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SEMI-GROUPS OF RANK-PRESERVING TRANSFORMERS ON MINIMAL NORM IDEALS IN $\mathcal{B}(\mathcal{H})$

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Abstract

Transformers on any minimal norm ideal of the C^* -algebra $\mathcal{B}(\mathcal{H})$, generated by left, right or two-sided multiplication with operators acting in \mathcal{H} , share many features with these operators. Thus, their spectra are related in a simple manner, semi-groups of such transformers have many properties in common with the corresponding operator semi-groups, their generators are algebraically related, etc. In particular, such transformer semi-groups can be unambiguously characterized by their property of preserving the rank of the operators in the norm ideal on which they act. Necessary and sufficient conditions under which their generators are spectral operators are derived.

1. Introduction

The present paper deals with the theory of a class of linear transformations in two-sided minimal norm ideals of completely continuous operators on a complex Hilbert space \mathcal{H} . As is well known [1–3], each such ideal \mathcal{I} can be equipped with a unitarily invariant cross-norm and the linear manifold \mathcal{R} of all completely continuous operators with finite rank is a dense subset of \mathcal{I} equipped with the appropriate cross-norm topology. Throughout this paper \mathcal{I}_α will denote the two-sided minimal norm ideal of completely continuous operators on \mathcal{H} with a given unitarily invariant cross-norm $\alpha(S) = |S|_\alpha$, $S \in \mathcal{I}_\alpha$. Since \mathcal{I}_α is a Banach space with respect to the norm α , it makes sense to consider linear transformations in this space. In order to distinguish between ordinary Hilbert space operators and operators on the Banach space \mathcal{I}_α ,

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we shall refer to the latter as ‘transformers’. (Other names in use are: transformators, super-operators and, in the more specialized context of statistical mechanics, Liouville operators.) We shall denote transformers on \mathcal{S}_α by A, B, \dots . In case A is bounded, we write $\|A\|_\alpha = \sup_{S \in \mathcal{S}_\alpha} |AS|_\alpha / |S|_\alpha$. Limits in the α -norm will be denoted by α -lim.

The organization of the present paper is as follows:

In section 2 a general Banach space \mathcal{X} is considered on which a one-to-one semi-linear mapping $X \rightarrow X^+$ ($X \in \mathcal{X}$), called conjugation, is defined. This mapping entails a corresponding mapping $A \rightarrow A^c$ of linear operators on \mathcal{X} , which is defined according to $(AX)^+ = A^cX^+$ ($X \in \mathcal{D}_A$). Various properties of this mapping c are investigated. These results are to be used frequently in the remaining part of this paper. In section 3 left and right multiplication transformers (in general unbounded) on an ideal \mathcal{S}_α are studied (defined according to $A_l S = (AS)^{**}$, $A_r S = (SA)^{**}$, where $S \in \mathcal{S}_\alpha$ and A is a densely defined operator on \mathcal{H}). In particular it is shown that the respective point, continuous and residual spectra of A and A_l coincide, and a similar result holds for A and A_r . It is also shown that if A generates a strongly continuous, uniformly bounded, semi-group on \mathcal{H} , then so do A_l and A_r on \mathcal{S}_α . It is also shown that if A_1 and A_2 generate strongly continuous, uniformly bounded semi-groups $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$, respectively, on \mathcal{H} , then $\mathcal{U}(t) = \mathcal{U}_1(t)\mathcal{U}_2(t)$ constitutes a strongly continuous, uniformly bounded, semi-group on \mathcal{S}_α with infinitesimal generator $A = A_{1l} + A_{2r}$. The semi-groups of transformers on \mathcal{S}_α considered thus far have the property of mapping finite-rank elements of \mathcal{S}_α into elements of the same rank. This raises the question whether this property is characteristic for operator-induced semi-groups of transformers on \mathcal{S}_α . This matter is settled in the affirmative sense in section 4 where it is shown that rank-preserving semi-groups of transformers on \mathcal{S}_α can be implemented by two semi-groups of operators on \mathcal{H} . This result makes it possible to obtain an unambiguous characterization of generalized inner derivation (defined through $\mathcal{D}_A(B) = (AB - BA)^{**}$, where A is not necessarily bounded) on both \mathcal{S}_α and $\mathcal{B}(\mathcal{H})$. In section 5 some attention is given to the question whether scalar spectral operators on \mathcal{H} induce scalar type transformers on \mathcal{S}_α . This is indeed the case for simple left and right multiplication transformers but in general such a result does not hold. This is due to the fact that two commuting uniformly bounded spectral measures on \mathcal{S}_α do not lead, in general, to a product spectral measure, which is uniformly bounded on \mathcal{S}_α .

In addition to throwing some light on a yet unexplored area in the theory of minimal norm ideals the above results are also of interest in quantum statistical mechanics [4] and quantum field theory [5]. This is

due to the fact that the behaviour in time of the statistical operator (density operator) ρ , which is a positive normal element of the traceclass $\mathcal{B}_1(\mathcal{H})$ of operators on \mathcal{H} , is governed by $\mathcal{U}(t) = \mathcal{U}_r(t)\mathcal{U}_r^*(t)$ with $\mathcal{U}(t) = \exp(-iHt)$, where $H = H$ is the full Hamiltonian of the system. Working with transformers enables us to derive [5] many results of scattering theory for the density operators which are analogous to well-known corresponding results for statevectors.

2. Conjugation of transformers on $\mathcal{B}(\mathcal{X})$

Let us consider in this section the general case of a complex Banach space \mathcal{X} with elements X, Y, \dots on which an operation of *conjugation*, denoted by $X \leftrightarrow X^+$, is defined, i.e. $X \leftrightarrow X^+$ is a one-to-one, norm-preserving, semi-linear mapping of \mathcal{X} into itself, for which $(X^+)^+ = X$. Let $A \in \mathcal{B}(\mathcal{X})$. Then the conjugation c in $\mathcal{B}(\mathcal{X})$ is introduced according to $(A\mathcal{X})^+ = A^c\mathcal{X}^+$. It is easily established that $A^c \in \mathcal{B}(\mathcal{X})$ and that the mapping $A \rightarrow A^c$ possesses the following properties ($A, B \in \mathcal{B}(\mathcal{X})$, I is the identity on $\mathcal{B}(\mathcal{X})$ and $\|A\|$ stands for the operator bound of A):

(1) $I^c = I$, (2) $A^{cc} \equiv (A^c)^c = A$, (3) $(\lambda A + B)^c = \bar{\lambda}A^c + B^c$, $\lambda \in \mathbb{C}$, (4) $(AB)^c = A^cB^c$, (5) $|A^c| = |A|$ and (6) If A^{-1} exists as an element of $\mathcal{B}(\mathcal{X})$, then $(A^{-1})^c = (A^c)^{-1}$, so that if $\rho(A)$ denotes the resolvent set of some $A \in \mathcal{B}(\mathcal{X})$, we have $\lambda \in \rho(A) \leftrightarrow \bar{\lambda} \in \rho(A^c)$.

It follows from property (5) that the mapping $\mathcal{B}(\mathcal{X}) \xrightarrow{c} \mathcal{B}(\mathcal{X})$ is continuous in the uniform topology on $\mathcal{B}(\mathcal{X})$. From the property $|X| = |X^+|$ for $X \in \mathcal{X}$ it follows that the same is true with respect to the strong operator topology.

If $\mathcal{X} = \mathcal{I}_\alpha$ then it is natural to choose the involution $A \rightarrow A^*$ in $\mathcal{B}(\mathcal{X})$ as the conjugation operation used to define conjugation c in $\mathcal{B}(\mathcal{I}_\alpha)$. Due to the special form of the bounded linear functionals on \mathcal{I}_α and the property $\overline{trST} = trS^*T^*$, $S \in \mathcal{I}_\alpha$, $T \in \mathcal{I}_\alpha^*$, it follows that the mapping $A \rightarrow A^c$ is continuous in the weak transformer topology.

We will say that $A \in \mathcal{B}(\mathcal{X})$ is *self-conjugate* if $A = A^c$. On \mathcal{I}_α the self-conjugate transformers are those that carry Hermitean operators into Hermitean operators. It is seen from the above property (6) that the spectrum $\sigma(A)$ of a self-conjugate transformer A is invariant under reflections with respect to the real axis (i.e. $\lambda \in \sigma(A) \leftrightarrow \bar{\lambda} \in \sigma(A)$). It is naturally desirable to extend the concept of the conjugate transformer A^c of A to the case when A is not bounded. The obvious definition is now $(AX)^+ = A^cX^+$, $X \in \mathcal{D}_A$ so that $X^+ \in \mathcal{D}_{A^c}$ if $X \in \mathcal{D}_A$. In this case property (5) loses its meaning and the other properties have to be modified insofar as domains of definition are concerned. The following lemma is easily established.

LEMMA 2.1:

- (a) A^c is densely defined on \mathcal{X} if and only if A is densely defined on \mathcal{X} .
 (b) A^c is closed on \mathcal{X} if and only if A is closed on \mathcal{X} .

THEOREM 2.1: Let A be a linear transformation in \mathcal{X} . A^c generates a uniformly bounded, strongly continuous one-parameter semi-group $\{\mathcal{V}(t)|t \geq 0$ on \mathcal{X} if and only if A generates such a semi-group $\{\mathcal{U}(t)|t \geq 0$. In that case $\mathcal{V}(t) = \mathcal{U}^c(t)$.

PROOF: Suppose that A generates the semi-group $\mathcal{U}(t)$ as stated above. This is the case if and only if (ref. [6], chapter 9) A is closed, densely defined on \mathcal{X} and in addition $n \in \rho(A)$ for $n = 1, 2, 3, \dots$ with $(I - n^{-1}A)^{-m} \leq K$ for all $n = 1, 2, 3, \dots, m = 1, 2, 3, \dots$.

According to lemma 2.1 A^c is closed and densely defined along with A . Furthermore property (6) remains true for unbounded, closed A . For, let $\lambda \in \rho(A)$. Then $\mathcal{R}_{\lambda-A} = \mathcal{X}$ and $B = B_\lambda = (\lambda - A)^{-1}$ exists as an element of $\mathcal{B}(\mathcal{X})$. Since Y^+ runs through all of \mathcal{X} if $Y = (\lambda - A)X$ does, it follows that $\mathcal{R}_{\lambda-A^c} = \mathcal{X}$. The relation $B(\lambda - A)X = X, X \in \mathcal{D}_A$ leads to

$$[B(\lambda - A)X]^+ = B^c(\bar{\lambda} - A^c)X^+ = X^+.$$

Thus $\bar{\lambda} - A^c$ has range \mathcal{X} and a bounded inverse $B^c = (\bar{\lambda} - A^c)^{-1}$ so that $\bar{\lambda} \in \rho(A^c)$. For real $n = 1, 2, 3, \dots$ it follows immediately that $n \in \rho(A)$ implies $n \in \rho(A^c)$. From the properties (4) and (5) above, we infer that $|(\mathbf{B}_n)^m| = |((\mathbf{B}_n)^m)^c| = |(\mathbf{B}_n^c)^m|$ and hence $|(I - n^{-1}A^c)^{-m}| = |(I - n^{-1}A)^{-m}| \leq K, n = 1, 2, 3, \dots, m = 1, 2, 3, \dots$. Thus A^c obeys the same conditions as A does and hence generates the semi-group $\mathcal{V}(t)$ as stated above. This proves the first part of the theorem; the 'only if' part being clear from Lemma 2.1 and equality (1). The relation $\mathcal{V}(t) = \mathcal{U}^c(t)$ follows the usual (ref. 8, chapter 9) construction of the semi-group from its infinitesimal generator. ■

REMARK: In Theorem 2.1 we considered uniformly bounded semi-groups. A similar theorem can be established under somewhat different conditions. For instance, one can obtain a bound for which

$$|\mathcal{U}(t)|_\alpha \leq \exp \beta t, \quad \beta > 0,$$

by redefining $\mathcal{U}(t)$.

It was shown in the proof of Theorem 2.1 that for closed A the resolvent sets, and hence the spectra of A and A^c are mapped onto each other by taking complex conjugates. In fact, more can be said about this relation-

ship. Denoting the points spectrum by σ_p , the residual spectrum by σ_r and the continuous spectrum by σ_c (where we follow the definition of the various components of the spectrum as given in ref. [6]) it is easily checked that:

LEMMA 2.2: *The spectra $\sigma_p(A^c)$, $\sigma_r(A^c)$ and $\sigma_c(A^c)$ consist of the complex conjugates of the elements in the spectra $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$, respectively.*

3. Semi-groups related to left and right multiplication transformers

Let $A \in \mathcal{B}(\mathcal{H})$. Then for $S \in \mathcal{I}_\alpha$, AS and SA are again contained in \mathcal{I}_α and we can define the transformers A_l and A_r by $A_l S = AS$ and $A_r S = SA$. Since \mathcal{I}_α is a two-sided ideal A_l and A_r are contained in $\mathcal{B}(\mathcal{I}_\alpha)$. In fact we have the following result, whose proof is an immediate consequence of cross-norm properties.

LEMMA 3.1: *Let A_1 and A_2 be contained in $\mathcal{B}(\mathcal{H})$. Then A , defined by $AS = A_1 S A_2$, $S \in \mathcal{I}_\alpha$ is a bounded linear transformer on \mathcal{I}_α and $\|A\|_\alpha = |A_1| |A_2|$. In particular, if $A \in \mathcal{B}(\mathcal{X})$, then $\|A_l\|_\alpha = \|A_r\|_\alpha = |A|$. The situation becomes more complicated for unbounded A . Then we define:*

DEFINITION 3.1: For a given densely defined linear operator A the left and right multiplication transformers, A_l and A_r respectively, induced by A , are given by

$$A_l S = (AS)^{**}, \quad \mathcal{D}_{A_l} = \{S \in \mathcal{I}_\alpha | (AS)^{**} \in \mathcal{I}_\alpha\}$$

$$A_r S = (SA)^{**}, \quad \mathcal{D}_{A_r} = \{S \in \mathcal{I}_\alpha | (SA)^{**} \in \mathcal{I}_\alpha\}.$$

LEMMA 3.2: *Suppose A is closed and densely defined on \mathcal{H} . Then $A_l^c = A_r^*$ and $A_r^c = A_l^*$, where A_l^* and A_r^* are defined by $A_l^* S = (A^* S)^{**}$ and $A_r^* S = (S A^*)^{**}$, with S in the proper domain.*

PROOF: If B is a linear operator in \mathcal{H} and $S \in \mathcal{B}(\mathcal{H})$ then (cf. ref. 7, p. 300) $(SB)^* = B^* S^*$. If, moreover, B is closed and densely defined, then $B^{**} = B$ and $(S^* B^*)^* = BS$. Suppose $S \in \mathcal{D}_{A_r}$. Then $S \in \mathcal{B}(\mathcal{H})$ and $A_l^c S^* = (A_r S)^* = (S A)^{***} = (A^* S^*)^{**} = A_l^* S^*$ i.e. $A_l^c \subset A_l^*$. For $S^* \in \mathcal{D}_{A_l^*}$ it follows from reading this sequence of relations from the right to the left that $A_l^* \subset A_l^c$ and hence $A_l^c = A_l^*$.

Consider next the case that $S \in \mathcal{D}_{A_l} \subset \mathcal{B}(\mathcal{H})$. Then $A_l S \in \mathcal{B}(\mathcal{H})$ and

$A_l^c S^* = (A_l S)^* = (AS)^{***} = (S^* A^*)^{**} = A_r^* S^*$ or $A_l^c \subset A_r^*$. In the same way as above it follows that $A_r^* \subset A_l^c$ and hence $A_l^c = A_r^*$. ■

REMARK: A more obvious definition of A_l and A_r would be $A_l S = AS$ and $A_r S = SA$ respectively. Indeed, if A is closed and S is bounded then AS is closed. Hence, if $(AS)^{**}$ exists then $(AS)^{**} = AS$ and both definitions of A_l coincide for closed A . On the other hand SA is only defined on the domain of A and, in order to obtain a useful definition of A_r , we have to require that SA can be extended to a bounded operator, contained in \mathcal{I}_α . This is achieved in definition 3.1 by demanding that $(SA)^{**}$, the smallest closed extension of SA (SA is in general not closed or closable, even if A is closed), is an element of \mathcal{I}_α .

Note that the symmetry expressed in Lemma 3.2 may be lost if the densely defined operator A is not closed. In that case AS need no longer be equal to $(S^* A^*)^* = A^{**} S$, which property was used in the second part of the proof of the lemma.

LEMMA 3.3: *Let A be closed and densely defined on \mathcal{H} . Then A_l and A_r are closed and densely defined on \mathcal{I}_α .*

PROOF: Suppose that A is densely defined on \mathcal{H} . Let $S \in \mathcal{I}_\alpha$, $S \neq 0$. For given $\varepsilon > 0$ there exists an $R \in \mathcal{B}$, $R \neq 0$, such that $|S - R|_\alpha < \varepsilon/2$. We write R according to its canonical decomposition

$$R = \sum_{k=1}^n \mu_k(\cdot, x_k) y_k, \quad \mu_1 \geq \mu_2 \cdots \geq \mu_n > 0.$$

As \mathcal{D}_A is dense in \mathcal{H} there exists for each $y_k \in \mathcal{H}$ a $z_k \in \mathcal{D}_A$ with $|y_k - z_k| < \varepsilon/(2\mu_1 n)$. Then the operator $R_0 = \sum_{k=1}^n \mu_k(\cdot, x_k) z_k$ is an element of \mathcal{I}_α , contained in \mathcal{D}_{A_l} . We obviously have that $|S - R_0|_\alpha < \varepsilon$. Hence we conclude that \mathcal{D}_{A_l} is dense in \mathcal{I}_α . Since A is closed and densely defined this is also true for A^* , so that $\mathcal{D}_{A_l^*}$ is dense in \mathcal{I}_α . Since $A_r = (A_l^*)^c$, it follows from lemma 2.1 that \mathcal{D}_{A_r} is dense in \mathcal{I}_α .

Let A be closed and densely defined in \mathcal{H} . Suppose that for certain $\{S_n\}_{n=1}^\infty \subset \mathcal{D}_{A_l}$ with $\alpha\text{-}\lim_{n \rightarrow \infty} S_n = S$ the limit $T = \alpha\text{-}\lim_{n \rightarrow \infty} A_l S_n$ exists. Let x be an arbitrary element of \mathcal{H} . Since

$$|S_n x - Sx| \leq |S_n - S| |x| \leq |S_n - S|_\alpha |x|$$

we see that $y \equiv Sx = s\text{-}\lim_{n \rightarrow \infty} y_n$, where $y_n = S_n x$. For $z = Tx$ we have in addition $(AS_n = A_l S_n$, since A is closed)

$$|A y_n - z| = |(AS_n - T)x| \leq |A_l S_n - T|_\alpha |x|$$

and hence $s\text{-}\lim_{n \rightarrow \infty} Ay_n = z$. As A is closed it follows that $y = Sx \in \mathcal{D}_A$ and $Ay = z$. Thus $ASx = Tx$ for arbitrary $x \in \mathcal{H}$. Hence $AS = T$. Since S and T are contained in \mathcal{I}_α it follows that $S \in \mathcal{D}_{A_l}$ and $A_l S = T$. Hence A_l is closed.

Starting from A_l and using Lemma 2.1 we show in the same way that A_r is closed. ■

THEOREM 3.1: *Let A be closed and densely defined on \mathcal{H} . Then $\rho(A_l) = \rho(A)$ and for $\lambda \in \rho(A_l)$, $(\lambda - A_l)^{-1}$ is the bounded left multiplication transformer $(\lambda - A)^{-1}$. In addition $\sigma_p(A_l) = \sigma_p(A)$, $\sigma_r(A_l) = \sigma_r(A)$ and $\sigma_c(A_l) = \sigma_c(A)$.*

PROOF: (a) Suppose that $\mathcal{R}_{\lambda-A}$ is dense in \mathcal{H} and that $(\lambda - A)^{-1}$ exists on $\mathcal{R}_{\lambda-A}$. By essentially the same method as used in part (a) of the proof of Lemma 3.3 it follows that $\mathcal{R}_{\lambda-A_l}$ is dense in \mathcal{I}_α . Let the transformer B on \mathcal{I}_α be defined by the relation $BS = (\lambda - A)^{-1}S$. Then $Sx \in \mathcal{D}_A$ ($A = A^{**}$) for every $x \in \mathcal{H}$ and we have $(\lambda - A)^{-1}(\lambda - A)Sx = Sx$ for all $x \in \mathcal{H}$, so that $(\lambda - A)^{-1}(\lambda - A)S = B(\lambda - A_l)S = S$, $S \in \mathcal{D}_{A_l}$. Thus $B = (\lambda - A_l)^{-1}$ exists on \mathcal{I}_α and has dense domain. Suppose that $\lambda \in \rho(A)$. In that case $(\lambda - A)^{-1}$ is bounded and everywhere defined on \mathcal{H} (A is closed). Consequently $B = (\lambda - A_l)^{-1}$ is bounded and everywhere defined on \mathcal{I}_α . Hence $\rho(A) \subset \rho(A_l)$.

Suppose now that $\lambda \in \sigma_c(A)$. Then $(\lambda - A)^{-1}$ exists as a densely defined, discontinuous operator. Since A is closed, $(\lambda - A)^{-1}$ is necessarily unbounded. Thus there is a sequence $\{y_n\}_{n=1}^\infty \subset \mathcal{D}_{(\lambda-A)^{-1}}$ with $y_n \neq 0$ and $|(\lambda - A)^{-1}y_n| > n|y_n|$. We show that $(\lambda - A_l)^{-1}$ is unbounded. Thus let S_n be defined by $S_n = (\cdot, x)y_n$, $x \neq 0$. Then $S_n \in \mathcal{D}_{(\lambda-A_l)^{-1}}$ and

$$(\lambda - A_l)^{-1}S_n = (\cdot, x)(\lambda - A)^{-1}y_n.$$

We now have

$$|(\lambda - A_l)^{-1}S_n|_\alpha = |(\cdot, x)(\lambda - A)^{-1}y_n|_\alpha = |x| |(\lambda - A)^{-1}y_n| > n|x||y_n| = n|S_n|_\alpha$$

and we conclude that $(\lambda - A_l)^{-1}$ is unbounded. Since A_l is closed, $(\lambda - A_l)^{-1}$ is closed and unbounded. Hence $\lambda \in \sigma_c(A_l)$ and $\sigma_c(A) \subset \sigma_c(A_l)$.

(b) Suppose that $\lambda \in \sigma_p(A)$. Then there exists an $x \in \mathcal{H}$, $x \neq 0$, with $Ax = \lambda x$. Defining S by $S = (\cdot, y)x$, $y \neq 0$ we see that $A_l S = (\cdot, y)Ax = \lambda S$ and hence $\lambda \in \sigma_p(A_l)$. Suppose, conversely, that $\lambda \in \sigma_p(A_l)$. Then there exists an $S \in \mathcal{I}_\alpha$, $S \neq 0$ with $A_l S = AS = \lambda S$. Since $S \neq 0$ there is at least one $x \in \mathcal{H}$, such that $y = Sx \neq 0$. For this x and y we have $Ay = ASx = \lambda Sx = \lambda y$ and hence $\lambda \in \sigma_p(A)$. We conclude that $\sigma_p(A) = \sigma_p(A_l)$.

(c) Suppose that $\lambda \in \sigma_r(A)$. Then $\mathcal{R}_{\lambda-A}$ is not dense in \mathcal{H} and there exists a non-zero vector x in \mathcal{H} which is orthogonal to $\mathcal{R}_{\lambda-A}$. This implies that $\mathcal{R}_{\lambda-A}$ is not dense in \mathcal{I}_α . In fact, if $\alpha - \lim_{n \rightarrow \infty} (\lambda - A_n)S_n = (\cdot, y)x$ for some $\{S_n\}_{n=1}^\infty \subset \mathcal{D}_{A_n}$ and $y \neq 0$ then we would have

$$\lim_{n \rightarrow \infty} (x, (\lambda - A)S_n y) = \lim_{n \rightarrow \infty} (x, [(\lambda - A_n)S_n]y) = (x, x)(y, y) > 0$$

which is impossible since $x \perp \mathcal{R}_{\lambda-A}$ and consequently $(x, (\lambda - A)z) = 0$ for any $z \in \mathcal{D}_A$.

Hence $\lambda \in \sigma_p(A_n)$ or $\lambda \in \sigma_r(A_n)$. The first possibility must be excluded on account of $\sigma_p(A_n) = \sigma_p(A)$, so that $\sigma_r(A) \subset \sigma_r(A_n)$. Using now the well known facts that ρ , σ_c , σ_p and σ_r are disjoint, that their union is \mathbb{C} and that the complement of ρ in \mathbb{C} is σ , we conclude that $\rho(A) = \rho(A_n)$, $\sigma_c(A) = \sigma_c(A_n)$, $\sigma_p(A) = \sigma_p(A_n)$, $\sigma_r(A) = \sigma_r(A_n)$. ■

COROLLARY 3.1: *Let A be closed and densely defined on \mathcal{H} . Then $\rho(A_r) = \rho(A)$ and for $\lambda \in \rho(A_r)$, $(\lambda - A_r)^{-1}$ is the bounded right multiplication transformer $(\lambda - A)_r^{-1}$. In addition*

$$\sigma_p(A_r) = \{\bar{\lambda} | \lambda \in \sigma_p(A^*)\}, \quad \sigma_r(A_r) = \{\bar{\lambda} | \lambda \in \sigma_r(A^*)\}$$

and $\sigma_c(A_r) = \sigma_c(A)$.

PROOF: As A is closed and densely defined on \mathcal{H} this is also the case for A^* . Applying first Theorem 3.1 to $A^* = A_r^c$ and afterwards Lemma 2.2 we get the desired result on the resolvent sets and spectra by noting that $\rho(A) = \rho(A^*)$ and $\sigma_c(A) = \sigma_c(A^*)$.

THEOREM 3.2: *Suppose A generates a uniformly bounded, strongly continuous one parameter semi-group on \mathcal{H} . Then A_l and A_r generate uniformly α -bounded and strongly continuous one parameter semi-groups on \mathcal{I}_α .*

PROOF: Since A generates a uniformly bounded, strongly continuous semi-group on \mathcal{H} , A is closed and densely defined on \mathcal{H} . Moreover (see the proof of Theorem 2.1) $n \in \rho(A)$ for $n = 1, 2, \dots$ and $|(1 - n^{-1}A)^{-m}| \leq K$ for all $n = 1, 2, \dots$, $m = 1, 2, 3, \dots$. Since A is closed and densely defined we have $\rho(A) = \rho(A_l) = \rho(A_r)$ and it follows at once that $n = 1, 2, \dots$ is also contained in the resolvent sets of the closed, densely defined (on \mathcal{I}_α) transformers A_l and A_r . Since

$$(\lambda - A_l)^{-1} = (\lambda - A)_l^{-1} \quad \text{and} \quad (\lambda - A_r)^{-1} = (\lambda - A)_r^{-1}$$

are bounded transformers on \mathcal{I}_α for $\lambda \in \rho(A)$ it is clear from Lemma 3.1 that $\|(1 - n^{-1}A_l)^{-m}\|_\alpha \leq K$, $\|(1 - n^{-1}A_r)^{-m}\|_\alpha \leq K$ for $n = 1, 2, \dots$ and $m = 1, 2, \dots$, and the theorem is established. \blacksquare

THEOREM 3.3: *Suppose that $\{\mathcal{U}(t)|t \geq 0\}$ is a uniformly bounded, strongly continuous one-parameter semi-groups on \mathcal{H} . Then:*

(1) $\{\mathcal{U}_l(t)|t \geq 0\}$ and $\{\mathcal{U}_r(t)|t \geq 0\}$ constitute uniformly bounded, one-parameter semi-groups on \mathcal{I}_α , continuous in the strong transformer topology, and their bounds coincide with the bound of $\mathcal{U}(t)$.

(2) Let A be the infinitesimal generator of $\mathcal{U}(t)$. Then A_l and A_r are the infinitesimal generators of $\mathcal{U}_l(t)$ and $\mathcal{U}_r(t)$, respectively.

PROOF: The proof of (1) is straightforward. According to theorem 3.2 A_l and A_r generate uniformly bounded, strongly continuous semi-groups on \mathcal{I}_α . Let us show then that B_r , the infinitesimal generator of $\mathcal{U}_r(t)$, is contained in A_r (the corresponding proof for $B_l \subset A_l$ is even simpler). Suppose that $S \in \mathcal{I}_\alpha$ is contained in \mathcal{D}_{B_r} . Let $x, y \in \mathcal{H}$, $x \neq 0, y \neq 0$. Then (cf. ref. 1) $R = (\cdot, z)y \in \mathcal{I}_\alpha^*$ and

$$0 = \lim_{t \rightarrow +0} \text{tr}[B_r S - t^{-1}(\mathcal{U}_r(t) - 1)S]R = \lim_{t \rightarrow +0} (y, (B_r S)^* x - t^{-1}(\mathcal{U}^*(t) - 1)S^* x).$$

Consequently $t^{-1}(\mathcal{U}^*(t) - 1)S^* x$ converges weakly towards $(B_r S)^* x$ in \mathcal{H} . However, since $\mathcal{U}^*(t)$ is a strongly continuous semi-group, this is sufficient to conclude that $S^* x$ is in the domain of A^* , the infinitesimal generator of $\mathcal{U}^*(t)$. Since x was chosen arbitrarily from \mathcal{H} , it follows from the equality $(B_r S)^* x = A^* S^* x$ that $(B_r S)^* = A^* S^*$ or, since $S \in \mathcal{B}(\mathcal{H})$, $(B_r S)^* = (SA)^*$. Now $B_r S$ is contained in \mathcal{I}_α and therefore is bounded. Hence $B_r S = (B_r S)^{**} = (SA)^{**} \in \mathcal{I}_\alpha$, i.e. $S \in \mathcal{D}_{A_r}$. Since S was chosen arbitrarily from \mathcal{D}_{B_r} we conclude that $\mathcal{D}_{B_r} \subset \mathcal{D}_{A_r}$.

THEOREM 3.4: *Suppose $\{\mathcal{U}_1(t)|t \geq 0\}$ and $\{\mathcal{U}_2(t)|t \geq 0\}$ are two uniformly bounded, strongly continuous semi-groups on \mathcal{H} with bounds K_1 and K_2 , respectively. Then $\{\mathcal{U}(t)|t \geq 0\}$, defined on \mathcal{I}_α by*

$$\mathcal{U}(t)S = \mathcal{U}_1(t)S\mathcal{U}_2(t), \quad t \geq 0, \quad S \in \mathcal{I}_\alpha,$$

constitutes a uniformly bounded, strongly continuous semi-group of transformers on \mathcal{I}_α and its bound does not exceed $K_1 K_2$. Denoting the infinitesimal generators of $\mathcal{U}_1(t)$, $\mathcal{U}_2(t)$ and $\mathcal{U}(t)$ by A_1 , A_2 and A , respectively, then, for $S \in \mathcal{D}_{A_1} \cap \mathcal{D}_{A_2}$ we have

$$S \in \mathcal{D}_A \quad \text{and} \quad AS = (A_1 S + SA_2)^{**} = A_{11} S + A_{2r} S.$$

PROOF: The semi-group properties of $\mathcal{U}(t)$ are evident, the continuity being clear from the strong continuity of $\mathcal{U}_1(t)$ and $\mathcal{U}_2^*(t)$. In addition Lemma 3.1 shows us that $|\mathcal{U}(t)|_\alpha = |\mathcal{U}_1(t)||\mathcal{U}_2(t)| \leq K_1 K_2$.

Suppose that $S \in \mathcal{D}_{A_{1l}} \cap \mathcal{D}_{A_{2r}}$. Then $(A_1 S + SA_2)^{**} \in \mathcal{I}_\alpha$ and

$$|t^{-1}(\mathcal{U}(t) - 1)S - (A_1 S + SA_2)^{**}|_\alpha \leq K_2 |t^{-1}(\mathcal{U}_{1l}(t) - 1)S - A_{1l}S|_\alpha \\ + |(\mathcal{U}_{2r}(t) - 1)(A_1 S)^{**}|_\alpha + |t^{-1}(\mathcal{U}_{2r}(t) - 1)S - A_{2r}S|_\alpha,$$

tends to zero for $t \rightarrow +0$. We conclude that

$$S \in \mathcal{D}_A \quad \text{and} \quad AS = (A_1 S + SA_2)^{**} = A_{1l}S + A_{2r}S. \quad \blacksquare$$

REMARK: $\mathcal{U}(t)$ can obviously be written as the product of the two commuting transformers $\mathcal{U}_{1l}(t)$ and $\mathcal{U}_{2r}(t)$. For the special case that $A_2 = A_1$ we have $\mathcal{U}_{2r}(t) = \mathcal{U}_{1l}^c(t)$, i.e. $\mathcal{U}(t)$ is self-conjugate.

COROLLARY 3.2: *Suppose that $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$ in Theorem 3.4 are two strongly continuous groups of unitary operators on \mathcal{H} (i.e., according to Stone's theorem, their infinitesimal generators can be written as $A_1 = iH_1$, $A_2 = iH_2$, where H_1 and H_2 are self-adjoint on \mathcal{H}). Then $\{\mathcal{U}(t) | t \in \mathbb{R}\}$ is a strongly continuous group of isometric transformers on \mathcal{I}_α .*

4. Semi-groups of rank-preserving transformers

DEFINITION 4.1: A bounded transformer A on \mathcal{I}_α is *rank-preserving* if it maps every finite-rank operator $S \in \mathcal{I}_\alpha$ into an operator AS of rank equal to the rank of S .

We note that a bounded transformer on \mathcal{I}_α is rank-preserving if it maps the set \mathcal{R}_1 of rank-one operators into itself and if it is invertible. If $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$ are bounded and invertible operators on \mathcal{H} then $\mathcal{U}(t)S = \mathcal{U}_1(t)S\mathcal{U}_2^*(t)$ is clearly a rank-preserving transformer. Moreover, if $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$ are uniformly bounded strongly continuous semigroups in $t \geq 0$ then we have seen in Section 3 that $\mathcal{U}(t)$ is strongly continuous in α -norm on \mathcal{I}_α . This section is devoted to the proof of the following theorem, which represents to a certain extent a converse of the above statement.

THEOREM 4.1: *Suppose that $\mathcal{U}(t)$, $t \geq 0$, is an α -continuous semigroup of rank-preserving transformers on a minimal norm ideal \mathcal{I}_α on the C^* -algebra $\mathcal{B}(\mathcal{H})$ of a Hilbert space \mathcal{H} of dimension greater than two. Then there are strongly continuous semigroups $\mathcal{U}_1(t)$ and $\mathcal{U}_r(t)$ of operators on \mathcal{H} such that $\mathcal{U}(t)S = \mathcal{U}_1(t)S\mathcal{U}_r(t)$, for all $S \in \mathcal{I}_\alpha$ and $t \geq 0$.*

We proceed now with the proof, in the course of which we derive several auxilliary lemmas.

First of all we note that since $\mathcal{U}(t)$ maps a rank-one operator $(\cdot, y)x$ into an operator which is also of rank one, we can write

$$\mathcal{U}(t)(\cdot, y)x = (\cdot, y(t))x(t).$$

Of course, $x(t)$ and $y(t)$ are not uniquely determined by x and y , since

$$(1) \quad (\cdot, \overline{\gamma(t)}y(t))\gamma(t)^{-1}x(t) = (\cdot, y(t))x(t)$$

for any nonzero $\gamma(t) \in \mathbb{C}$.

LEMMA 4.1: *For any two normalized vectors $x_0, y_0 \in \mathcal{H}$ there are vector-valued functions $x_0(t)$ and $y_0(t)$ defined for all $t \in [0, \tau_0(x_0, y_0)] = I_0$, $\tau_0(x_0, y_0) > 0$, such that $(y_0(t), y_0)$ is nonzero and continuous in $t \in I_0$, $|y_0(t)| \equiv 1$, $t \in I_0$, and $\mathcal{U}(t)(\cdot, y_0)x_0 = (\cdot, y_0(t))x_0(t)$ for all $t \in I_0$, where $x_0 = x_0(0)$, $y_0 = y_0(0)$.*

PROOF: Since $\mathcal{U}(0) = \mathbf{1}$ and $\mathcal{U}(t)$ is α -continuous in a right neighbourhood of $t = 0$, there is a number $\tau_0 > 0$ and there are vector-valued functions $x_1(t)$ and $y_1(t)$ such that $\mathcal{U}(t)(\cdot, y_0)x_0 = (\cdot, y_1(t))x_1(t) \neq 0$,

$$h(t) = \text{tr}\{[\mathcal{U}(t)(\cdot, y_0)x_0](\cdot, x_0)y_0 = (y_0, y_1(t))(x_1(t), x_0) \neq 0,$$

for all $t \in [0, \tau_0] = I_0$. This last relation follows from the fact that the functional $\phi(R) = \text{tr}\{R(\cdot, y_0)x_0\}$, $R \in I_\alpha$, is an element of \mathcal{S}_α^* (cf. ref. [1]) and therefore it is α -continuous in $R \in \mathcal{S}_\alpha$. Since, $\phi[\mathcal{U}(0)(\cdot, y_0)x_0] = 1$, the continuity of this functional implies that $\phi[\mathcal{U}(t)(\cdot, y_0)x_0] \neq 0$ for sufficiently small values of $t \geq 0$. By setting $\gamma(t) = (x_1(t), x_0)$ and

$$x_2(t) = \gamma(t)^{-1}x_1(t), \quad y_2(t) = \overline{\gamma(t)}y_1(t),$$

we get by using (1)

$$\mathcal{U}(t)(\cdot, y_0)x_0 = (\cdot, y_2(t))x_2(t), \quad h(t) = (y_2(t), y_0).$$

Hence $(y_2(t), y_0)$ is continuous in I_0 . Since $\gamma(0) = 1$, we have $y_2(0) = y_0$. Furthermore, the α -continuity of $\mathcal{U}^c(t)(\cdot, y_0)x_0$ implies that

$$|y_2(t)|^2 = \text{tr}\{[\mathcal{U}^c(t)(\cdot, y_0)x_0](\cdot, x_0)x_0[\mathcal{U}(t)(\cdot, y_0)x_0]\}$$

is continuous in $t \in I_0$. By virtue of the fact that $y_2(t) \neq 0$ for all $t \in I_0$, since $(y_2(t), y_0) \neq 0$ for all such t , we conclude that $y_0(t) = |y_2(t)|^{-1}y_2(t)$, $x_0(t) = |y_2(t)|x_2(t)$, are the vector-valued functions satisfying all the conditions of the lemma. ■

LEMMA 4.2: *Let W be a rank-preserving transformer and let x_0, y_0, x'_0, y'_0 be nonzero vectors in \mathcal{H} such that $W(\cdot, y_0)x_0 = (\cdot, y'_0)x'_0$. Then either there is a bounded linear operator W'_l on \mathcal{H} such that $W(\cdot, y_0)x = (\cdot, y'_0)W'_l x$, or there is a bounded semi-linear operator W'_r such that $W(\cdot, y_0)x = (\cdot, W'_r x)x'_0$ for all $x \in \mathcal{H}$. Similarly, either there is a bounded linear operator W''_l such that $W(\cdot, y)x_0 = (\cdot, W''_l y)x'_0$ or there is a bounded semi-linear operator W''_r such that $W(\cdot, y)x_0 = (\cdot, y'_0)W''_l y$ for all $y \in \mathcal{H}$.*

PROOF: We shall prove only the first part of the lemma. The proof of the second part is analogous.

Let $x_0, x_1, x_2 \in \mathcal{H}$ be linearly independent, and let us set

$$W(\cdot, y_0)x_k = (\cdot, y_0^{(k)})x_k^{(k)}, \quad k = 1, 2.$$

Since W is linear and rank-preserving we can write

$$\begin{aligned} (\cdot, y'_0)x'_0 + \alpha_1(\cdot, y_0^{(1)})x_1^{(1)} + \alpha_2(\cdot, y_0^{(2)})x_1^{(2)} \\ = W(\cdot, y_0)(x_0 + \alpha_1 x_1 + \alpha_2 x_2) = (\cdot, y'_0(\alpha_1, \alpha_2))(x_0 + \alpha_1 x_1 + \alpha_2 x_2) \end{aligned}$$

for some vectors $(x_0 + \alpha_1 x_1 + \alpha_2 x_2)'$, $y'_0(\alpha_1, \alpha_2)$ in \mathcal{H} which depend on $\alpha_1, \alpha_2 \in \mathbb{C}$. The left-hand side of the above equation can be equal to the right-hand side if and only if either $y'_0 = \omega_1 y_0^{(1)} = \omega_2 y_0^{(2)}$ or if $x'_0 = \zeta_1 x_1^{(1)} = \zeta_2 x_2^{(2)}$, where $\omega_1, \omega_2, \zeta_1, \zeta_2$ are nonzero complex numbers since W is strictly rank-preserving. In the first case we have

$$W(\cdot, y_0)x_k = (\cdot, y'_0)x'_k, \quad x'_k = \overline{\omega_k} x_k^{(k)}, \quad k = 1, 2,$$

and then obviously

$$W(\cdot, y_0)(x_0 + \alpha_1 x_1 + \alpha_2 x_2) = (\cdot, y'_0)(x'_0 + \alpha_1 x'_1 + \alpha_2 x'_2)$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$. Thus $x_1 \rightarrow x'_1 = W'_l x_1$ determines a linear operator when x_1 varies over all vectors linearly independent of x_0 . By setting $W'_l x_0 = x'_0$, we have completed the construction of W'_l . Its boundedness follows from the fact that $|W'_l x| = |y_0|^{-1} |W(\cdot, y_0)x|_\alpha$.

In the second case $W(\cdot, y_0)x_k = (\cdot, y'_k)x'_0$, $y'_k = \overline{\zeta_k}^{-1} y_0$, $k = 1, 2$ and

clearly $W(\cdot, y_0)(x_0 + \alpha_1 x_1 + \alpha_2 x_2) = (\cdot, y'_0 + \overline{\alpha_1} y'_1 + \overline{\alpha_2} y'_2) x'_0$. Hence the construction of a semi-linear W'_r is obtained in the same manner by setting $y'_1 = W'_r x_1$, and the proof of the lemma is complete. ■

In view of the assumption that $\mathcal{U}(t)$ is rank-preserving in some right neighbourhood $[0, \tau_1]$ of zero, we infer by using the preceding lemma that for each such value of t there is either a linear $\mathcal{U}'_l(t)$ or an antilinear $\mathcal{U}'_r(t)$ as defined in that lemma.

LEMMA 4.3: For given normalized $x_0, y_0 \in \mathcal{H}$ there is a right neighbourhood I'_0 of $t = 0$ and a family $\{\mathcal{U}'_l(t) | t \in I'_0\}$ of bounded linear operators which is strongly continuous in $t \in I'_0$ and such that

$$(2) \quad \mathcal{U}(t)(\cdot, y_0)x = (\cdot, y_0(t))\mathcal{U}'_l(t)x$$

for all $x \in \mathcal{H}$ where $y_0(t)$ has been chosen as in Lemma 4.1.

PROOF: We must first eliminate the possibility that $\mathcal{U}(t)$ and $x_0, y_0 \in \mathcal{H}$ determine a semi-linear $\mathcal{U}'_r(t)$ rather than linear $\mathcal{U}'_l(t)$ for values of t in arbitrarily small neighbourhoods of zero. Hence assume that there is a sequence $t_1, t_2, \dots \rightarrow +0$ such that $\mathcal{U}(t_n)(\cdot, y_0)x = (\cdot, \mathcal{U}'_r(t_n)x)x_0(t_n)$. Then the expression

$$tr\{[\mathcal{U}(t_n)(\cdot, y_0)x]z(\cdot, x_0)\} = (z, \mathcal{U}'_r(t_n)x)(x_0(t_n), x_0)$$

converges to $(z, y_0)(x, x_0)$ due to the α -continuity of $\mathcal{U}(t)$. Setting above $x = x_0$ and $z = y_0$ we conclude that

$$(y_0, y_0(t_n))(x_0(t_n), x_0) \rightarrow tr\{[\mathcal{U}(0)(\cdot, y_0)x_0]y_0(\cdot, x_0)\} = 1.$$

Since by Lemma 4.1 $(y_0, y_0(t_n)) \rightarrow 1$, we infer that $(x_0(t_n), x_0) \rightarrow 1$ and therefore $\lim_{n \rightarrow \infty} (z, \mathcal{U}'_r(t_n)x) = (z, y_0)(x, x_0)$. This would imply that

$$|\{\mathcal{U}(t_n)(\cdot, y_0)x\}z| = |(z, \mathcal{U}'_r(t_n)x||x_0(t_n)|) \rightarrow |(z, y_0)(x, x_0)|$$

when, in fact, $\mathcal{U}(t_n)$ converges in α -norm to the identity transformer in \mathcal{I}_α , and therefore

$$\lim_{n \rightarrow \infty} |\{\mathcal{U}(t_n)(\cdot, y_0)x\}z| = |\{(\cdot, y_0)x\}z| = |(z, y_0)||x|.$$

Thus, we have established the existence of a right-neighbourhood I'_0 of $t = 0$ in which (2) holds.

The strong continuity of $\mathcal{U}'_i(t)$ follows from the inequalities

$$\begin{aligned} |\mathcal{U}'_i(t')x - \mathcal{U}'_i(t)x| &= |(\cdot, y_0(t'))(\mathcal{U}'_i(t')x - \mathcal{U}'_i(t)x)|_\alpha \\ &\leq |(\mathcal{U}(t') - \mathcal{U}(t))(\cdot, y_0)x|_\alpha + |(\cdot, y_0(t') - y_0(t))\mathcal{U}'_i(t)x|_\alpha \end{aligned}$$

which are the consequence of elementary properties of the α -norm and of $|y_0(t)| = 1$ (cf. Lemma 4.1). Since $\mathcal{U}(t)$ is α -continuous and

$$\lim_{t' \rightarrow t} |(\cdot, y_0(t') - y_0(t))\mathcal{U}'_i(t)x|_\alpha = |\mathcal{U}'_i(t)x| \lim_{t' \rightarrow t} |y_0(t') - y_0(t)|$$

this continuity of $\mathcal{U}'_i(t)$ is obtained by noting that $y_0(t)$ is strongly continuous since $|y_0(t)| \equiv 1$ and

$$(z, y_0(t)) = (x_0(t), x_0)^{-1} \text{tr}\{[\mathcal{U}(t)(\cdot, y_0)x_0]z(\cdot, x_0)\}$$

is continuous for any $z \in \mathcal{H}$.

Now, having established the existence of a strongly continuous $\mathcal{U}'_i(t)$ satisfying (2), we proceed by proving the existence of a continuous non-zero complex function $\omega_0(t)$ defined for t in a right neighbourhood I''_0 of zero, for which $\mathcal{U}_i(t) = \omega_0(t)^{-1}\mathcal{U}'_i(t)$, $t \in I''_0$ is not only strongly continuous but also additive:

$$\mathcal{U}_i(t_1 + t_2) = \mathcal{U}_i(t_1)\mathcal{U}_i(t_2), \quad t_1, t_2, t_1 + t_2 \in I''_0.$$

First of all, we note that

$$\begin{aligned} (3) \quad \mathcal{U}_i(t_1 + t_2)(\cdot, y_0)x &= (\cdot, y_0(t_1 + t_2))\mathcal{U}'_i(t_1 + t_2)x \\ &= \mathcal{U}(t_1)\mathcal{U}(t_2)(\cdot, y_0)x = \mathcal{U}(t_1)(\cdot, y_0(t_2))\mathcal{U}'_i(t_2)x. \end{aligned}$$

On the other hand

$$(4) \quad \mathcal{U}(t_1)(\cdot, y_0)\mathcal{U}'_i(t_2)x = (\cdot, y_0(t_1))\mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2)x.$$

Applying the second part of Lemma 4.2 to $\mathcal{U}'_i(t_2)x$, $y_0(t_2) \in \mathcal{H}$ (instead of x_0 , $y_0 \in \mathcal{H}$), we infer by using (4) and setting

$$\mathcal{U}(t_1)(\cdot, y_0(t_2))\mathcal{U}'_i(t_2)x = (\cdot, y_0(t_2))(\mathcal{U}'_i(t_2)x)'$$

that either we can choose $y_0(t_2)' = y_0(t_1 + t_2)$, and then $\mathcal{U}''_i(t_1, t_2, x)$ defined by

$$\mathcal{U}(t_1)(\cdot, y)\mathcal{U}'_i(t_2)x = (\cdot, y_0(t_1 + t_2))\mathcal{U}''_i(t_1, t_2; x)y$$

is an semi-linear operator, or $[\mathcal{U}_i(t_2)x]' = \mathcal{U}'_i(t_1+t_2)x$, in which case $\mathcal{U}'_r(t_1, t_2; x)$ defined by

$$(5) \quad \mathcal{U}(t_1)(\cdot, y)\mathcal{U}'_i(t_2)x = (\cdot, \mathcal{U}'_r(t_1, t_2; x)y)\mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2)x$$

is a linear operator. The first case can be eliminated by a reasoning similar to the one used for $\mathcal{U}'_r(t)$ in the proof of Lemma 4.3. Thus (5) must hold true. Setting in (5) $y = y_0(t_2)$ and comparing with (3), we conclude that for some nonzero $\omega(t_1, t_2; x) \in \mathbb{C}$

$$\begin{aligned} \mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2)x &= \omega(t_1, t_2; x)\mathcal{U}'_i(t_1+t_2)x, \\ y_0(t_1+t_2) &= \omega(t_1, t_2; x)\mathcal{U}'_r(t_1, t_2)y_0(t_2). \end{aligned}$$

Since $\mathcal{U}(t)$ is rank-preserving for $t \in I'_0$, it follows that $\mathcal{U}'_i(t)$ is invertible, and therefore

$$\omega(t_1, t_2; x)x = \mathcal{U}'_i(t_1+t_2)^{-1}\mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2)x.$$

Hence $\omega(t_1, t_2; x)$ does not depend on x :

$$(6) \quad \mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2) = \omega(t_1, t_2)\mathcal{U}'_i(t_1+t_2). \quad \blacksquare$$

LEMMA 4.4: *The complex-valued function $\omega(t_1, t_2)$ defined by (6) for $t_1, t_2, t_1+t_2 \in I'_0$ is continuous, nonzero, and such that $(0, 0) = 1$ and*

$$(7) \quad \omega(t_1, t_2)\omega(t_1+t_2, t_3) = \omega(t_1, t_2+t_3)\omega(t_2, t_3).$$

PROOF: The continuity of $\omega(t_1, t_2)$ follows from the strong continuity of $\mathcal{U}'_i(t)$ by noting that in $(\mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2)x_0, x_0) = \omega(t_1, t_2)(\mathcal{U}'_i(t_1+t_2)x_0, x_0)$ both sides of the relation are nonzero. Furthermore, $\mathcal{U}'_i(0) = 1$ and therefore we get $\omega(0, 0) = 1$ by inserting in (6) $t_1 = t_2 = 0$. Finally, (7) follows from

$$\begin{aligned} \omega(t_2, t_3)\omega(t_1, t_2+t_3)\mathcal{U}'_i(t_1+(t_2+t_3)) &= \mathcal{U}'_i(t_1)\mathcal{U}'_i(t_2)\mathcal{U}'_i(t_3) \\ &= \omega(t_1, t_2)\omega(t_1+t_2, t_3)\mathcal{U}'_i((t_1+t_2)+t_3). \quad \blacksquare \end{aligned}$$

The above lemma tells us that the complex function $\omega(t_1, t_2)$ has all the properties of a local factor (cf. ref. 8, p. 8) except one, namely the property $|\omega(t_1, t_2)| \equiv 1$ might be lacking. However, since $\omega(t_1, t_2)$ does not vanish, there are uniquely determined real-valued functions $\rho(t_1, t_2)$ and $\psi(t_1, t_2)$, $0 \leq \psi(t_1, t_2) < 2\pi$, such that

$$\omega(t_1, t_2) = \exp \{ \rho(t_1, t_2) + i\psi(t_1, t_2) \}.$$

We deduce from Lemma 4.4 that $\rho(0, 0) = \psi(0, 0) = 0$,

$$\begin{aligned}\rho(t_1, t_2) + \rho(t_1 + t_2, t_3) &= \rho(t_1, t_2 + t_3) + \rho(t_2, t_3), \\ \psi(t_1, t_2) + \psi(t_1 + t_2, t_3) &= \psi(t_1, t_2 + t_3) + \psi(t_2, t_3),\end{aligned}$$

and that $\rho(t_1, t_2)$ and $\psi(t_1, t_2)$ are continuous for $t_1, t_2 \in I''_0$, where $I''_0 \subset I'_0$ is an adequately chosen right neighbourhood of $t = 0$. Thus ρ and ψ are local exponents (cf. ref. 8, p. 20) and there are continuous functions $\rho_0(t)$ and $\psi_0(t)$, $t \in I''_0$ (see ref. 8, pp. 19–21; the statement of this result is presented in this reference for the case when ρ and ψ are defined not only for right but also for left neighbourhoods of $t_1 = 0$ and $t_2 = 0$, but its proof does not require this assumption) such that

$$\rho(t_1, t_2) = \rho_0(t_1) + \rho_0(t_2), \quad \psi(t_1, t_2) = \psi_0(t_1) + \psi_0(t_2).$$

Hence we can write $\omega(t_1, t_2) = \omega_0(t_1)\omega_0(t_2)$, $\omega_0(t) = \exp [\rho_0(t) + i\psi_0(t)]$, and therefore the relation $\mathcal{U}_l(t) = \omega_0(t)^{-1}\mathcal{U}'_l(t)$, $t \in I''_0$ is established.

Let us employ again Lemma 4.2 in defining a family of bounded linear operators $\mathcal{U}_r(t)$ by

$$(8) \quad \mathcal{U}(t)(\cdot, y)x_0 = (\cdot, \mathcal{U}_r(t)y)x_0(t).$$

Using an appropriate version of Lemma 4.3 to eliminate the possibility of semi-linear operators occurring in arbitrarily small right-neighbourhoods of $t = 0$, we infer that (8) must hold at least in some right-neighbourhood I_1 of $t = 0$. In order to show now that

$$(9) \quad \mathcal{U}(t)(\cdot, y)x = (\cdot, \mathcal{U}_r(t)y)\mathcal{U}_l(t)x, \quad x, y \in \mathcal{H},$$

at least for t in some neighbourhood $I'_1 \subset I_1 \cap I''_0$, we define $V(x; t)$ by $(\cdot, V(x, t)y)\mathcal{U}_l(t)x = \mathcal{U}(t)(\cdot, y)x$, where obviously $V(x_0, t) = \mathcal{U}_r(t)$, and the possibility of $V(x, t)$ being anti-linear is eliminated as before by setting above $y = y_0$.

Choose x orthogonal to x_0 , and note that the linearity of $\mathcal{U}(t)$ implies:

$$(10) \quad \begin{aligned}\mathcal{U}(t)(\cdot, y)(\alpha x_0 + x) &= (\cdot, V(\alpha x_0 + x, t)y)\mathcal{U}_l(t)(\alpha x_0 + x) \\ &= \alpha(\cdot, V(x_0, t)y)\mathcal{U}_l(t)x_0 + (\cdot, V(x, t)y)\mathcal{U}_l(t)x.\end{aligned}$$

Since $\mathcal{U}(t)$ is rank-preserving, the operator $\mathcal{U}_l(t)$ is invertible and consequently $\mathcal{U}_l(t)x_0$ is linearly independent of x . Hence (10) can be true if and only if $V(\alpha x_0 + x, t) = V(x, t) = V(x_0, t) = \mathcal{U}_r(t)$. This establishes that (9) holds for all $t \in [0, \tau]$, where $\tau > 0$ is adequately chosen. Hence

$\mathcal{U}(t)R = \mathcal{U}_l(t)R\mathcal{U}_r^*(t)$, $0 \leq t \leq \tau$ for all finite $R \in \mathcal{R}$, and since \mathcal{R} is dense in \mathcal{S}_α , and $|\mathcal{U}(t)|_\alpha$, $|\mathcal{U}_l(t)|$ and $|\mathcal{U}_r(t)|$ are finite, this relation holds for all $R \in \mathcal{S}_\alpha$. Furthermore, $\mathcal{U}_l(t)$ and $\mathcal{U}_r(t)$ can be extended to semigroups defined for all $t \geq 0$ by setting $\mathcal{U}_l(t) = \mathcal{U}_l(t')\mathcal{U}_l(\tau)^n$, $\mathcal{U}_r(t) = \mathcal{U}_r(t')\mathcal{U}_r(\tau)^n$, $t = t' + n\tau$, $0 \leq t' < \tau$, $n = 1, 2, \dots$. Thus we are dealing with two one-parameter strongly continuous semigroups of transformers which coincide on some right neighbourhood of zero. Hence, they have to coincide everywhere. This establishes that Theorem 4.1 is true.

REMARK: It is interesting to note that Theorem 4.1 generalizes some well-known results (cf. ref. 8) on ray representations of one-parameter Lie groups. In fact, it is easy to see that the following definition of a ray symmetry mapping is equivalent to the conventional one (cf. ref. 9).

DEFINITION 4.2: A ray symmetry mapping \hat{T} in \mathcal{H} is a bijective mapping of \mathbb{R}_1 onto itself, which maps hermitian projectors $S \in \mathbb{R}_1$ into hermitian projectors $\hat{T}S$, and is such that $tr\{(\hat{T}S_1)^*(\hat{T}S_2)\} = tr\{S_1^*S_2\}$ for any two orthogonal projectors $S_1, S_2 \in \mathbb{R}_1$.

Wigner's well-known theorem (cf. ref. 9) implies that every ray symmetry mapping \hat{T} is a transformer of \mathcal{R} onto itself which preserves the Schmidt inner product $(\cdot, \cdot)_2 : (S_1, S_2)_2 = tr\{S_1S_2^*\}$. On the other hand, this theorem does not preclude the possibility that the transformer \hat{T} is semilinear rather than linear. However, in either case \hat{T} can be extended to, respectively, a linear or semilinear transformer T on the Schmidt class $\mathcal{B}_2(\mathcal{H})$. In case of a continuous semi-group of ray symmetry mappings $\hat{T}(t)$ with $\hat{T}(0) = I$ the possibility of semilinearity can be eliminated by using the type of argument employed in the course of proving Lemma 4.3. Furthermore, in this case we easily deduce that $\mathcal{U}_l(t) \equiv \mathcal{U}_r(t)$ from the requirement that $\hat{T}(t)$ maps the set of self-adjoint one-dimensional projectors into itself.

5. Spectral transformers induced by spectral operators

In this section we consider left and right multiplication transformers, which are derived from spectral operators [10]. The central question is whether such transformers and their linear combinations are again spectral. We recall that a homomorphic map E from a Boolean algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ of subsets of a set \mathcal{S} into a Boolean algebra of commuting projections in the complex Banach space \mathcal{X} is called a spectral measure in \mathcal{X} , provided that it is bounded, that $E(\mathcal{S}) = I$, the identity on \mathcal{X} , and countably additive, i.e. E has the properties $(\gamma, \delta \in \mathcal{A})$: (1) $E(\gamma \cap \delta) =$

$E(\gamma) \wedge E(\delta) = E(\gamma)E(\delta)$; (2) $E(\gamma \cup \delta) = E(\gamma) \vee E(\delta) = E(\gamma) + E(\delta) - E(\gamma)E(\delta)$,
 (3) $E(\gamma') = I - E(\gamma)$, $E(\phi) = 0$, $E(\mathcal{S}) = I$, (4) $|E(\gamma)| \leq K$ and (5)

$$\sum_{i=1}^{\infty} x^* E(\gamma_i) x = x^* E(\gamma) x$$

for any family of mutually disjoint γ_i of \mathcal{S} with union γ and for any $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. A closed densely defined operator A on \mathcal{X} is said to be a spectral operator if there exists a spectral measure E on the family $\mathcal{A}(\mathbb{C})$ of Borel sets in the complex plane \mathbb{C} , such that (6) $\mathcal{D}_A \supset E(\gamma)\mathcal{X}$ for bounded $\gamma \in \mathcal{A}(\mathbb{C})$, (7) $E(\gamma)\mathcal{D}_A \subset \mathcal{D}_A$ and $AE(\gamma)x = E(\gamma)Ax$, $x \in \mathcal{D}_A$, $\gamma \in \mathcal{A}(\mathbb{C})$, (8) $\sigma(A, E(\gamma)\mathcal{X}) \subset \tilde{\gamma}$ where $\gamma \in \mathcal{A}(\mathbb{C})$ and $\sigma(A, E(\gamma)\mathcal{X})$ is the spectrum of the restriction of A to the subspace $E(\gamma)\mathcal{X}$, while $\tilde{\gamma}$ denotes the closure of the set γ . In this case $E(\gamma)$ is said to be the resolution of the identity for A . The scalar part, A_s , of A is the closed operator, defined by the Riemann-Stieltjes integral (improper for unbounded A)

$$A_s x = \int A E(d\lambda) x$$

with domain consisting of those $x \in \mathcal{X}$ for which the above integral exists. A is said to be of scalar type (briefly; scalar) if $A = A_s$. We will only be concerned with scalar operators and transformers.

LEMMA 5.1: $\{E(\gamma)|\gamma \in \mathcal{A}\}$ is a spectral measure in \mathcal{H} if and only if this is the case for $\{E^*(\gamma)|\gamma \in \mathcal{A}\}$. If $\{E(\gamma)|\gamma \in \mathcal{A}(\mathbb{C})\}$ is a resolution of the identity for the scalar operator A on \mathcal{H} , then A^* is a scalar operator and its resolution of the identity is $\{F(\gamma)|\gamma \in \mathcal{A}(\mathbb{C})\}$ where $F(\gamma) = E^*(\delta)$ with $\gamma = \{\bar{a}|a \in \delta\}$.

PROOF: (a) If $E(\gamma)$ is a spectral measure in \mathcal{H} then the properties (1) to (5) evidently hold for $E^*(\gamma)$ and vice versa.

(b) Suppose A is scalar on \mathcal{H} with resolution of the identity $E(\gamma)$, i.e. $Ax = \int \lambda E(d\lambda)x$, $x \in \mathcal{D}_A$. Let $x \in \mathcal{D}_A$ and $y \in \mathcal{D}_{A^*}$. Then we have $AE(\gamma)x = E(\gamma)Ax$ and

$$(x, E^*(\gamma)A^*y) = (AE(\gamma)x, y) = (E(\gamma)Ax, y) = (Ax, E^*(\gamma)y).$$

i.e. $E^*(\gamma)y \in \mathcal{D}_{A^*}$ and, since \mathcal{D}_A is dense in \mathcal{H} , $A^*E^*(\gamma)y = E^*(\gamma)A^*y$. Let $\gamma \in \mathcal{A}(\mathbb{C})$ be bounded and let $y \in \mathcal{D}_{A^*}$. Then, according to what has just been proven, $E^*(\gamma)y \in \mathcal{D}_{A^*}$ and for each $x \in \mathcal{D}_A$:

$$(x, A^*E^*(\gamma)y) = (x, E^*(\gamma)A^*y) = (E(\gamma)Ax, y) = (AE(\gamma)x, y).$$

However, since γ is bounded, $E(\gamma)x \in \mathcal{D}_A$ for every $x \in \mathcal{H}$ and it follows that $AE(\gamma)$ is closed and everywhere defined and hence bounded. Consequently, the closed, densely defined operator $A^*E^*(\gamma)$ is bounded since it is contained in the bounded operator $(AE(\gamma))^*$. Thus $E^*(\gamma)\mathcal{H} \subset \mathcal{D}_{A^*}$. In addition it is not difficult to verify that $\sigma(A^*, E^*(\gamma)\mathcal{H}) \subset \bar{\delta}$ with $\delta = \{a \in \mathbb{C} | \bar{a} \in \gamma\}$ and we conclude that A^* is spectral with resolution of the identity $F(\gamma)$ as defined in the lemma. Let A_s^* be its scalar part. Then, for $x \in \mathcal{D}_A$ and $y \in \mathcal{D}_{A^*}$:

$$\begin{aligned} (x, A^*y) &= (Ax, y) = \lim_{\rho \rightarrow \infty} \left(\int_{|\lambda| \leq \rho} \lambda E(d\lambda)x, y \right) \\ &= \lim_{\rho \rightarrow \infty} \left(x, \int_{|\lambda| \leq \rho} \bar{\lambda} E^*(d\lambda)y \right) = (x, A_s^*y), \quad \text{i.e. } A^* = A_s^*. \quad \blacksquare \end{aligned}$$

LEMMA 5.2: Suppose $\{E_p(\gamma) | \gamma \in \mathcal{A}\}$, $p = 1, 2$ are two spectral measures on \mathcal{A} in \mathcal{H} . Then $\{E_{pl}(\gamma) | \gamma \in \mathcal{A}\}$ and $\{E_{qr}(\gamma) | \gamma \in \mathcal{A}\}$, $p, q = 1, 2$, are two commuting spectral measures on \mathcal{A} in \mathcal{I}_α . If $E_p(\gamma)$ is strongly countably additive on \mathcal{H} , then so are $E_{pl}(\gamma)$ and $E_{pr}(\gamma)$ on \mathcal{I}_α .

PROOF: (a) For $E_{pl}(\gamma)$ and $E_{pr}(\gamma)$ the properties (1) to (3) and the commutation property are clear, whereas property (4) follows after the application of lemma 3.1. Since $E_p(\gamma)$ is uniformly bounded on \mathcal{H} it follows (cf. [9] p. 34) that $E_p(\gamma)$ is countably additive on $\mathcal{B}(\mathcal{H})$ equipped with the ultra-weak operator topology and hence (cf. ref. 11, p. 35): $\sum_{i=1}^\infty tr E_p(\gamma_i)R = tr E(\gamma)R$, $R \in \mathcal{B}_1(\mathcal{H})$ where the γ_i are mutually disjoint sets of $\mathcal{A}(\mathbb{C})$ with union γ and $\mathcal{B}_1(\mathcal{H})$ denotes the trace-class of operators on \mathcal{H} . Since any continuous linear functional Φ on \mathcal{I}_α has the form $\Phi(S) = tr ST$, where $T \in \mathcal{I}_\alpha^*$, and ST and TS are contained in $\mathcal{B}_1(\mathcal{H})$ it follows that:

$$\sum_{i=1}^\infty tr(E_{pl}(\gamma_i)S)T = \sum_{i=1}^\infty tr E_p(\gamma_i)ST = tr E(\gamma)ST = tr(E_{pl}(\gamma)S)T.$$

Since a similar argument can be applied to $E_{pr}(\gamma)$ we conclude that $E_{pl}(\gamma)$ and $E_{pr}(\gamma)$ are countably additive on \mathcal{I}_α .

(b) Suppose next that $E_p(\gamma)$ is strongly countably additive on \mathcal{H} , i.e. $\sum_{i=1}^\infty E_p(\gamma_i)x = E(\gamma)x$. A direct application of Lemma 3.1 then shows that $E_{pl}(\gamma)$ is strongly countably additive on \mathcal{I}_α , whereas the same result for $E_{pr}(\gamma)$ is obtained by first considering $E_{pl}^*(\gamma)$ and the application of Lemma 5.1. Since $E_{pr}(\gamma) = (E_{pl}^*(\gamma))^c$ and the conjugation is α -continuous, the result for $E_{pr}(\gamma)$ follows. \blacksquare

THEOREM 5.1: *Suppose A is a spectral operator of scalar type on \mathcal{H} with resolution of the identity $\{E(\gamma)|\gamma \in \mathcal{A}(\mathbb{C})\}$. Then A_l and A_r are spectral transformers of scalar type on \mathcal{I}_α with resolutions of the identity $\{E_l(\gamma)|\gamma \in \mathcal{A}(\mathbb{C})\}$ and $\{E_r(\gamma)|\gamma \in \mathcal{A}(\mathbb{C})\}$, respectively.*

PROOF: (a) Suppose that γ is bounded. Then (see part (b) of the proof of Lemma 5.1. $AE(\gamma)$ is bounded and it follows that $AE(\gamma)S \in \mathcal{I}_\alpha$ for $S \in \mathcal{I}_\alpha$, i.e. $E_l(\gamma)\mathcal{I}_\alpha \in \mathcal{D}_{A_l}$ for bounded γ .

(b) Suppose $S \in \mathcal{D}_{A_l}$. Then $Sx \in \mathcal{D}_A$ for each $x \in \mathcal{H}$. Thus $E(\gamma)Sx \in \mathcal{D}_A$ and $AE(\gamma)Sx = E(\gamma)ASx$, or since x may be chosen arbitrarily from \mathcal{H} , $AE(\gamma)S = E(\gamma)AS \in \mathcal{I}_\alpha$. It follows that for $S \in \mathcal{D}_{A_l}$, we have $E_l(\gamma)S \in \mathcal{D}_{A_l}$ and $A_l E_l(\gamma)S = E_l(\gamma)A_l S$.

(c) Define $\mathcal{H}(\gamma) = E(\gamma)\mathcal{H}$ and $\mathcal{I}_\alpha(\gamma) = E_l(\gamma)\mathcal{I}_\alpha$. Let $A(\gamma)$ be the restriction of A to $\mathcal{H}(\gamma)$ and $A_l(\gamma)$ the restriction of A_l to $\mathcal{I}_\alpha(\gamma)$. It follows that for $S \in \mathcal{I}_\alpha(\gamma)$, $A_l(\gamma)S = A(\gamma)S$. Then by Theorem 3.1 property (8) for the pair $(A, E(\gamma))$ implies this same property for the pair $(A_l, E_l(\gamma))$.

(d) We conclude from (a), (b) and (c) that A_l is spectral with resolution of the identity $E_l(\gamma)$. Define $A(\rho) = \int_{|\lambda| \leq \rho} \lambda E(d\lambda)$ and $B_l(\rho) = \int_{|\lambda| \leq \rho} \lambda E_l(d\lambda)$, where the first integral is a Riemann integral with respect to the uniform topology on $\mathcal{B}(\mathcal{H})$ and the second a Riemann integral with respect to the uniform topology on $\mathcal{B}(\mathcal{I}_\alpha)$, respectively.

Let $C, C_0 \in \mathcal{B}(\mathcal{H})$. Then C_l converges uniformly to C_{0l} on \mathcal{I}_α if and only if C converges to C_0 uniformly on $\mathcal{B}(\mathcal{H})$, since, according to Lemma 3.1, $\|C_l - C_{0l}\|_\alpha = \|C - C_0\|$. Consequently $B_l(\rho) = (A(\rho))_l$. Let B_l be defined according to $B_l S = \alpha - \lim_{\rho \rightarrow \infty} B_l(\rho)S$, where \mathcal{D}_{B_l} consists of those $S \in \mathcal{I}_\alpha$ for which the above limit exists in \mathcal{I}_α . Then B_l is spectral of scalar type on \mathcal{I}_α with resolution of the identity $E_l(\gamma)$. Suppose $x \in \mathcal{H}$ and $S \in \mathcal{D}_{B_l}$. Then

$$\begin{aligned} |B_l Sx - A(\rho)Sx| &= |B_l Sx - B_l(\rho)Sx| \leq |B_l S - B_l(\rho)S| \|x\| \\ &\leq |B_l S - B_l(\rho)S|_\alpha \|x\| \end{aligned}$$

and it follows that $Sx \in \mathcal{D}_A$ and $ASx = B_l Sx$. Since x was chosen arbitrarily from \mathcal{H} and $A^{**} = A$ we have $B_l \subset A_l$. However (cf. [12], p. 378) a spectral operator has no proper closed extension which is spectral. Consequently $A_l = B_l$ is of scalar type. Thus, the theorem is proven for the case of A_l . The corresponding results for A_r can be easily derived from those for A_l by using Lemma 3.2 and Theorem 3.1. ■

At this stage it should be pointed out that the results stated in Theorem 5.1 are not quite as obvious as it might seem at the first sight. Otherwise we would expect that A_l and A_r would be in general spectral transformers on the entire C^* -algebra $\mathcal{B}(\mathcal{H})$. In fact, properties (1) to (4) above follow

straightforwardly for $E_1(\gamma)$ and $E_r(\delta)$ on $\mathcal{B}(\mathcal{H})$. Property (5), however, requires the positive functionals on $\mathcal{B}(\mathcal{H})$ to be normal [10], and will in general only hold for the linear functionals on $\mathcal{B}(\mathcal{H})$, contained in its predual $\mathcal{B}_*(\mathcal{H}) = \mathcal{B}_1(\mathcal{H})$ (see part (a) of the proof of Lemma 5.2) which is a total family. In fact we can state [13, Theorem 2.1] that A_1 and A_r are prespectral on $\mathcal{B}(\mathcal{H})$ of class $\mathcal{B}_1(\mathcal{H})$. Concerning spectral operators on $\mathcal{B}(\mathcal{H})$ (i.e. prespectral operators on $\mathcal{B}(\mathcal{H})$ of class $\mathcal{B}^*(\mathcal{H})$) there exist some recent results [14], which can be restated in the following form:

If N_1 and N_2 are two normal operators in \mathcal{H} then $N_{1l} + N_{2r}$ is a spectral transformer in $\mathcal{B}(\mathcal{H})$ if and only if $\sigma(N_1) \cup \sigma(N_2)$ is a finite set, i.e. if and only if the spectra of N_1 and N_2 are pure point spectra and finite. Furthermore, if $N_{1l} + N_{2r}$ is a spectral transformer on $\mathcal{B}(\mathcal{H})$, then it is of scalar type.

The above quoted result indicates that only by restricting our considerations to minimal norm ideals \mathcal{I}_α with a corresponding α -norm topology can we obtain that Theorem 5.1 holds. However, as will be proven presently, an extension of this theorem to $A_{1l} + A_{2r}$, where A_1 and A_2 are any two spectral operators, does exist in general only for the Hilbert space $\mathcal{B}_2(\mathcal{H})$ constituted by the Schmidt class of operators on \mathcal{H} .

Suppose that A_1 and A_2 are two bounded spectral operators of scalar type on \mathcal{H} with resolutions of the identity $E_1(\gamma)$ and $E_2(\delta)$, respectively. Then (cf. [10], Ch. XV-6) there exist bounded, invertible, Hermitean operators B_1 and B_2 , with bounded inverses, such that $B_j E_j(\gamma) B_j^{-1}$, $j = 1, 2$ are spectral measures consisting of Hermitean projections and $\hat{A}_j = B_j A_j B_j^{-1}$, $j = 1, 2$ are normal. Consider $A = A_{1l} + A_{2r}$ and $\hat{A} = \hat{A}_{1l} + \hat{A}_{2r}$ on \mathcal{I}_α . It is easily seen that B , defined by $BS = B_1 S B_2^{-1}$ for $S \in \mathcal{I}_\alpha$, is a bounded invertible transformer on \mathcal{I}_α with bounded inverse. Since $A = B^{-1} \hat{A} B$ and $\hat{A} = B A B^{-1}$ it is clear that A is spectral of scalar type on \mathcal{I}_α if and only if such is the case for A . Thus it is sufficient to consider transformers A generated by normal operators A_1 and A_2 . In particular we may assume in the sequel that the corresponding spectral measures $E_1(\gamma)$ and $E_2(\delta)$ consist of Hermitean projectors.

THEOREM 5.2: *Let A_1 and A_2 be two bounded spectral operators of scalar type on the infinite dimensional complex Hilbert space \mathcal{H} . A necessary and sufficient condition for the product $E(\Delta)$, $\Delta \in \mathcal{A}(\mathbb{C}^2)$, of the spectral measures $E_{1l}(A_1)$ and $E_{2r}(A_2)$ to be uniformly bounded and for $A_{1l} + A_{2r}$ to be spectral on the minimal norm ideal \mathcal{I}_α is that either \mathcal{I}_α is identical to the Schmidt class $\mathcal{B}_2(\mathcal{H})$ and that the α -norm topology is equivalent to the Schmidt-norm topology or that at least one of the spectra $\sigma(A_1)$ and $\sigma(A_2)$ consists of a finite number of points.*

PROOF: The sufficiency is evident from [15, Theorem 1]. We assume now that A is spectral on \mathcal{S}_α and that both A_1 and A_2 are normal and have infinite spectra. If $E(A)$ is uniformly bounded, then (cf. [13, Ch. XVII-2]) $\|\sum_{i=1}^n a_i E(\Delta_i)\|_\alpha \leq K \sup_{1 \leq i \leq n} |a_i|$ where K is independent of n , the a_i are arbitrary complex numbers and the Δ_i are mutually disjoint Borel subsets of \mathbb{C}^2 , so that $E(\Delta_i)E(\Delta_j) = E(\Delta_i)\delta_{ij}$.

Since A_1 and A_2 both have infinite spectra there exist countably infinite families $\{\gamma_k\}_{k=1}^\infty$ and $\{\delta_l\}_{l=1}^\infty$ of mutually disjoint non-empty Borel subsets of the plane, such that the corresponding projectors $E_1(\gamma_k)$ and $E_2(\delta_l)$ are non-zero for $k, l = 1, 2, 3, \dots$. Thus the family $\{E(\Delta_{kl})\}$ with $E(\Delta_{kl}) = E_{1l}(\gamma_k)E_{2r}(\delta_l)$ is a countably infinite collection of mutually disjoint projections in \mathcal{S}_α . Since $E_1(\gamma_k)$ and $E_2(\delta_l)$ are non-vanishing, there exist two (in general non-unique) orthonormal systems $\{u_k\}_{k=1}^\infty$ and $\{v_l\}_{l=1}^\infty$ in \mathcal{H} , such that $u_k \neq 0, v_l \neq 0$ and $E_1(\gamma_k)u_k = u_k, E_2(\delta_l)v_l = v_l$. Next we consider the transformers

$$D^{(n)} = \sum_{k,l=1}^n d_{kl}^{(n)} E(\Delta_{kl})$$

with $d_{kl}^{(n)} = \exp(2\pi i n^{-1}kl)$, so that

$$\sum_{k=1}^n \overline{d_{kr}^{(n)}} d_{ks}^{(n)} = n \delta_{rs}, \quad |d_{kl}^{(n)}| = 1.$$

For any positive numbers μ_1, \dots, μ_n let us introduce the finite-rank operators

$$S_1 = \sum_{k,l=1}^n \mu_l(\cdot, v_l)u_k \quad S_2 = \sum_{k,l=1}^n \mu_l \overline{d_{kl}^{(n)}}(\cdot, v_l)u_k.$$

We then easily compute that

$$\begin{aligned} |S_1|_\alpha &= |[S_1^* S_1]^{\frac{1}{2}}|_\alpha = |[n(\cdot, \sum_{k=1}^n \mu_k v_k) \sum_{l=1}^n \mu_l v_l]^{\frac{1}{2}}|_\alpha \\ &= n^{\frac{1}{2}} |\sum_{l=1}^n \mu_l v_l| = n^{\frac{1}{2}} |\sum_{l=1}^n \mu_l^2|^{\frac{1}{2}} \\ &= n^{\frac{1}{2}} \phi_2(\mu_1, \dots, \mu_n, 0, 0, \dots). \end{aligned}$$

A similar calculation yields

$$\begin{aligned} |S_2|_\alpha &= |[S_2^* S_2]^{\frac{1}{2}}|_\alpha = |n^{\frac{1}{2}} \sum_{l=1}^n \mu_l(\cdot, v_l)v_l|_\alpha = n^{\frac{1}{2}} \phi_\alpha(\mu_1, \dots, \mu_n, 0, \dots), \\ |D^{(n)} S_1|_\alpha &= n^{\frac{1}{2}} \phi_\alpha(\mu_1, \dots, \mu_n, 0, 0, \dots), \\ |D^{(n)} S_2|_\alpha &= n^{\frac{1}{2}} \phi_2(\mu_1, \dots, \mu_n, 0, 0, \dots). \end{aligned}$$

The above introduced functions $\phi_2(\mu_1, \dots, \mu_n, \dots)$ and $\phi_\alpha(\mu_1, \dots, \mu_n, \dots)$ are the symmetric gauge norming functions [1–3] on $\mathcal{B}_2(\mathcal{H})$ and \mathcal{I}_α , respectively. The condition of uniform boundedness implies that $|\mathcal{S}_j|_\alpha^{-1} |\mathcal{D}^{(n)} \mathcal{S}_j|_\alpha \leq K$ for $j = 1, 2$, and consequently

$$\phi_2(\mu_1, \dots, \mu_n, 0, \dots) \leq K \phi_\alpha(\mu_1, \dots, \mu_n, 0, \dots) \leq K^2 \phi_2(\mu_1, \dots, \mu_n, 0, \dots).$$

A general operator R of finite rank has the form $R = \sum_{k=1}^n \mu_k(\cdot, x_k) y_k$, where $\{x_k\}$ and $\{y_k\}$ are arbitrary finite orthonormal systems and μ_1, \dots, μ_n are positive numbers. Since

$$|R|_\alpha = \phi_\alpha(\mu_1, \dots, \mu_n, 0, \dots), \{tr R^* R\}^{\frac{1}{2}} = \phi_2(\mu_1, \dots, \mu_n, 0, \dots),$$

we infer from the above inequality that on the family \mathcal{R} of finite rank operators the topology induced by the α -norm is equivalent to the topology induced by the Schmidt norm. In view of the fact that \mathcal{R} is dense in \mathcal{I}_α , we conclude that \mathcal{I}_α is identical to $\mathcal{B}_2(\mathcal{H})$, and that its topology is equivalent to the topology of the Schmidt norm $\|\cdot\|_2$. ■

REMARK: Although A_1 and A_2 in Theorem 6.2 were assumed bounded in order to avoid technical complications with domains of definition, it is evident from the proof that this restriction is not necessary, since even in the case of unbounded operators it is still possible to transform the resolutions of the identity to Hermitean ones.

The matter of the uniform boundedness of the product spectral measure was considered earlier by C. A. McCarthy [16] for \mathcal{I}_α restricted to the Carlemann classes $\mathcal{B}_p(\mathcal{H})$. Unfortunately the choice made for $d_{ki}^{(n)}$ does not lead to the results stated. However, as kindly pointed out to us by Professor McCarthy, the present form of $d_{ki}^{(n)}$ indeed leads to the desired results.

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