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JOHN BORIS MILLER

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SIMULTANEOUS LATTICE
AND TOPOLOGICAL COMPLETION
OF TOPOLOGICAL POSETS

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1. Introduction

Let X be a nonempty set, carrying a directed partial order $\leqslant$ and a uniformity generating a topology $\mathscr{U}$ on X. Then X has two conceptually quite distinct completions: the lattice completion $(\mathfrak{X}, \subseteq)$ of $(X, \leqslant)$ due to H. M. MacNeille [6] generalizing the Dedekind completion of the rationals by use of sections; and the topological completion $(\mathfrak{Y}, \mathscr{V})$ due to A. Weil [13] generalizing the completion of the rationals by means of Cauchy sequences. Both generalizations date from about 1937.

We discuss in this paper the delineation of a class of spaces $(X, \leqslant, \mathscr{U})$ for which the two completions can be compared. This problem was dealt with by B. Banaschewski in 1957, for the case where X is also a partially ordered group and $\mathscr{U}$ is obtained from $\leqslant$ by using topological identities; there is the convenient simplification that $(X, \mathscr{U})$ is a topological group, so that $\mathscr{U}$ is necessarily generated by a uniformity and there is no need to make that fact an additional postulate. In our case the construction is carried through without assumption of any group structure on X. Of course the completion of the rationals $\mathbb{Q}$ by Cauchy sequences makes explicit use of both the group and order structures, in the actual completion process. The group structure is not used in the construction of the lattice completion of $\mathbb{Q}$.

The topology $\mathscr{U}$ which we place on X is related to the order $\leqslant$ but not uniquely determined by it. When $(X, \leqslant)$ is also a pogroup our topology coincides with Banaschewski's, in the sense that $\mathscr{U}$ is generated by a set of topological identities iff it is an open-interval topology as used here; this follows from a remark due to N. R. Reilly [10]. However, our construction is rather different from Banaschewski's even in the group case.

The principal results are Theorems 9, 12, 13 and 15.

For some results about completion in a rather similar setting see

R. H. Redfield [8] and B. F. Sherman [12]. The paper [9] by Redfield reached the author after the present paper was completed; the two papers discuss quite similar problems. There are interesting discussions of order-theoretic completions of pogroups in L. Fuchs [4], Ch. V, and of relations between order completeness and topological completeness in ordered topological vector spaces in A. L. Peressini [7], Ch. IV.

2. Definitions of $(\mathfrak{X}, \preceq, \subseteq, \mathcal{T})$ and $(X, \leqq, \leqslant, \mathcal{U})$; the embedding θ

2.1. The construction is based on the lattice completion $\mathfrak{X}$ rather than the topological completion. We recall the definition of $\mathfrak{X}$. Given the poset $(X, \leqslant)$, directed to left and right, and any $A \subseteq X$, write

$$(A \leqslant) = \{x : a \leqslant x \text{ for every } a \in A\}$$

for the set of all upper bounds of A, $(\leqslant A)$ for the set of all lower bounds of A, and

$$A^* = (\leqslant (A \leqslant)) = \{x : x \leqslant u \text{ for every } u \geqslant A\}.$$

$\mathfrak{X}$ will denote the set of all sets A^* obtained from nonempty subsets A bounded above. Assume that $(X, \leqslant)$ has no extremal elements; then $\emptyset$ and X do not belong to $\mathfrak{X}$. With $\subseteq$ as usual denoting set inclusion, $(\mathfrak{X}, \subseteq)$ is the MacNeille completion of $(X, \leqslant)$: it is a conditionally complete lattice, and the map

$$\theta : x \mapsto \{x\}^* = (\leqslant x), X \to \mathfrak{X}$$

is an order isomorphism, i.e. $x \leqslant y$ iff $\theta(x) \subseteq \theta(y)$, preserving all meets and joins which may exist in X; and $\theta(X)$ is lattice-dense in $\mathfrak{X}$ in the sense that every element of $\mathfrak{X}$ is expressible both as a meet and as a join of elements from $\theta(X)$. Moreover, if $\phi : (X, \leqq) \to (\mathfrak{Z}, \preceq)$ is any other map into a conditionally complete lattice, with these properties, then there exists a map $s : (\mathfrak{X}, \subseteq) \to (\mathfrak{Z}, \preceq)$ such that $s \circ \theta = \phi$, and s is a complete lattice isomorphism into $\mathfrak{Z}$. So in this sense $(\mathfrak{X}, \subseteq)$ is minimal.

The lattice operations in $\mathfrak{X}$ are given by

$$\bigwedge_{i \in I} A_i^* = \bigcap_{i \in I} A_i^*,$$

$$\bigvee_{i \in I} A_i^* = (\bigcup_{i \in I} A_i^*)^*$$

for an arbitrary family $\{A_i^* : i \in I\}$ bounded below or above respectively in $(\mathfrak{X}, \subseteq)$. (See for example G. Birkhoff [2], p. 126; P. Ribenboim [11], pp 82–85; L. Fuchs [4], pp 92–95.)

2.2. We return to the space X, supposed given, with partial order $\leqslant$ but so far without a topology; on it we place a topology $\mathcal{U}$ defined in terms of a second partial order $\leqq$. It will be supposed that some $\leqq$ is given on X satisfying the following conditions. *These conditions are assumed to hold throughout the paper.*

(i) $(X, \leqq)$ is a directed poset with no minimal and no maximal elements, and with the properties $TR(1, 2)$ and $TR(2, 1)$. (For natural numbers m, n, $TR(m, n)$ denotes the property: For every set of $m+n$ elements $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n$ in X satisfying $a_i < b_j$ for all i, j there exists x in X such that $a_i < x < b_j$ for all i, j.)

(ii) For all $a, b \in X$,

$$a \leqslant b \quad \text{iff} \quad (\forall u \in X)[u < a \Rightarrow u < b] \text{ and } (\forall v \in X)[v > b \Rightarrow v > a].$$

(iii) $(X, \leqq)$ has the property (Σ); i.e. for all $a, b \in X$,

$$(\forall u \in X)[u < a \Rightarrow u < b] \quad \text{iff} \quad (\forall v \in X)[v > b \Rightarrow v > a].$$

Property (ii) asserts that $\leqslant$ is the *associated preorder* of $\leqq$ (by assumption it is in fact a partial order), and $\leqq$ is a *determining order* of $\leqslant$. Property (Σ) and hence (iii) holds in any pogroup.

The topology $\mathcal{U}$ is defined to be the open-interval topology of $\leqq$, having as subbase the set of all subsets of the form

$$(a <) = \{x \in X : a < x\}, \qquad (< b) = \{x \in X : x < b\}.$$

The assumptions (i) ensure that in fact the set of all intervals $(a, b) = (a <) \cap (< b)$, $a < b$, form a base for $\mathcal{U}$.

2.3. Here are some examples of spaces in which these conditions are met.

(1) $\mathbb{Q}^n$ and $\mathbb{R}^n$, with $x < y$ iff $x_i < y_i$ for $i = 1, 2, \cdots, n$ (the strong pointwise order), $x \leqslant y$ iff $x_i \leqq y_i$ for $i = 1, 2, \cdots, n$ (the weak pointwise order); $\mathcal{U}$ is the euclidean topology.

(2) $C(\Omega)$, the space of all real-valued continuous functions on a compact topological space Ω, with $f < g$ iff $f(\omega) < g(\omega)$ for all $\omega \in \Omega$, $f \leqslant g$ iff $f(\omega) \leqq g(\omega)$ for all $\omega \in \Omega$ (respectively the strong and weak pointwise orders); $\mathcal{U}$ is the topology of uniform convergence.

(3) $\mathbb{Q}^n$, $\mathbb{R}^n$ and $C(\Omega)$, with $\leqq$ taken to be a hybrid order. This is defined by fixing upon a proper non-empty subset A of $1, 2, \cdots, n$ or of Ω and defining $x < y$ to mean $x \leqslant y$ and $x_i < y_i$ for all $i \in A$, and $f < g$ to mean $f \leqslant g$ and $f(\omega) < g(\omega)$ for all $\omega \in A$. The associated order in each case is $\leqslant$. The topology is strictly stronger than in the cases (1) and (2).

(4) More generally, let $\{X_i : i \in I\}$ be a family of $TR(1, 1)$ ($\equiv$ order-dense) fully ordered groups $(X_i, \leqslant_i)$, and let $\mathscr{F}$ be a given filter of subsets of the index set I. The cartesian product $X = \prod\{X_i : i \in I\}$ is given two partial orders $\leqq$, $\leqslant$ as follows: $\leqslant$ is the weak pointwise order, and

$$x < y \quad \text{iff} \quad x \prec y \quad \text{and} \quad \{i \in I : x_i \prec_i y_i\} \in \mathscr{F}.$$

2.4. Assumptions (i)–(iii) have the following consequences:

(a) $(X, \mathscr{U})$ has no isolated points, so that $\mathscr{U}$ is not discrete. $\mathscr{U}$ is Hausdorff.

(b) $\leqslant$ is closed, i.e. $\{\langle a, b \rangle : a \leqslant b\}$ is a closed subset of $(X \times X, \mathscr{U} \times \mathscr{U})$; $\leqslant$ is semiclosed, i.e. $(\leqslant a)$ and $(a \leqslant)$ are closed subsets of $(X, \mathscr{U})$ for all $a \in X$.

(c) For all $a, b, c \in X$,

$$a < b \Rightarrow a \prec b,$$
$$a < b \prec c \Rightarrow a < c, \qquad a \prec b < c \Rightarrow a < c.$$

(d) If $(X, \leqq)$ is also a pogroup, then $(X, \mathscr{U})$ is a topological group. If $(X, \leqslant)$ is an l-group, then $(X, \leqslant, \mathscr{U})$ is a topological lattice, and $\leqq$ and $\leqslant$ are isolated orders.

(e) $(X, \leqslant)$ is directed to left and right, with no maximal or minimal elements (as previously required).

For these results see N. Cameron and J. B. Miller [3], R. J. Loy and J. B. Miller [5], pp. 227, 235.

It is clear that $\mathscr{U}$ is intrinsic to the partial order $\leqq$, and indirectly so to the partial order $\leqslant$. $\mathfrak{X}$ is the lattice completion of $(X, \leqslant)$, not of $(X, \leqq)$; however, it is reasonably kind to $\leqq$. A partial order $\leqslant$ may have several TR determining orders (cf. 2.3(3) above), and distinct determining orders for $\leqslant$ define distinct open-interval topologies. In fact, under assumptions 2.2 (i)-(iii), for two determining orders $\leqq_1$ and $\leqq_2$ for the same $\leqslant$ and corresponding topologies $\mathscr{U}_1$ and $\mathscr{U}_2$ we have $\mathscr{U}_1 \subseteq \mathscr{U}_2$ iff $(\forall x, y \in X) \left[x \leqq_1 y \Rightarrow x \leqq_2 y \right]$, so the set of open-interval topologies and the set of determining orders are order-isomorphic.

2.5. *An order for topological spaces.* Let $(\Omega, \mathscr{W})$ be any topological space. The power set $\mathscr{P} = \mathscr{P}(\Omega)$ can be ordered by containment $\subseteq$, and also as follows: for $A, B \in \mathscr{P}$, write

(2.1) $A \relbar\mkern-9mu\prec B$ iff $\bar{A} \subset B°$.

Here $^-$ and $°$ denote closure and interior respectively with respect to $\mathscr{W}$, and $\subset$ will always mean $\subsetneqq$. It can be verified that:

(a) $\mathrel{=\mkern-9mu\prec}$ (meaning '$\relbar\mkern-9mu\prec$ or $=$') is a partial order on $\mathscr{P}$.

(b) The $\mathrel{=\mkern-9mu\prec}$-minimal elements are precisely the subsets with empty interior; the $\mathrel{=\mkern-9mu\prec}$-maximal elements are precisely the dense subsets.

(c) If $(\Omega, \mathscr{W})$ is T_1 with no isolated points then

$$(\forall U \in \mathscr{P})[U \relbar\mkern-9mu\prec A \Rightarrow U \relbar\mkern-9mu\prec B]   \text{ iff }   A° \subseteq B°,$$

$$(\forall V \in \mathscr{P})[V \succ\mkern-9mu\relbar B \Rightarrow V \succ\mkern-9mu\relbar A]   \text{ iff }   \bar{A} \subseteq \bar{B}.$$

So if $\mathrel{\unlhd}$ denotes the associated preorder of $\mathrel{=\mkern-9mu\prec}$ on $\mathscr{P}$,

(2.2) $A \mathrel{\unlhd} B$ iff $A° \subseteq B°$ and $\bar{A} \subseteq \bar{B}$;

therefore $\mathrel{\unlhd}$ is in general not a partial order. We have

(2.3) $A \relbar\mkern-9mu\prec B \Rightarrow A \subset B$, $A \subset B \Rightarrow A \vartriangleleft B$.

(d) When $(\Omega, \mathscr{W})$ is normal and connected, if $A_1, A_2 \relbar\mkern-9mu\prec B_1, B_2$ and at least one of A_1, A_2 is not $\emptyset$ and at least one of B_1, B_2 is not Ω then there exists $C \in \mathscr{P}$ such that $A_1, A_2 \relbar\mkern-9mu\prec C \relbar\mkern-9mu\prec B_1, B_2$; $(\mathscr{P}^*, \mathrel{=\mkern-9mu\prec})$ is $TR(2, 2)$, where $\mathscr{P}^*$ denotes $\mathscr{P} \backslash \{\emptyset, \Omega\}$. (We will not use this result, since the connectedness condition is too strong.)

2.6. Starting with $(X, \leq, \leqslant, \mathscr{U})$, form MacNeille's lattice completion $(\mathfrak{X}, \subseteq)$ of $(X, \leqslant)$, as in 2.1.

We note some properties of the subsets A^*. (To simplify the notation we can write A in place of A^*; since $*$ is a closure operator on $P(X)$, $A^* = A$ for all $A \in \mathfrak{X}$, so no ambiguity arises.)

1. LEMMA: *For every* $A \in \mathfrak{X}$,

(a) $A°$ *is decreasing with respect to* $\leqslant$ (*and so* $\leq$),

$$A° = \{x : \theta(x) \relbar\mkern-9mu\prec A\},   (A°)^- = A,$$
$$(\leqslant x)° = (< x)   \text{ for all } x \in X;$$

(b) $(A <) = \{x : A \relbar\mkern-9mu\prec \theta(x)\}$.

PROOF:

(a) Let $x \in A°$; then $u < x < v$, $(u, v) \subseteq A$ for some $u, v \in X$. We have

$(< v) \subseteq A^\circ$; for if $t < v$ then by $TR(2, 1)$ there exists $s \in X$, u, $t < s < v$, so $s \in A$ and since A is decreasing, $t \in A$. So $y \leqslant x \Rightarrow y < v \Rightarrow y \in A^\circ$. Thus A° is decreasing. Clearly $\theta(x) \mathbin{\underrightarrow{\prec}} A \Leftrightarrow (\leqslant x) \subset A^\circ \Rightarrow x \in A^\circ$. Conversely, if $x \in A^\circ$ then $(\leqslant x) \subset (< v) \subseteq A^\circ$ so $\theta(x) \mathbin{\underrightarrow{\prec}} A^\circ$.

A is closed; for $(A \leqslant) = \bigcap \{(a \leqslant) : a \in A\}$ is closed since each $(a \leqslant)$ is closed (2.4(b)), and $A = A^* = \bigcap\{(\leqslant x) : x \in (A \leqslant)\}$.

Thus $(A^\circ)^- \subseteq A$. If $x \in A$ and $x \in (u, v)$ then for any w satisfying $u < w < x$ we have $w \in A$, $u \in (< w) \subseteq A$, so $u \in A^\circ$. By the same token $w \in A^\circ \cap (u, v)$, so $x \in (A^\circ)^-$. The last assertion of (a) is easily verified.

(b) If $A \mathbin{\underrightarrow{\prec}} \theta(x)$, then $A \subset (< x)$ (by (a)), i.e. $A < x$. Conversely suppose $A < x$, i.e. $A \subseteq (< x)$. If $A = (< x)$, then since every neighbourhood of x meets $(< x)$, $x \in (< x)^- = A^- = A = (< x)$ contradiction; so $A \mathbin{\underrightarrow{\prec}} \theta(x)$. //

Since $\mathfrak{X} \subseteq \mathscr{P}(X)$, $\mathbin{\underrightarrow{\prec}}$ induces a relative partial order on $\mathfrak{X}$; we denote this temporarily by $\mathbin{\underrightarrow{\prec}}'$. Likewise $\trianglelefteq$, the associated preorder of $\mathbin{\underrightarrow{\prec}}$, induces a preorder $\trianglelefteq'$ on $\mathfrak{X}$. Note that $\trianglelefteq'$ need not a priori be the associated preorder of $\mathbin{\underrightarrow{\prec}}'$. The first step is to identify the associated preorder of $\mathbin{\underrightarrow{\prec}}'$, and prove that in fact $\mathbin{\underrightarrow{\prec}}'$ does determine $\trianglelefteq'$.

2. PROPOSITION: *The associated preorder of* $(\mathfrak{X}, \mathbin{\underrightarrow{\prec}}')$ *is* $\subseteq$, *and is therefore a partial order.* $(\mathfrak{X}, \mathbin{\underrightarrow{\prec}}')$ *has property* (Σ).

PROOF: Let A, B be chosen from $\mathfrak{X}$. First, suppose

(2.6) $(\forall U \in \mathfrak{X})[U \mathbin{\underrightarrow{\prec}}' A \Rightarrow U \mathbin{\underrightarrow{\prec}}' B]$.

Since each U is closed (1. (a)), this asserts that for all $U \in \mathfrak{X}$,

(2.7) $U \subset A^\circ \Rightarrow U \subset B^\circ$.

Let $x \in A^\circ$; there exist a, t, $b \in X$, such that $a < x < t < b$ and $(a, b) \subseteq A$, and then $(\leqslant x) \subset (< b) \subseteq A^\circ$. Take $U = \{x\}^* = (\leqslant x)$ in (2.7), to deduce $x \in B^\circ$. Thus $A^\circ \subseteq B^\circ$. This conversely implies (2.6), so the two are equivalent.

Next, suppose instead that

(2.8) $(\forall V \in \mathfrak{X})[V \mathbin{\underrightarrow{\succ}}' B \Rightarrow V \mathbin{\underrightarrow{\succ}}' A]$

i.e. for all $V \in \mathfrak{X}$,

(2.9) $B \subset V^\circ \Rightarrow A \subset V^\circ$.

There exists $x > B$ (i.e. $x > b$ for all $b \in B$); for B is bounded above with

respect to $\leqslant$, say $u \geqslant B$, and since u is not $\leqq$-maximal we can take any $x > u$. Take $V = (\leqslant x)$, so that $V^\circ = (< x)$. We have $B \subseteq (< x)$ in fact $B \subset (< x)$ by the argument used to prove 1.(b); so $B \subset V^\circ$ and (2.9) gives $A \subset V^\circ$. Thus (2.8) implies

$$(2.10) \qquad (\forall x \in X)[x > B \Rightarrow x > A].$$

This in turn implies

$$(2.11) \qquad (\forall y \in X)[y \geqslant B \Rightarrow y \geqslant A].$$

For let $B \leqslant y$ and $a \in A$. If $y < v$ then $B < v$ so (2.10) gives $A < v$, $a < v$. Thus $(\forall v \in X)[v > y \Rightarrow v > a]$, and assumption 2.2.(iii) gives $a \leqslant y$. Thus $A \leqslant y$. This proves (2.11). From this we deduce $A \subseteq B$. For let $x \notin B$; since $B^* = B$, there must exist y for which $B \leqslant y$ and $x \nleqslant y$, and (2.11) gives $A \leqslant y$, so $y \notin A$.

Conversely, $A \subseteq B$ implies (2.9). Thus (2.8) is equivalent to $A \subseteq B$.

We have now shown that in $\mathfrak{X}$, A is less than or equal to B in the associated preorder of $\prec'$ iff $A^\circ \subseteq B^\circ$ and $A \subseteq B$. By 1.(a), each of these implies the other. The proposition is proved.

The proof also shows that $\subseteq'$ does indeed coincide with the associated preorder of $\prec'$. Henceforth it suffices to write more briefly $\prec$ for $\prec'$. We now have that $\mathfrak{X}$ is a conditionally complete lattice under $\subseteq$, and $\prec$ is a determining order for $\subseteq$: for all $A, B \in \mathfrak{X}$,

$$(2.12) \qquad A \prec B \quad \text{iff} \quad A \subset B^\circ.$$

Since the operation $^\circ$ is with respect to $\mathcal{U}$ on X, $\prec$ depends indirectly on $\leqq$; a previous remark shows that the map from $\leqq$ to $\prec$ is order-preserving. Let $\mathcal{T}$ denote the open-interval topology of $\prec$. This puts a topology on $\mathfrak{X}$, as we wished to do.

Before considering the possible generation of $\mathcal{U}$ and $\mathcal{T}$ by uniformities, we examine

$$(2.13) \qquad \theta : x \mapsto (\leqslant x), \qquad (X, \leqq, \leqslant, \mathcal{U}) \to (\mathfrak{X}, \prec, \subseteq, \mathcal{T})$$

as a structure-preserving map. First, as remarked in 2.1, θ is one-one and an order isomorphism for $\langle \leqslant, \subseteq \rangle$. It is also an order isomorphism for $(\leqq, \prec)$; for if $a < b$ there exists $c \in X$, $a < c < b$, and so $(\leqslant a) \subset (< b)$ since $c \in (< b) \backslash (\leqslant a)$; conversely $(\leqslant a) \subset (< b)$ implies $a < b$.

Next, note that $\theta : X \to \theta(X)$ is an open map. For if $U = (a, b)$, $a < b$, is a base open subset of X then

$$\theta(U) = \theta(X) \cap \{A \in \mathfrak{X} : \theta(a) \,\text{---}\!\!\!\!\prec\, A \,\text{---}\!\!\!\!\prec\, \theta(b)\},$$

an open subset of $(\theta(X), \mathcal{T}|\theta(X))$.

Continuity of θ is less straightforward. It is necessary to make further assumptions, about the extent of the penetration of $\mathfrak{X}$ by $\theta(X)$. Two such, expressed directly in terms of X, are as follows:

   I. For all $A \in \mathfrak{X}$, $(A <)$ is open.
  II. For all $A, B \in \mathfrak{X}$, if $A \subset B^{\circ}$ then $(A <) \cap B^{\circ} \neq \emptyset$.

We note some equivalences.

3. Conditions I, I′ and I″ are pairwise equivalent, where the second and third conditions are:

   I′. For all $A \in \mathfrak{X}$ and $x \in X$, if $A < x$ then there exists $y \in X$ such that $A < y < x$.
   I″. For all $A \in \mathfrak{X}$ and $x \in X$, if $A \,\text{---}\!\!\!\!\prec\, \theta(x)$ then there exists $y \in X$ such that $A \,\text{---}\!\!\!\!\prec\, \theta(y) \,\text{---}\!\!\!\!\prec\, \theta(x)$.

Each of I, I′ and I″ implies:

(2.14)   If $A_1, A_2 \,\text{---}\!\!\!\!\prec\, \theta(x)$, then there exists $y \in X$

such that $A_1, A_2 \,\text{---}\!\!\!\!\prec\, \theta(y) \,\text{---}\!\!\!\!\prec\, \theta(x)$.

PROOF: The equivalence of I′ and I″ follows from 1.(b). It is clear that I is equivalent to I′.

Assume I′, and $A_1, A_2 \,\text{---}\!\!\!\!\prec\, \theta(x)$ with $A_1, A_2 \in \mathfrak{X}$. There exist $y_1, y_2, t, y \in X$ such that $A_1 < y_1 < x$ and $A_2 < y_2 < x$ and

$$y_1, y_2 < t < y < x.$$

Then $A_1, A_2 \leqslant t$ so $A_1 \vee A_2 \subseteq (\leqslant t)$, whence $A_1, A_2 < y < x$, which is (2.14).//

The following also follows simply from earlier remarks.

4. II is equivalent to II′, and implies I; where

II′. For all $A, B \in \mathfrak{X}$, if $A \,\text{---}\!\!\!\!\prec\, B$ then $A \,\text{---}\!\!\!\!\prec\, \theta(x) \,\text{---}\!\!\!\!\prec\, B$ for some $x \in X$.

Since we do not know that $(\mathfrak{X}, \,\text{=}\!\!\!\!\prec\,)$ has any TR properties, the base

sets of $\mathscr{T}$ must be assumed to have the form

$$(2.15) \qquad\qquad V = \bigcap_{i \in I} (A_i \relbar\!\!\!\prec) \cap \bigcap_{j \in J} (\relbar\!\!\!\prec B_j),$$

with I and J finite. Since $(\mathfrak{X}, \subseteq)$ is directed with no extremals, any base neighbourhood of a point can be assumed to be of the form V with $I = J \neq \emptyset$.

5. For any $A \in \mathfrak{X}$, $\theta^{-1}(\relbar\!\!\!\prec A)$ is open in $(X, \mathscr{U})$. If I holds, then also $\theta^{-1}(A \relbar\!\!\!\prec)$ is open, and so $\theta : X \to \mathfrak{X}$ is continuous.

PROOF: Let $x \in \theta^{-1}(\relbar\!\!\!\prec A)$. Then $(\leqslant x) \subset A^\circ$, so there exist $a, b \in X$ with $x \in (a, b) \subseteq A$, and also $a, b \in A$. If $y \in (a, b)$ then also $y \in A^\circ$, and 1.(a) gives $\theta(y) \relbar\!\!\!\prec A$, so $(a, b) \subseteq \theta^{-1}(\relbar\!\!\!\prec A)$. Thus $\theta^{-1}(\relbar\!\!\!\prec A)$ is open.
   Suppose instead that $x \in \theta^{-1}(A \relbar\!\!\!\prec)$, so $A \relbar\!\!\!\prec \theta(x)$. By I″ there exists y such that $A \relbar\!\!\!\prec \theta(y) \relbar\!\!\!\prec \theta(x)$; then $x \in (y <) \subseteq \theta^{-1}(A \relbar\!\!\!\prec)$. Thus $\theta^{-1}(A \relbar\!\!\!\prec)$ is open. //

6. COROLLARY: *When I holds, θ is a topological embedding of $(X, \mathscr{U})$ in $(\mathfrak{X}, T)$.*

Next, we need conditions which imply interpolation properties for $\relbar\!\!\!\prec$. These are partially associated with interpolation by elements of $\theta(X)$.

7. (i) Let $(X, \leqslant)$ be a $\vee$-semilattice. If $x_1, x_2 \in A^\circ$, $A \in \mathfrak{X}$, then $x_1 \vee x_2 \in A^\circ$.
   (ii) Let $(X, \leqslant)$ be a $\wedge$-semilattice, and let I hold. If $A < x_1, x_2$, $A \in \mathfrak{X}$, then $A < x_1 \wedge x_2$.

PROOF: (i) There exist $y_1, y_2 \in A^\circ$ such that $x_1 < y_1$ and $x_2 < y_2$. Then $x_1 \vee x_2 < y_1 \vee y_2 \in A$, so $(< y_1 \vee y_2)$ is a neighbourhood of $x_1 \vee x_2$ contained in A; $x_1 \vee x_2 \in A^\circ$.
   (ii) By I, there exist $y_1, y_2 \in X$ such that $A < y_1 < x_1$ and $A < y_2 < x_2$. Then $A \leqslant y_1 \wedge y_2 < x_1 \wedge x_2$, so $A < x_1 \wedge x_2$.

8. PROPOSITION: *If $(X, \leqslant)$ is a lattice then between the properties:*
   (a) *II holds;*
   (b) *$\theta(X)$ is dense in $(\mathfrak{X}, \mathscr{T})$ and $(\mathfrak{X}, \relbar\!\!\!\prec)$ is $TR(1, 1)$;*
   (c) *$(\mathfrak{X}, \relbar\!\!\!\prec)$ is $TR(2, 2)$;*
   (d) *The intervals $(\theta(a), \theta(b))$, $a < b$, form a base for $(\mathfrak{X}, \mathscr{T})$, and $\mathscr{T}$ is Hausdorff,*

there are the implications

(a) $\Leftrightarrow$ (b),   (a) $\Rightarrow$ (c),   (a) $\Rightarrow$ (d).

PROOF: (a) $\Rightarrow$ (c): Let $A_1, A_2 \prec B_1, B_2$, in $\mathfrak{X}$. By II there exist $x_1, x_2 \in X$ such that $x_1 \in (A_1 <) \cap B_1^\circ$, $x_2 \in (A_2 <) \cap B_1^\circ$. By 7.(i), the element $y_1 = x_1 \vee x_2$ satisfies $y_1 \in (A_1 <) \cap (A_2 <) \cap B_1^\circ$. Similarly there exists $y_2 \in (A_1 <) \cap (A_2 <) \cap B_2^\circ$. By 7.(ii), we have

$$A_1, A_2 < y_1 \wedge y_2 \leqslant y_1 < B_1, B_2,$$

giving

$$A_1, A_2 \prec \theta(y_1 \wedge y_2) \prec B_1, B_2.$$

Thus $(\mathfrak{X}, \prec)$ is $TR(2, 2)$.

(a) $\Rightarrow$ (b) Since $(X, \prec)$ is $TR(2, 2)$ and has no extremals the $\prec$-intervals (A, B), $A \prec B$, form a base for $\mathscr{T}$, so II immediately implies that $\theta(X)$ is dense in $(\mathfrak{X}, \mathscr{T})$.

(b) $\Rightarrow$ (a) If $A \prec B$, the nonempty open subset (A, B) must meet $\theta(X)$. (This does not use the lattice assumption.)

(a) $\Rightarrow$ (d) Again, the intervals (A, B), $A \prec B$, form a base for $\mathscr{T}$; II then implies that the intervals $(\theta(a), \theta(b))$, $a < b$, form a base. That $\mathscr{T}$ is Hausdorff is an application to $(\mathfrak{X}, \prec, \mathscr{T})$ of 2.4(a). (Actually, it suffices to know that $\prec$ is $TR(1, 1)$, has property (Σ), and its associated order is partial.) //

If $(\mathfrak{X}, \prec)$ is $TR(2, 2)$ and $\theta(X)$ is dense in $(\mathfrak{X}, \mathscr{T})$ then II holds, whether or not $(X, \leqslant)$ is a lattice.

The following theorem summarizes the position we have reached.

9. THEOREM: *Let* $(X, \leq, \leqslant, \mathscr{U})$ *have the properties* 2.2(i)–(iii). *Let* $(\mathfrak{X}, \prec, \subseteq, \mathscr{T})$ *be as previously constructed, so that* $(\mathfrak{X}, \subseteq)$ *is the lattice completion of* $(X, \leqslant)$ *and is a conditionally complete lattice,* $\prec$ *is defined by* (2.12) *and is a determining order for* $\subseteq$, *and* $\mathscr{T}$ *is the open-interval topology of* $\prec$. *Let* θ *be the map* (2.13).

Then θ *is an order isomorphism for both orderpairs* $\langle \leq, \prec \rangle, \langle \leqslant, \subseteq \rangle$, *preserving all existing meets and joins for the second pair; and* $\theta : X \to \theta(X)$ *is open. If* I *holds,* θ *is a topological embedding of* $(X, \mathscr{U})$ *in* $(\mathfrak{X}, \mathscr{T})$. *If* II *holds and* $(X, \leqslant)$ *is a lattice, then* $\theta(X)$ *is dense in* $(\mathfrak{X}, \mathscr{T})$ *and* $(\mathfrak{X}, \prec)$ *is* $TR(2, 2)$.

3. Uniformities generating the topologies

3.1. If $(X, \leq)$ is a partially ordered group in addition to what has been

assumed in 2.2, then $(X, \mathcal{U})$ is a topological group and therefore has right and left uniformities generating its topology. If also $(X, \leqslant)$ is para-archimedean, the lattice completion $\mathfrak{X}$ is a group, necessarily commutative. (See e.g. [4], pp 92–95). If also $(\mathfrak{X}, =\!\!\prec)$ is $TR(1, 1)$, then $(\mathfrak{X}, \mathcal{T})$ is a topological group and therefore its topology is also generated by a uniformity. So broadly speaking, when X is a commutative group we can use the natural uniformity on it. This is done in Section 5, below.

If no group structure is assumed on X we must look elsewhere. There is no preordained uniformity generating $\mathcal{U}$ in general, and therefore we suppose its existence as an extra hypothesis: throughout this section we assume that $\mathcal{U}$ is generated by a uniformity having base $\mathcal{B}$. The problem is to obtain from $\mathcal{B}$ a suitable uniformity on $\mathfrak{X}$. This can be done if $\mathcal{B}$ satisfies a number of conditions asserting compatibility with $\subseteq$ and $\leqslant$, and provided $\theta(X)$ is dense in $(\mathfrak{X}, \mathcal{T})$; we arrive eventually at Theorem 13 asserting that $\mathfrak{X}$ is both the lattice and topological completion of X.

Recall that $\mathcal{B} \subseteq X \times X$ is a base for a uniformity on X iff $\mathcal{B}$ is a filter base of $\mathcal{P}(X \times X)$ containing the diagonal $\Delta = \{\langle x, x \rangle : x \in X\}$, and such that if $\boldsymbol{B} \in \mathcal{B}$ then there exist $\boldsymbol{C}, \boldsymbol{D} \in \mathcal{B}$ such that $\boldsymbol{C} \subseteq \boldsymbol{B}^{-1}$ and $\boldsymbol{D} \circ \boldsymbol{D} \subseteq \boldsymbol{B}$. The uniformity generated by $\mathcal{B}$ is the set of all its supersets in $X \times X$; and the sets $\boldsymbol{B}[x] = \{y : \langle x, y \rangle \in \boldsymbol{B}\}$, $\boldsymbol{B} \in \mathcal{B}$, form a base for the neighbourhood system at x in $(X, \mathcal{U})$. The sets $\boldsymbol{B}, \boldsymbol{B}[x]$ are not assumed open in $\mathcal{U} \times \mathcal{U}$, $\mathcal{U}$ respectively.

With a uniformity assumed for $(X, \mathcal{U})$, the uniformity for $(\mathfrak{X}, \mathcal{T})$ is constructed as follows. For each $\boldsymbol{B} \in \mathcal{B}$, embed $\boldsymbol{B}$ in $\mathfrak{X} \times \mathfrak{X}$, as

$$\theta(\boldsymbol{B}) = \{\langle \theta(x), \theta(y) \rangle : \langle x, y \rangle \in \boldsymbol{B}\};$$

let $\overline{\overline{\boldsymbol{B}}}$ denote the closure $\overline{\theta(\boldsymbol{B})}$ of $\theta(\boldsymbol{B})$ in $(\mathfrak{X} \times \mathfrak{X}, \mathcal{T} \times \mathcal{T})$, and let $\overline{\overline{\mathcal{B}}}$ be the collection of all subsets $\overline{\overline{\boldsymbol{B}}}$.

We introduce the following conditions on $\mathcal{B}$:

III. For every $Z \in \mathfrak{X}$ and $\boldsymbol{E} \in \mathcal{B}$ there exist elements $p, q, x \in X$ such that

$$\theta(p) \longrightarrow\!\!\!\!< Z \longrightarrow\!\!\!\!< \theta(q), \qquad (p, q) \subseteq \boldsymbol{E}[x]$$

IV. For all $x, y \in X$ and $\boldsymbol{B} \in \mathcal{B}$,
   (i) $x < y$ implies $(\boldsymbol{B}[x] \leqslant) \supseteq (\boldsymbol{B}[y] \leqslant)$,
   (ii) $x < y$ implies $(\leqslant \boldsymbol{B}[x]) \subseteq (\leqslant \boldsymbol{B}[y])$.

These conditions are discussed further in Section 4, and for pogroups in Section 5.

10. Proposition: *If* III *holds and* $\theta(X)$ *is dense in* $(\mathfrak{X}, \mathscr{T})$, *then* $\bar{\bar{\mathscr{B}}}$ *is a base for a uniformity on* $\mathfrak{X}$, *and the uniform topology* $\mathscr{W}$ *generated by* $\bar{\bar{\mathscr{B}}}$ *is weaker than* $\mathscr{T}$. *If furthermore* II *and* IV *hold, and* $(X, \leqslant)$ *is a lattice, then* $\mathscr{W} = \mathscr{T}$.

The proof of this proposition is broken down into several parts. First note that the condition that $\theta(X)$ be dense in $\mathfrak{X}$ cannot be avoided, since every $\bar{\bar{B}} \in \bar{\bar{\mathscr{B}}}$ must contain the diagonal in $\mathfrak{X} \times \mathfrak{X}$; it is a necessary and sufficient condition for this property. Further, it is easily verified that $\bar{\bar{\mathscr{B}}}$ is a filter base on $\mathfrak{X} \times \mathfrak{X}$, since $\mathscr{B}$ is a filter base, and that $\overline{\overline{B^{-1}}} = \bar{\bar{B}}^{-1}$ for each $B \in \mathscr{B}$. It remains to prove that

(3.1)   If $\bar{\bar{B}} \in \bar{\bar{\mathscr{B}}}$, there exists $\bar{\bar{D}} \in \bar{\bar{\mathscr{B}}}$ such that $\bar{\bar{D}} \circ \bar{\bar{D}} \subseteq \bar{\bar{B}}$,

and that the topology of the uniformity generated by $\bar{\bar{\mathscr{B}}}$ is $\mathscr{T}$.

11. Lemma: *Suppose* $\theta(X)$ *is dense in* $(\mathfrak{X}, \mathscr{T})$. *If* $p, q \in X$, $A \in \mathfrak{X}$ *and*

$$\theta(p) \prec\!\!\!- A \prec\!\!\!- \theta(q),$$

then $\theta(p, q)^-$, *the closure in* $(\mathfrak{X}, \mathscr{T})$ *of* $\theta\{x : p < x < q\}$, *is a neighbourhood of* A.

Proof: We show that

(3.2) $(\theta(p), \theta(q)) \subseteq \theta(p, q)^-$

where $(\cdot, \cdot)$ denotes the open interval in $(\mathfrak{X}, \prec\!\!\!-)$ and $(X, \leq)$ respectively. Let $B \in (\theta(p), \theta(q))$. If V is any $\mathscr{T}$-neighbourhood of B then $V \cap (\theta(p), \theta(q))$ is also a neighbourhood, and meets $\theta(X)$. Thus V meets $\theta(p, q)$.

It follows from (3.2) that $(\theta(p), \theta(q))^- = \theta(p, q)^-$, whenever $p < q$. //

Using this lemma and III we prove (3.1). Given $\bar{\bar{B}} \in \bar{\bar{\mathscr{B}}}$, find $D, E \in \mathscr{B}$ such that $D \circ E \circ D \circ D \subseteq B$ and $E \subseteq D^{-1}$. Suppose $\langle A, B \rangle \in \bar{\bar{D}} \circ \bar{\bar{D}}$. There exists $Z \in \mathfrak{X}$ such that $\langle A, Z \rangle, \langle Z, B \rangle \in \bar{\bar{D}}$, and hence nets $(a_i)_{i \in I}$, $(z_i')_{i \in I}$, $(b_j)_{j \in J}$, $(z_j'')_{j \in J}$ such that

$$\theta(a_i) \to A, \qquad \theta(z_i') \to Z, \qquad \theta(b_j) \to B, \qquad \theta(z_j'') \to Z$$

and $\langle a_i, z_i' \rangle \in D$, $\langle z_j'', b_j \rangle \in D$ for all i, j. Let p, q, x be elements corresponding to this Z and E, as in III. Then there exist $i_0 \in I$ and $j_0 \in J$ such that $p < z_i', z_j'' < q$ and hence $z_i', z_j'' \in E[x]$ for $i \geq i_0$, $j \geq j_0$. This leads by composition of relations to $\langle a_i, b_j \rangle \in D \circ E \circ D \circ D \subseteq B$, so

$\langle A, B \rangle = \lim \langle \theta(a_i), \theta(b_j) \rangle \in \theta(B)^- = \bar{\bar{B}}$, proving (3.1).

Thus $\bar{\bar{\mathscr{B}}}$ is a base for a uniformity on $\mathfrak{X}$. Let $\mathscr{W}$ denote the uniform topology generated by this uniformity. To prove that $\mathscr{W} \subseteq \mathscr{T}$ it suffices to show that $\bar{\bar{B}}[A]$ is a $\mathscr{T}$-neighbourhood of A, for every $A \in \mathfrak{X}$, $B \in \mathscr{B}$.

So let A, B be given. Find successively C, D and E in $\mathscr{B}$ such that $C \circ C \subseteq B$, $D \subseteq C^{-1}$, $E \subseteq C \cap D$. For $Z = A$ and this E let p, q, x be elements as in III. We prove

(3.3) $\theta(p, q)^- \subseteq B[A].$

By 11, this implies that $\bar{\bar{B}}[A]$ is a neighbourhood of A, as required.

Let $F \in \theta(p, q)^-$. There exists a net $(f_i)_{i \in I}$ such that $\theta(f_i) \to F$ and $p < f_i < q$. Since $\theta(X)$ is dense in $\mathfrak{X}$, there is also a net $(a_j)_{j \in J}$ for which $\theta(a_j) \to A$, and some $j_0 \in J$ such that $\theta(p) \prec\!\!\!-\!\!\!\prec \theta(a_j) \prec\!\!\!-\!\!\!\prec \theta(q)$, i.e. $p < a_j < q$ for $j \geqq j_0$. So $f_i, a_j \in E[x]$ for all $i \in I$, $j \geqq j_0$. This implies $\langle a_j, f_i \rangle \in B$ and so $\langle A, F \rangle \in \theta(B)^-$, $F \in \bar{\bar{B}}[A]$, proving (3.3).

Finally, we prove $\mathscr{W} \supseteq \mathscr{T}$ under the further stated assumptions. Let $T \in \mathscr{T}$ and $A \in T$. By 8, the intervals $(\theta(p), \theta(q))$, $p < q$, form a base for $\mathscr{T}$, so there exist $p, q \in X$ such that

$$\theta(p) \prec\!\!\!-\!\!\!\prec A \prec\!\!\!-\!\!\!\prec \theta(q), \qquad (\theta(p), \theta(q)) \subseteq T.$$

Since II holds, so does I; because of 1(b) we can without loss of generality assume that

$$[\![\theta(p), \theta(q)]\!] \equiv \{C \in \mathfrak{X} : \theta(p) \subseteq C \subseteq \theta(q)\} \subseteq T$$

and that p_1, q_1 exist in X satisfying

$$\theta(p) \prec\!\!\!-\!\!\!\prec \theta(p_1) \prec\!\!\!-\!\!\!\prec A \prec\!\!\!-\!\!\!\prec \theta(q_1) \prec\!\!\!-\!\!\!\prec \theta(q).$$

Then $p_1, q_1 \in \theta^{-1}(T)$, an open subset of X by 5, so there exist $P, Q \in \mathscr{B}$ such that $P[p_1]$ and $Q[q_1]$ are contained by $\theta^{-1}(T) \cap (p, q)$. Let $R \in \mathscr{B}$, $R \subseteq P \cap Q$; then $R[p_1], R[q_1] \subseteq \theta^{-1}(T) \cap (p, q)$. We prove that

(3.4) $\bar{\bar{R}}[A] \subseteq T.$

Let $F \in \bar{\bar{R}}[A]$, i.e. $\langle A, F \rangle \in \theta(R)^-$. There exist nets $(a_i)_{i \in I}$, $(f_i)_{i \in I}$ such that $\theta(a_i) \to A$, $\theta(f_i) \to F$, $\langle a_i, f_i \rangle \in R$. For $i \geqq$ some i_0, $p_1 < a_i < q_1$; then IV gives $(\leqslant R[p_1]) \subseteq (\leqslant R[a_i])$. Since $p < R[p_1]$, we deduce $p \leqslant R[a_i]$, and similarly $R[a_i] \leqslant q$, for $i \geqq i_0$. Therefore $f_i \in R[a_i] \subseteq [\![p, q]\!]$, so

$$\theta(f_i) \in [\![\theta(p), \theta(q)]\!] \subseteq T \qquad \text{for} \quad i \geqq i_0.$$

Now $\subseteq$ is a closed partial order on $(\mathfrak{X}, \mathscr{T})$ (this follows from the fact (8, 2) that $(\mathfrak{X}, \underline{\underline{\prec}})$ is $TR(1, 1)$ and has property (Σ); cf. 2.4(b)), so $[\![\theta(p), \theta(q)]\!]$ is closed and therefore it contains $F = \lim \theta(f_i)$, so $F \in T$. This proves (3.4). Since $\bar{\bar{R}}[A]$ is a base $\mathscr{W}$-neighbourhood of A, we have proved $\mathscr{W} \supseteq \mathscr{T}$. This concludes the proof of 10. //

3.2. Having now accumulated the necessary structure on $\mathfrak{X}$ we are in a position to ask if $(\mathfrak{X}, \mathscr{T})$ is complete. For this we will apply to $(\mathfrak{X}, \subseteq, \mathscr{T})$ the following general result about topological completion in a conditionally complete lattice.

12. THEOREM: *Let Y be a set carrying a partial order $\leqslant$ and a topology $\mathscr{V}$. Suppose that*

(a) *$(Y, \leqslant)$ is conditionally lattice complete;*

(b) *$\mathscr{V}$ is generated by a uniformity having a base $\mathscr{B}$ such that all subsets $B[y]$ are order-bounded, and the subsets*

$$B[y]\sharp = [\inf B[y], \sup B[y]] \qquad (y \in Y, B \in \mathscr{B})$$

form a uniform neighbourhood base; i.e. for every $C \in \mathscr{B}$ there exists $B \in \mathscr{B}$ such that

(3.5) $$B[y]\sharp \subseteq C[y] \qquad \text{for all } y \in Y.$$

Then $(Y, \mathscr{V})$ is a complete topological space.

Here $[\cdot, \cdot]$ denotes the closed order interval in $(Y, \leqslant)$, so that $B[y]\sharp$ is the complete sublattice of Y generated by $B[y]$. Note that in (3.5), the one B must serve for all y. A stronger but simpler condition than (b) is:

(b') *$\mathscr{V}$ is generated by a uniformity having a base $\mathscr{B}$ such that $B[y]$ is a complete sublattice, for every $y \in Y$, $B \in \mathscr{B}$.*

Clearly, (b') is the case (b) with $B[y]\sharp = B[y]$ for all y.

PROOF OF THE THEOREM: Let $(y_i)_{i \in I}$ be a Cauchy net in $(Y, \mathscr{V})$. Given $B \in \mathscr{B}$, there exists i_0 such that

$$\langle y_i, y_j \rangle \in B \qquad \text{for } i, j \geqq i_0.$$

Then $y_i \in B[y_{i_0}]$ for $i \geqq i_0$, the net $(y_i)_{i \geqq i_0}$ is bounded and elements

$$\xi_i = \sup_{j \geqq i} y_j, \qquad \eta_i = \inf_{j \geqq i} y_j$$

exist for $i \geqq i_0$, and belong to $\boldsymbol{B}[y_{i_0}]\sharp$. For $j \geqq i$ we have $\xi_j \leqslant \xi_i$ and $\eta_j \geqslant \eta_i$. It is easily deduced that elements

$$\xi = \inf_{i \geqq i_0} \xi_i, \qquad \eta = \sup_{i \geqq i_0} \eta_i$$

exist in Y and are independent of i_0. Indeed

$$\xi, \eta \in \boldsymbol{B}[y_i]\sharp \qquad \text{for all } i \geqq i_0.$$

Given any $\boldsymbol{D} \in \mathscr{B}$, find $\boldsymbol{C} \subseteq \boldsymbol{D}^{-1}$ and then $\boldsymbol{B}$ as in (3.5)', and i_0 for this $\boldsymbol{B}$. By (b') we have $i \geqq i_0 \Rightarrow \xi \in \boldsymbol{B}[y_i]\sharp \Rightarrow \xi \in \boldsymbol{C}[y_i] \Rightarrow y_i \in \boldsymbol{D}[\xi]$. Thus $(y_i)_{i \in I}$ converges to ξ; so $(Y, \mathscr{V})$ is complete. (The net also converges to η; and $\eta = \xi$ if $\mathscr{V}$ is Hausdorff.)

To invoke 12 for $(\mathfrak{X}, \subseteq, \mathscr{T})$ we therefore need a further condition; say:

V'. $\overline{\boldsymbol{B}}[A]$ is a complete sublattice of $(\mathfrak{X}, \subseteq)$ for all $A \in \mathfrak{X}$, $\boldsymbol{B} \in \mathscr{B}$ (using (b') rather than (b)). Then we have:

13. THEOREM: *Let $(X, \leqq, \leqslant, \mathscr{U})$, $(\mathfrak{X}, \underline{\prec}, \subseteq, \mathscr{T})$ and θ be as in the first paragraph of Theorem 9. Suppose in addition that $(X, \leqslant)$ is a lattice, and $\mathscr{U}$ is generated by a uniformity with base $\mathscr{B}$. Suppose that II, III, IV and V' (and therefore I) hold.*

Then $(\mathfrak{X}, \mathscr{T})$ is a complete Hausdorff topological space, θ is a topological embedding of X as a dense subset of $\mathfrak{X}$, and $(\mathfrak{X}, \mathscr{T})$ is, to within a uniform isomorphism, the topological completion of $(X, \mathscr{U})$.

4. Remarks on the various conditions

4.1. Condition I, and hence II, is not a consequence of 2.2(i)–(iii) even when $(X, \leqq)$ is also a pogroup and $(X, \leqslant)$ is a para-archimedean l-group. For take $X = C[0, 1]$, ordered as in 2.3(2). This X meets the requirements just stated, but not I. Every lower-semicontinuous function on $[0, 1]$ can be written as the supremum of a subset of $C[0, 1]$; take k to be the lower-semicontinuous function.

$$k(t) = \begin{cases} (2^n - 1)2^{-n} & \text{if } (2^n - 1)2^{-n} < t \leqq (2^{n+1} - 1)2^{-n-1}, \\ & \qquad\qquad\qquad\qquad n = 0, 1, 2, \cdots, \\ 0 & \text{if } t = 1, \end{cases}$$

and $A = A^* = \{f \in C[0, 1] : f \leqslant k\}$, $g(t) = 1$ for all $t \in [0, 1]$. Then

$A < g$ but $A < h < g$ for no $h \in C[0, 1]$.

4.2. Condition III can be reformulated in terms of a statement about uniform neighbourhoods of subsets A°. Given any $S \subseteq X$, by a (base) uniform neighbourhood of S is meant a subset of X of the form

$$B[S] = \bigcup \{B[x] : x \in S\}$$

for some $B \in \mathscr{B}$. In particular we can form $B[A^\circ]$ for $A \in \mathfrak{X}$. Note that

(4.1) $\bar{A} = A \subseteq B[A^\circ]$ for all $A \in \mathfrak{X}$.

For let $x \in A$. Since $\overline{A^\circ} = A$, $C[x]$ meets A° for all $C \in \mathscr{B}$. Choose $C \subseteq B^{-1}$; if $y \in C[x] \cap A^\circ$ then $x \in B[y]$, $y \in A^\circ$, so $x \in B[A^\circ]$. Introduce the conditions
   III*. For every $A \in \mathfrak{X}$ and $B \in \mathscr{B}$, $B[A^\circ]$ meets $(A <)$.
   VI.   Every subset $B[x]$ is $\leqq$-convex.
Then

14. (i) III implies III*. (ii) III* and VI imply III.

PROOF: (i) Given $A \in \mathfrak{X}$ and $B \in \mathscr{B}$, find $C, D \in \mathscr{B}$ such that $C \circ C^{-1} \subseteq B$, $D^{-1} \circ D \subseteq C$, and then $p, q, x \in X$ such that

$$\theta(p) \prec A \prec \theta(q), \qquad (p, q) \subseteq D[x].$$

Since $p \in D[x]^-$, $D[p]$ meets $D[x]$ and hence $p \in C[x]$. Similarly $q \in C[x]$, so $q \in B[p]$; and $p \in A^\circ$, so $q \in B[A^\circ] \cap (A <)$.
   (ii) Let $q \in B(A^\circ) \cap (A <)$; there exists $p \in A^\circ$ such that $q \in B[p]$ and $\theta(p) \prec A \prec \theta(q)$. Since $p \in B[p]$, we have $(p, q) \subseteq B[p]$. //

4.3. Condition IV can be interpreted as saying that the sets $B[x]$ are 'similar in their order-theoretic shape', as x varies through X. Note that IV (i) is equivalent to: $x < y$ implies $B[x]^* \subseteq B[y]^*$. Conditions IV (i) and IV (ii) are equivalent under the further assumption:

(4.2)   For all $x, y \in X$, $x \leqslant B[y]$ iff $B[x] \leqslant y$.

This can be interpreted as saying that x is 'centrally placed' in $B[x]$, for each x.

4.4. Conditions I and II, although they refer to the penetration of $\theta(X)$ in $\mathfrak{X}$, are formulated directly in terms of X. Conditions III and IV

can similarly be regarded as statements about X and $\mathscr{B}$. However, V′ appears to involve essential properties of $\mathfrak{X}$, not capable of simple reformulation in terms of X and its structure.

V′ could be weakened to a condition V based on 12. (b) rather than 12. (b′), without necessitating a change in the statement of Theorem 13. In this connection it is worth remarking that a weaker version of 12. (b) can be proved to hold in $\mathfrak{X}$, in the presence of the other assumptions; namely:

VII. For every $\bar{\bar{C}} \in \bar{\bar{\mathscr{B}}}$ and $A \in \mathfrak{X}$ there exists $\bar{\bar{B}} \in \bar{\bar{\mathscr{B}}}$ such that $\bar{\bar{B}}[A]\sharp \subseteq \bar{\bar{C}}[A]$. But here $\bar{\bar{B}}$ depends upon A. The proof of VII rests on the fact that every interval $[\![P, Q]\!]$ in $(\mathfrak{X}, \subseteq)$ is lattice complete. But the author has not succeeded in deducing V′ or V from the other conditions, or from further reasonable conditions placed on X and $\mathscr{B}$.

5. Completion of tight Riesz groups

5.1. The conditions to date are mostly fulfilled when X is a group. More precisely, let $(X, \leqq)$ be a pogroup satisfying 2.2 (i) and (ii), i.e. a tight Riesz group with associated partial order $\leqslant$. Then (cf [5], § 2) $(X, \leqslant)$ is a pogroup, 2.2 (iii) is satisfied, $(X, \mathscr{U})$ is a Hausdorff topological group, and the intervals $(-a, a)$, $a > 0$, form a base at 0 for $\mathscr{U}$. Give X the two-sided uniformity generating $\mathscr{U}$: for this we can take $\mathscr{B}$ to be the set of all vicinities of the forms

$$\begin{aligned} {}_a\boldsymbol{B} &= \{\langle x, y\rangle : -a < -x+y < a\}, \\ \boldsymbol{B}_a &= \{\langle x, y\rangle : -a < x-y < a\} \end{aligned}$$

with $a > 0$. Then for example $\boldsymbol{B}_a[x] = (-a+x, a+x)$,

$$(\leqslant \boldsymbol{B}_a[x]) = (\leqslant -a+x),$$

and IV certainly holds. Moreover, VI holds, since $\boldsymbol{B}_a[x]$ and ${}_a\boldsymbol{B}[x]$ are clearly convex.

If II holds and X is an l-group, then $\boldsymbol{B}_a[A^\circ]$ and ${}_a\boldsymbol{B}[A^\circ]$ meet $(A <)$, for all $a > 0$, $A \in \mathfrak{X}$, so III* holds. For given a and A, $x \in \boldsymbol{B}_a[A^\circ]$ iff $-a+y < x < a+y$ for some $y \in A^\circ$. Choose $0 < c < b < a$ and y in X so that $A - \theta(c) \prec\!\!\!- \theta(y) \prec\!\!\!- A$; then $y \in A^\circ$. Let $x = b+y$; we have $x \in \boldsymbol{B}_a[A^\circ] \cap (A <)$. Since III* and VI hold, III holds.

Under reasonable assumptions it can be proved that

(5.1) $\bar{\bar{\boldsymbol{B}}}_a[A] = [\![A - \theta(a), A + \theta(a)]\!]$ for all $a > 0$, $A \in \mathfrak{X}$,

so V′ holds. Sufficient conditions to ensure (5.1) are: $(\mathfrak{X}, \subseteq)$ is an *l*-group and $(\mathfrak{X}, =\!\!\prec)$ is $TR(2, 2)$. (One uses the fact that $\subseteq$ is closed and that $[\![P, Q]\!] = (P, Q)^-$ for all $P\!\!-\!\!\prec Q$ in $\mathfrak{X}$.) Thus V′ holds, for example, if $(X, \leqslant)$ is a para-archimedean *l*-group and II holds. The para-archimedean property ensures that $\mathfrak{X}$ has a group structure; it also implies that X and $\mathfrak{X}$ are commutative groups. $\mathfrak{X}$ being a group, $(\mathfrak{X}, \subseteq)$ is certainly an *l*-group.

To sum up, we have

15. THEOREM: *Let $(X, \leq, \mathcal{U})$ be a tight Riesz group whose associated order $\leqslant$ is a para-archimedean lattice order, and let $(\mathfrak{X}, \subseteq)$ be the lattice completion of $(X, \leqslant)$ with topology $\mathcal{T}$ as previously defined. If II holds, then $(\mathfrak{X}, \mathcal{T})$ is a complete Hausdorff topological group, the completion of $(X, \mathcal{U})$.*

16. COROLLARY: *If $(X, \leq, \mathcal{U})$ is a tight Riesz group such that the associated order $\leqslant$ is a para-archimedean and $(X, \leqslant)$ is a complete lattice, then $(X, \mathcal{U})$ is a complete Hausdorff topological group.*

REFERENCES

[1] B. BANASCHEWSKI: Über die Vervollständigung geordneter Gruppen. *Math. Nach-richten 16* (1957) 51–71

[2] G. BIRKHOFF: *Lattice theory.* (Amer. Math. Soc., Providence, 3rd ed. 1967).

[3] N. CAMERON and J. B. MILLER: Topology and axioms of interpolation in partially ordered spaces. *J. für die reine u. angewandte Math.* (to appear).

[4] L. FUCHS: *Partially ordered algebraic systems.* (Pergamon, 1963).

[5] R. J. LOY and J. B. MILLER: Tight Riesz groups. *J. Australian Math. Soc. 13* (1972) 224–240.

[6] H. M. MACNEILLE: Partially ordered sets. *Trans Amer. Soc. 42* (1937) 416–460.

[7] A. L. PERESSINI: *Ordered topological vector spaces.* (Harper and Row, New York, 1967).

[8] R. H. REDFIELD: Ordering uniform completions of partially ordered sets. *Canadian J. Math. 26* (1974) 644–664.

[9] R. H. REDFIELD: *Uniform subcompletions of Dedekind completions.* (preprint, Simon Fraser University, Burnaby, B.C.)

[10] N. R. REILLY: Representations of ordered groups with compatible tight Riesz orders. *J. Aust. Math. Soc.* (to appear).

[11] P. RIBENBOIM: *Théorie des groupes ordonnés.* (Universidad Nacional del Sur, Bahia Blanca, 1960).

[12] B. F. SHERMAN: Cauchy completion of partially ordered groups. *J. Aust. Math. Soc.* (to appear).

[13] A. WEIL: *Sur les espaces à structure uniforme et sur la topologie générale.* Actualités Sci. Ind. 551 (Paris, 1937).