) Then

$$Q_h(z, w) = \sum h(s_j) \varphi_j(z) \otimes \varphi_j(w)^*$$

where * is the anti-linear map of V into its dual space determined by the Hermitian structure. For reasons explained in [8] this is absolutely convergent. Hence we can set $z = w$, take the trace in $\mathrm{End}\,(V)$ and integrate over $G \backslash \boldsymbol{H}$,

$$\sum_{g \in G} \int_{G\backslash \boldsymbol{H}} \mathrm{Tr}\,(\chi(g)) \cdot j(g, z)^k q_h(z, g(z)) d\sigma(z) = \sum_j h(s_j).$$

The sum on the left can be transformed into a sum over conjugacy classes in G (as in [10]). The term $I \in G$ behaves differently to all the others and it is for this reason that we need (16). Suppose, for simplicity, that these are no elliptic elements in G. The evaluation is now standard and we find

$$(22)\quad \sum_j h(s_j) = \frac{\sigma(G\backslash \boldsymbol{H})}{4\pi} \dim(V) \int_{-\infty}^{\infty} \frac{r \cdot \sinh 2\pi r}{\cosh 2\pi r + \cos 2\pi k} h(\tfrac{1}{2}+ir)dr$$

$$+ \frac{\sigma(G\backslash \boldsymbol{H})}{4\pi} \dim(V) \sum_{0 \leqq m < |k|-\frac{1}{2}} (|k|-m-\tfrac{1}{2}) h(|k|-m)$$

$$+ (\tfrac{1}{2}) \sum_{\gamma \in H} \sum_{j=0}^{\infty} \frac{\operatorname{Tr}(\chi(\gamma^j)) \log N(\gamma)}{N(\gamma)^{j/2} - N(\gamma)^{-j/2}} g(j \cdot \log N(\gamma)).$$

In this formula H is a set of primitive hyperbolic elements representing the primitive hyperbolic conjugacy classes ($N(\gamma)$ is the multiplier of $\gamma \in H$). $g(\cdot)$ is the Fourier transform of h; it is given by

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-irx} \cdot h(\tfrac{1}{2}+ir)dr.$$

(22) is the Selberg trace formula and our discussion is substansially the same as that given by Selberg [10]. It would also be quite possible to give a representation theoretic discussion (as in [5]) using the ideas of [7]. We have only shown (22) under the conditions (i′), (ii′), (iii′) of Section 3. However it follows from the theory of Maass operators that it in fact holds only under the conditions (i), (ii), (iii). This theory shows that the spectra of $-\Delta_k$ and $-\Delta_{k-1}$ (with multiplier $\bar{\chi}$) are the same except for the point $k(1-k)$. Thus we can prove (22) first for $|k| \leqq 1$ and afterwards extend it to arbitrary k.

Now let $s = |k|-m$ for some m ($m \in \boldsymbol{Z}, 0 \leqq m \leqq |k|$). Let $N(s, \chi)$ be the dimension of the space of analytic automorphic forms of weight s and multiplier χ. First suppose that $s > 1$. Then we can set

$$\begin{aligned} h(t) &= 1 \qquad (t = s, 1-s) \\ &= 0 \qquad (t \neq s, 1-s). \end{aligned}$$

This satisfies conditions (i′), (ii′), (iii′) of Section 3). From (22) we obtain that

$$(23)\qquad N(s, \chi) = (s-\tfrac{1}{2})\sigma(G\backslash \boldsymbol{H}) \dim(V)/2\pi.$$

Next take $\frac{1}{2} \leqq k \leqq 1$. Then $\bar{\chi}$ is a multiplier of weight $1-k$. The spectrum of Δ_k with multiplier χ is the same as the spectrum of Δ_{-k} with multiplier $\bar{\chi}$ (by taking the complex conjugate). The theory of Maass operators shows that the spectrum of $-\Delta_k$ and $-\Delta_{1-k}$ are the same outside the point $k(1-k)$. Comparing the corresponding versions of (22) and using the remarks made at the beginning of the paragraph we see that

$$N(k, \chi) - N(1-k, \bar{\chi}) = (k - \tfrac{1}{2})\sigma(G \backslash \boldsymbol{H}) \dim(V)/2\pi \tag{24}$$

In fact, if $k < 0$, $N(k, \chi) = 0$. From (23) we see that (24) is valid for all k. This is the general form of the Riemann-Roch theorem for Riemann surfaces.

5. Groups of the second kind

In this section we turn our attention to Fuchsian groups which are 'small' as opposed to those considered in Section 4 which were as 'large' as possible. So let G be a Fuchsian group of the second kind and let L_G be its limit set. Let $\mathrm{Re}(s) = \delta(G)$ be the abscissa of convergence of the Dirichlet series

$$\sum_{g \in G} \sigma(g(z_1), z_2)^{-s}$$

where still

$$\sigma(z_1, z_2) = |z_1 - \bar{z}_2|^2/4 \,\mathrm{Im}(z_1)\,\mathrm{Im}(z_2).$$

This does not depend on z_1, z_2. Clearly $0 \leqq \delta(G) \leqq 1$; one can say more – see [6] and the references there.

Let χ, V, $L(G, V)$ be as in Section 4. For $\zeta \in \boldsymbol{O}(G) = (\boldsymbol{R} \cup \{\infty\}) \smallsetminus L_G$ we define the Eisenstein series

$$E_\zeta(z, s) = \sum_{g \in G} \chi(g)^{-1} j(g, z)^k P(g(z), \zeta)^s (g(z), \zeta)^k \tag{25}$$

This converges uniformly on compact subsets of $\boldsymbol{H}$ if $\mathrm{Re}(s) > \delta(G)$.

One verifies that, if $g \in G$,

$$E_\zeta(g(z), s) \cdot j(g, z)^k = \chi(g) E_\zeta(z, s) \tag{26}$$

and

$$E_{g(\zeta)}(z, s)\varepsilon(g(\zeta), g(\infty))|g'(\zeta)|^s = E_\zeta(z, s)\chi(g)^{-1}. \tag{27}$$

Also

$$-\Delta_k E(z, s) = s(1-s)E_\zeta(z, s). \tag{28}$$

Here $\varepsilon(\cdot, \cdot)$ is the function defined in Section 2.

We now make the following two assumptions:

(i) $m(L_G) = 0$ (m = Lebesgue measure)
(ii) $\delta(G) < \frac{1}{2}$.

It is quite possible that the second of these implies the first; this is so if G is finitely generated.

Let h be a function satisfying conditions (i′), (ii′), (iii′) of Section 3. We form $q_h(z, w)$ by (12). Now we need the following lemma:

LEMMA: *The following expressions are absolutely convergent:*

(i) $$\sum_{g\in G}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_{\boldsymbol{R}}\chi(g)P(z,\zeta)^s P(g(w),\zeta)^{1-s}(z,\zeta)^k(g(w),\zeta)^{-k}$$
$$\times j(g, w)^{-k}\frac{(2s-1)\cdot\sin 2\pi s}{\sin\pi(s+k)\cdot\sin\pi(s-k)}h(s)d\zeta ds.$$

(ii) *Let* $\alpha > \frac{1}{2}$, $m \geqq 0$, $m \in \boldsymbol{Z}$, $k \in \boldsymbol{R}$,

$$\sum_{g\in G}\chi(g)\sigma(z, g(w))^{-\alpha}j(g, w)^{-k}((z, g(w)))^k F(2\alpha+m, -m; 2\alpha; \sigma(z, gw)^{-1}).$$

Furthermore both of these expressions represent continuous functions of z and w.

PROOF: (i) is majorised by

$$\sum_{g\in G}\int_{\boldsymbol{R}}P(z,\zeta)^{\frac{1}{2}}P(g(w),\zeta)^{\frac{1}{2}}d\zeta\int|h(s)(2s-1)||ds|.$$

The last term is convergent by our assumptions on h. The first integral clearly converges. We can evaluate it by (1) and we see from the properties of the hypergeometric function that the series is majorised by

$$\sum_{g\in G} \min\,(1, \log \sigma(z, g(w))/\sigma(z, g(w))^{\frac{1}{2}}).$$

This converges as $\delta(G) < \frac{1}{2}$. The continuity either follows from this argument or directly from (15).

(ii) is majorised by

$$\sum_{g\in G} \sigma(z, g(w))^{-\alpha}$$

which converges again as $\delta(G) < \frac{1}{2} < \alpha$. The continuity either follows from this or (15). This proves the lemma.

As in Section 4 we define

$$Q_h(z, w) = \sum_{g\in G} \chi(g) j(g, w)^{-k} q_h(z, g(w))$$

which is again a continuous function satisfying (18) and (19). From the lemma and (12) we see that

$$\begin{aligned}(29)\quad & \frac{1}{16\pi^2 i} \sum_{g\in G} \chi(g) \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_{-\infty}^{\infty} P(z, \zeta)^s (z, \zeta)^k j(g, w)^{-k} P(g(w), \zeta)^{1-s} \\ & \times (g(w), \zeta)^{-k} \frac{(2s-1)\cdot \sin 2\pi s \cdot h(s)}{\sin \pi(s+k)\cdot \sin \pi(s-k)}\, d\zeta ds \\ & + \frac{e^{-2\pi i k}}{4\pi} \sum_{g\in G} \sum_{0\leq m<|k|-\frac{1}{2}} \chi(g) j(g, w)^{-k}((z, g(w))) \frac{\Gamma(2|k|-m)(-1)^m}{m!\Gamma(2|k|-2m-1)} \\ & \times \sigma(z, g(w))^{-|k|+m} F(2|k|-m, -m; 2|k|-2m; \sigma(z, g(w))^{-1})\end{aligned}$$

converges and is equal to $Q_h(z, w)$.

$\boldsymbol{O}(G)$ is open and so we can find $B \subset \boldsymbol{O}(G)$ so that B is an open fundamental domain for the action of G on $\boldsymbol{O}(G)$. From $m(L_G) = 0$ it follows that

$$m(\boldsymbol{R} \smallsetminus \bigcup_{g\in G} g(B)) = 0.$$

Hence for any integrable function f on $\boldsymbol{R}$ we have

$$\int_{\boldsymbol{R}} f(x)dx = \int_B \sum_{g\in G} |g'(x)| f(gx)dx.$$

Applying this transformation to the first term in (29) we see that it is equal to

$$(16\pi^2 i)^{-1} \int_B \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E_\zeta(z, s)\cdot E_\zeta(w, s)^* \frac{\sin 2\pi s\ (2s-1)h(s)}{\sin \pi(s+k)\sin \pi(s-k)}\, dsd\zeta$$

where * means the adjoint with respect to the Hermitian structure on V.

Let

$$R_{k,m}(z, w) = \frac{e^{-2\pi ik}}{4\pi} \sum_{g\in G} \chi(g) j(g, w)^{-k}((z, g(w)))^k \frac{\Gamma(2|k|-m)(-1)^m}{m!\Gamma(2|k|-2m-1)}$$
$$\times\, \sigma(z, g(w))^{-|k|+m} F(2|k|-m,\ -m;\ 2|k|-2m;\ \sigma(z, g(w))^{-1}).$$

Now we have shown that

$$\text{(30)}\quad Q_h(z, w) = \frac{1}{16\pi^2 i} \int_L \int_B E(z, s)\cdot E(w, s)^* \frac{(2s-1)\cdot \sin 2\pi s h(s)}{\sin \pi(s+k)\cdot \sin \pi(s-k)}\, d\zeta ds$$
$$+ \sum_{0\leqq m<|k|-\frac{1}{2}} h(|k|-m) R_{k,m}(z, w).$$

Here L is the line $\mathrm{Re}\,(s) = \frac{1}{2}$. As h is bounded Q_h represents a linear operator on $L(G, V)$. Let $E_{k,m}(G, V)$ be the eigenspace with eigenvalue $(|k|-m)(1+m-|k|)$ $(0 \leqq m < |k|-\frac{1}{2})$. Then let $L_e(G, V)$ be the orthogonal complement to

$$\bigoplus_{0\leqq m<|k|-\frac{1}{2}} E_{k,m}(G, V).$$

The study of the spaces $E_{k,m}(G, V)$ can clearly be carried out by means of the $R_{k,m}(z, w)$ which represent the projections onto the $E_{k,m}(G, V)$. Furthermore from (30) it follows that $L_e(G, V)$ is spanned by the 'Eigenpakete'

$$\int_{\frac{1}{2}}^{s} \int_B E_\zeta(z, s')\varphi(\zeta) d\zeta ds'$$

(φ is an L^2 function on B; in this connection see [1] especially Satz 4.4). This could also be phrased in terms of generalised eigenfunctions. This gives the spectral decomposition of $L(G, V)$ and answers a question put by Elstrodt [1; p. 91].

The Eisenstein series $E_\zeta(z, s)$ even satisfies a functional equation. From (3) we see that, for $\mathrm{Re}\,(s) > \frac{1}{2}$,

$$\int_R P(z, \zeta)^s (z, \zeta)^k \varepsilon(\zeta, \eta) |\zeta - \eta|^{-2(1-s)} d\zeta$$
$$= \pi e^{\pi i k} 2^{2(1-s)} P(z, \eta)^{1-s} (z, \eta)^k \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)}.$$

From this it follows that if $\frac{1}{2} < \mathrm{Re}\,(s) < 1 - \delta(G)$,

$$(31) \quad \int_R E_\zeta(z, s) \varepsilon(\zeta, \eta) |\zeta - \eta|^{-2(1-s)} d\zeta$$
$$= \pi e^{\pi i k} 2^{2(1-s)} \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} E_\eta(z, 1-s).$$

By the same transformation as before we can convert the integral in (31) into an integral over B. We introduce the function

$$(32) \qquad S(\zeta, \eta; s) = \sum_{g \in G} \frac{\varepsilon(g(\zeta), \eta)\varepsilon(g(\infty), g(\zeta))^{-1} |g'(\zeta)|^s}{|g(\zeta) - \eta|^{2s}} \chi(g)^{-1}.$$

On carrying out the calculations we see that

$$(33) \quad \int_B E_\zeta(z, s) S(\zeta, \eta; 1-s) d\zeta = \pi e^{-\pi i k} 2^{2(1-s)} \frac{\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} E_\eta(z, 1-s).$$

This equation holds true for $\frac{1}{2} < \mathrm{Re}\,(s) < 1 - \delta(G)$. However the divergence at $s = \frac{1}{2}$ is caused only by the singularity of $S(\zeta, \eta; 1-s)$ at the point $\zeta = \eta$. As $E_\zeta(z, s)$ is differentiable as a function of ζ it is then possible to give the analytic continuation of the integral on the left hand side into the region $\delta(G) < \mathrm{Re}\,(s) < 1 - \delta(G)$.

The function $S(\zeta, \eta; s)$ plays the same in our theory as the 'constant term' of the Eisenstein series associated to a parabolic vertex does in Selberg's theory.

There is an interpretation of (33) that is sometimes very suggestive. We suppose that $k = 0$ and $\chi = I$. The points of B should really be regarded as points of $G\backslash \boldsymbol{O}(G)$. This last set is the set of directions at a point $x \in G\backslash \boldsymbol{H}$ which lead to infinity. Then, in quantum mechanical terms $E_\zeta(z, s)$ represents a beam of particles of energy $s(1-s)$ (we presume that $\mathrm{Re}\,(s) = \frac{1}{2}$) coming from the direction represented by ζ. Then (33) shows how these particles are scattered to infinity again after passing through the finite part of $G\backslash \boldsymbol{H}$. In other words $S(\cdot, \cdot; s)$ is the S-matrix in this situation. This is closely connected with some remarks of Gel'fand [4].

It follows also that the integral operators represented by $S(\cdot, \cdot; s)$ and $S(\cdot, \cdot; 1-s)$ are, up to multiplication by a function of s, inverse to one another. However $S(\cdot, \cdot; s)$ is defined for $\mathrm{Re}\,(s) > \delta(G)$ and so one would hope that by inverting the corresponding operator to obtain an analytic continuation of $S(\cdot, \cdot; s)$ to the whole s-plane. If this were possible then (33) would define an analytic continuation of $E_\zeta(z, s)$ also to the whole plane. Unfortunately there seem to be considerable difficulties in carrying out this programme. However in the case that G is finitely generated this programme is feasible. This is because there is a good description of the geometry involved. We shall briefly sketch what can be done in the next section.

6. Finitely generated groups of the second kind

This section contains only a sketch of the theory. I hope to publish the details at some time in the future. Unfortunately the technical details become rather involved although the basic ideas are straightforward enough. So this summary will aid the reader to find his way through the later work. As a word of caution I should say that I have only checked the calculations under simplifying assumptions so that the reader, if he chooses, may regard the contents of this section as conjecture. There is no reason however to believe that the extra complications will present any serious problems.

The new tool which we have now is a much better description of the geometry involved. Let G be a Fuchsian group of the second kind. We will also suppose it to be non-elementary. One then knows that

$$0 < \delta(G) < 1$$

and that, if G has parabolic elements,

$$\tfrac{1}{2} < \delta(G) < 1.$$

Also $m(L_G) = 0$. (See [6]). Let $\boldsymbol{O}(G)$ be as in Section 5. $\boldsymbol{O}(G)$ is an open set. There is a (non-empty) finite set $\{\Omega_\alpha : \alpha \in A\}$ of connected components of $\boldsymbol{O}(G)$ so that

(i) if $\alpha, \beta \in A,\ \alpha \neq \beta,\ g \in G$ then

$$g(\Omega_\alpha) \cap \Omega_\beta = \emptyset,$$

(ii) $\boldsymbol{O}(G) = \bigcup_{g \in G} \bigcup_{\alpha \in A} g(\Omega_\alpha)$

Further the group

$$G_\alpha = \{g : g \in G, g(\Omega_\alpha) = \Omega_\alpha\}$$

is an infinite cyclic group of hyperbolic elements. For proofs of these statements see [6].

We now construct the θ-lens ($\theta \leqq \pi/2$) Λ_α over Ω_α (see [6]). Let P be a maximal set of inequivalent parabolic vertices. Then we can construct, for $p \in P$, a horocycle C_p at p so that

(i) if $g \in G,\ p, q \in P,\ p \neq g(q)$
$C_p \cap g(C_q) = \emptyset,$

(ii) $C_p \cap (\bigcup_{\alpha \in A} \bigcup_{g \in G} g(\Lambda_\alpha) = \emptyset.$

It is also true that if $\alpha, \beta \in A$, $\Omega_\alpha \neq g(\Omega_\beta)$ then

$$\Lambda_\alpha \cap g(\Lambda_\beta) = \emptyset.$$

We can find a relatively compact set D_0 in $\boldsymbol{H}$ so that

$$\bigcup_{g \in G} g(\bar{D}_0) \cup \bigcup_{g \in G} \bigcup_{p \in P} g(C_p) \cup \bigcup_{g \in G} \bigcup_{\alpha \in A} g(\Lambda_\alpha) = \boldsymbol{H}.$$

Again for the details see [6]. This description shows that associated with each infinite part of $G \backslash \boldsymbol{H}$ there is a cyclic elementary subgroup of G. This means usually that we only have to solve the problem for an elementary subgroup and then fit together the different groups correctly. This philosophy dates back to Selberg who uses it in his treatment of what happens when parabolic elements are present. We shall now give a simple example to show how this technique is applied.

PROPOSITION: *Suppose G is as above, V, as in the preceeding sections.*

Suppose $\varphi \in L(G, V)$ *be such that*

$$-\Delta_k \varphi = \lambda \varphi.$$

Then $\lambda < \frac{1}{4}$.

PROOF: We may assume that G is replaced by a conjugate group; thus in particular we may assume that for some $\alpha \in A$, $\Omega_\alpha =]0, \infty[$. Then there is $\kappa > 1$ so that G_α is generated by $z \mapsto \kappa z$. Here Λ_α is $\{z : \arg(z) < \theta\}$.

The automorphy of φ implies that it satisfies a functional equation of the form

$$\varphi(\kappa z) = T\varphi(z)$$

where T is a unitary matrix. We shall show that there are no functions satisfying such an equation which are also eigenfunctions of $-\Delta_k$ with eigenvalue $\lambda \geqq \frac{1}{4}$, square integrable on $G_\alpha \backslash \Lambda_\alpha$.

Let us suppose to the contrary that φ is such a function. We can diagonalize T and by doing this we see that we can assume that $V = \boldsymbol{C}$ and that $T = e^{\eta i}$. We now consider the Fourier expansion of φ with respect to G_α. This has the form

$$\varphi(z) = \sum_{n \in \boldsymbol{Z}} c_n(\arg(z)) e^{2\pi i(n+\eta)\cdot \log|z|/\log \kappa}$$

By the technique of separating variables we see that there is a second order differential operator D_n so that

$$-D_n(c_n(\psi)) = \lambda c_n(\psi).$$

This is essentially a confluent hypergeometric equation but we do not need this fact here. We shall show that if $\lambda \geqq \frac{1}{4}$ then for any solution of

$$-D_n f = \lambda f \tag{34}$$

the integral

$$\int_0^\theta |f(\theta')|^2 (\sin\theta')^{-2} d\theta' \tag{35}$$

is divergent. Since, by Parseval's theorem

$$\int_{G_\alpha \backslash \Lambda_\alpha} |\varphi|^2 d\sigma = (\log \kappa) \sum_{n \in \boldsymbol{Z}} \int_0^\theta |c_n(\theta')|^2 (\sin\theta')^{-2} d\theta'$$

this will be sufficient to prove the proposition.

As D_n is of the second order there are precisely two linearly independent solutions we have only to find two different solutions. To this end we study the function

$$F(z, s, t) = \int_{-\infty}^{0} P(z, \zeta)^s (z, \zeta)^k |\zeta|^{s+t-1} d\zeta$$

which exists when $\mathrm{Re}\,(s+t) > 0$, $\mathrm{Re}\,(s-t) > 0$. This is clearly an eigenfunction of $-\Delta_k$ with eigenvalue $s(1-s)$. Further F also satisfies, for $r > 0$,

$$F(rz, s, t) = r^t F(z, s, t).$$

Thus one solution of (34) is

$$F(e^{i\theta}, s, i(n+\eta)/\log \kappa)$$

where $s(1-s) = \lambda$. Clearly another solution is

$$F(e^{i\theta}, 1-s, i(n+\eta)/\log \kappa).$$

It is easy to check that, as $\theta \to 0$,

$$F(e^{i\theta}, s, t) \sim B(s+t, s-t)(\sin \theta)^s.$$

Here $B(\cdot, \cdot)$ is the Euler Beta-function. Thus if $\lambda > 0$, $\lambda \neq \frac{1}{4}$ we see that there are constants A, B so that a solution f of (34) satisfies, as $\theta \to 0$,

$$f(\theta) \sim A \cdot (\sin \theta)^s + B(\sin \theta)^{1-s}$$

and both A and B vanish only if $f = 0$. Thus if $\lambda > \frac{1}{4}$ we see at once that (35) diverges. If $\lambda = \frac{1}{4}$ the argument is more complicated but follows the same lines. In this case we find that there are constants A, B so that

$$f(\theta) \sim (A \cdot \log (\sin \theta) + B)(\sin \theta)^{\frac{1}{2}}$$

but we shall omit the details. This completes the proof.

This proposition shows that there is a marked difference between the cases of G of the first and of the second kind.

We shall consider first the case of G with $\delta(G) < \frac{1}{2}$. There are no parabolic elements. If we expand $E_\zeta(z, s)$ and $S(\zeta, \eta; s)$ as Fourier series with respect to the G_α we see first that the operator represented by $S(\cdot, \cdot; s)$ has the form

$$\Delta(s)(I+\Sigma(s))$$

where $\Delta(s)$ is a diagonal operator independent of G and $\Sigma(s)$ is a compact operator whose entries decrease exponentially rapidly (thus it is represented by a smooth kernel). Thus $\Delta(s)(I+\Sigma(s))$ can be inverted if and only if

$$\varphi(s) = \det(I+\Sigma(s))$$

(det = Fredholm determinant) is non-zero. The function φ exists as an analytic function in $\operatorname{Re}(s) > \delta(G)$. It is not difficult to see that φ is not identically 0 and hence, by the remarks at the end of Section 5, the operator represented by $S(\cdot, \cdot; s)$ can be continued as a meromorphic function for all values of s. It follows that $E_\zeta(z, s)$ and $S(\zeta, \eta; s)$ can be continued as meromorphic functions.

Now let $\delta(G) \geqq \frac{1}{2}$. Let D^θ be a fundamental domain for the action of G on

$$\boldsymbol{H} \setminus \Big(\bigcup_{\alpha\in A}\bigcup_{g\in G} g(\Lambda_\alpha) \cup \bigcup_{p\in P}\bigcup_{g\in G} g(C_p)\Big).$$

From the remarks at the beginning of this section we can choose this to be relatively compact in $\boldsymbol{H}$. By an application of Stokes' theorem we can find an expression for

$$\int_{D^\theta} (E_\zeta(z, s), E_\zeta(z, t))\, d\sigma(z) \tag{36}$$

which involves the Fourier coefficients of $E(z, \cdot)$ and known functions. In this formula $(\cdot, \cdot)$ is the Hermitian product on End (V) induced from that on V.

Now let $E_{\zeta,\alpha}(z, s)$ be the Eisenstein series belonging to G_α instead of G. Define

$$\begin{aligned} E_\zeta^{(\theta)}(z, s) &= E_\zeta(z, s) - E_{\zeta,\alpha}(z, s) && \text{if } z \in \Lambda_\alpha,\ \zeta \in \bigcup_{g\in G} g(\Omega_\alpha) \\ &= E_\zeta(z, s) && \text{otherwise.} \end{aligned}$$

Then we can deduce an expression for

$$\int_{G\setminus \boldsymbol{H}} (E^{(\theta)}(z, s), E^{(\theta)}(z, t))\, d\sigma(z) \tag{37}$$

which depends only *linearly* on the Fourier coefficients of $E(\cdot, \cdot)$. We can now follow the line of argument indicated by Selberg in [11] and show that $E_\zeta(z, s)$ can be continued to $\{s : \mathrm{Re}\,(s) > \frac{1}{2}, s \notin]\frac{1}{2}, \delta(G)]\}$.

Using the positivity of (36) we deduce that if there is given a sequence $s_j \to s_0 \in \{s : \mathrm{Re}\,(s) = \frac{1}{2}, s \neq \frac{1}{2}\}$ then $\lim E_\zeta(z, s_j)$ exists. We also obtain a quadratic relation between the Fourier coefficients of $E_\zeta(z, s)$ on $\mathrm{Re}\,(s) = \frac{1}{2}$. This proves to be exactly the same as the functional equation considered above. This can then be proved for all s by an application of the reflection principle. It then follows that $E_\zeta(z, s)$ can be continued as a meromorphic function over the whole plane except for the interval $[\delta(G), 1-\delta(G)]$.

The same method shows that the Eisenstein series $E_p(z, s)$ associated to a parabolic vertex can be continued to the same region.

The part of the theory we have just described is the technically most involved but it should be emphasized that it uses no more ideas than can be found in [11]. It seems to the author that Selberg's ideas should find their proper place in general differential geometry rather than the theory of symmetric spaces.

Now let h be a function satisfying (i′), (ii′), (iii′) of Section 3 and also $h(\frac{1}{2}) = 0$.Let q_h, Q_h be as above. Now by analogy with (30) form

$$Q_h^{(1)}(z, w) = \frac{1}{16\pi^2 i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_B E_\zeta(z, s) E_\zeta(w, s)^*$$

$$\times \frac{(2s-1)\cdot \sin 2\pi s h(s)}{\sin \pi(s+k) \sin \pi(s-k)} d\zeta ds + \sum_{0 \leqq m < |k|-\delta(G)} h(|k|-m) R_{k,m}(z, w)$$

$$+ \sum_{p \in P} \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E_p(z, s) E_p(z, s)^* h(s) ds.$$

Using the series $E_{\zeta,\alpha}(z, s)$ as intermediaries we show that, at least if $k = 0$

$$Q_h^{(2)}(z, w) = Q_h(z, w) - Q_h^{(1)}(z, w)$$

is of trace class. The proof is a matter of estimates and follows a by now well-troden path. There is thus an at most countable set of eigenfunctions which also form the set of eigenfunctions of $-\Delta_k$. Let the set of these eigenfunctions be $\{\varphi_\mu\}$. Suppose that

$$-\Delta_k \varphi_\mu = \lambda_\mu \varphi_\mu.$$

By what has gone before, if $k = 0$

$$\delta(G)(1-\delta(G)) \leqq \lambda_\mu < \tfrac{1}{4}.$$

It is not difficult to see that λ_μ has no limit point in $[\delta(G)(1-\delta(G)), \frac{1}{4}[$. From this we deduce that $E_\zeta(z, s)$ and $E_p(z, s)$ can be continued as meromorphic functions to the region $\boldsymbol{C} \smallsetminus \{\frac{1}{2}\}$. It also follows that

$$\sum (\tfrac{1}{4}-\lambda_\mu)^{\frac{1}{2}}$$

converges. I do not know whether in general there are an infinite number of eigenvalues or not. In [6] we showed that if $k = 0$, $\chi = I$ then $\lambda = \delta(G)(1-\delta(G))$ is actually an eigenvalue of multiplicity 1.

In this case we mention again the quantum mechanical analogy. The $E_\zeta(z, s)$ correspond to the free (unbound) states. The eigenfunctions correspond to bound states. So we see that there is an 'escape energy' of $\frac{1}{4}$. A particle with energy less than $\frac{1}{4}$ is bound whereas otherwise it is free. This is analogous to the theory of the hydrogen atom. One knows that the problem of establishing the existence of an infinite number of bound states is a difficult one.

The above sketch shows also how to find the spectral decomposition of $-\Delta_k$ on $G\backslash \boldsymbol{H}$. The success of Selberg's ideas here is quite striking. The fact that $Q_h^{(2)}$ is of trace class shows that we can even write down a trace formula. This looks much the same as the ordinary trace formula but differs chiefly by the appearance of a new term of the form

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} (\varphi'(s)/\varphi(s))h(s)ds$$

where φ is a slightly modified version of the Fredholm determinant introduced above. This is just as one would expect by analogy with the trace formula for groups of the first kind with parabolic elements. The area of the fundamental domain is replaced by $\sigma(D^{\pi/2})$. The details must wait for a future publication.

REFERENCES

[1] Elstrodt, J.: *Über das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene bei Fuchsschen Gruppen zweiter Art.* Dissertation, München, 1969.

[2] Elstrodt, J.: Die Resolvente zum Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I. *Math. Ann. 203* (1973) 295–330. II. *Math. Z. 132* (1973) 99–134. III. *Math. Ann. 208* (1974) 99–132.

[3] Elstrodt, J. und Roelcke, W.: Über das wesentliche Spektrum zum Eigenwertproblem der automorphen Formen. *Manuscripta Math. 11* (1974) 391–406.
[4] Gel'fand, I. M.: Automorphic functions and the theory of representations. *Proc. Int. Cong. Math. Stockholm, 1962.*
[5] Gel'fand, I. M., Graev, M. I. and Pyatetskii-Shapiro, I. I.: *Representation theory and automorphic functions.* W. B. Saunders, 1969.
[6] Patterson, S. J.: *The limit set of a Fuchsian group.* To appear in 'Acta Mathematica'.
[7] Patterson, S. J.: *On the cohomology of Fuchsian groups.* To appear in 'Glascow M.J.
[8] Patterson, S. J.: *A lattice point problem in hyperbolic space.* To appear in 'Mathematika' 22 (1975) 81–88.
[9] Roelcke, W.: Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I. *Math. Ann. 167* (1966) 292–337. II. *Math. Ann. 168* (1967) 261–324.
[10] Selberg, A.: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. 20* (1956) 47–87.
[11] Selberg, A.: Discontinuous groups and harmonis analysis. *Proc. Int. Cong. Math. Stockholm, 1962.*
[12] Watson, G. N.: Asymptotic expansions of hypergeometric functions. *Camb. Phil. Trans. XXII* (1918) 277–308.

(Oblatum 20–XII–1974)

Mathematisches Institut
34 Göttingen
Bunsenstrasse 3/5
BRD