COMPOSITIO MATHEMATICA

S. COEN

Factoriality of a ring of holomorphic functions

Compositio Mathematica, tome 29, nº 2 (1974), p. 191-196
<http://www.numdam.org/item?id=CM_1974__29_2_191_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 2, 1974, pag. 191–196
Noordhoff International Publishing
Printed in the Netherlands

FACTORIALITY OF A RING OF HOLOMORPHIC FUNCTIONS

S. Coen*

A non negative *Cousin II distribution* $\mathscr{D} = \{U_i, f_i\}_{i \in I}$ on a complex (reduced) space X is the datum of an open covering $\{U_i\}_{i \in I}$ of X and, for every $i \in I$, of a holomorphic function $f_i : U_i \to \mathbb{C}$ with the property that for $U_i \cap U_j \neq \emptyset$ one has, on $U_i \cap U_j$, $f_i = \eta_{ij} f_j$ where η_{ij} is a holomorphic function on $U_i \cap U_j$ and η_{ij} is nowhere zero. A solution of $\mathscr{D}$ is a holomorphic function $f : X \to \mathbb{C}$ such that $f = \eta_i f_i$ on every U_i when $\eta_i : U_i \to \mathbb{C}$ is a nowhere zero, holomorphic function. One says that X is a *Cousin-II space* when every non negative Cousin-II distribution on X has a solution; if $K \subset X$ we say that K is a *Cousin-II set* when K has, in X, an open neighborhood base $\mathscr{U}$ and every $U \in \mathscr{U}$ is a Cousin-II space.

The object of this note is to prove the following

THEOREM: *Let X be a complex manifold; let K be a semianalytic connected compact Cousin-II subset of X; then the ring of the holomorphic functions on K is a unique factorization domain.*

Observe that, in the preceding conditions, K is not necessarily a Stein set.

In the proof we shall use some results of J. Frisch [1].

The proof is based on two lemmas. We begin with some notations and definitions.

Let X be a complex space; the structure sheaf of X is denoted by $\mathcal{O}_X$ or $\mathcal{O}$. Let Y be a (closed) analytic subset of X; we say that a *function* $f \in H^0(X, \mathcal{O})$ is *associated*, in X, *to* Y if for every $x \in X$ the germ f_x generates the ideal of Y in x, Id Y_x. If $g \in H^0(X, \mathcal{O})$ we set $Z(X, g) := \{x \in X | g(x) = 0\}$; we write also $Z(g)$ to denote $Z(X, g)$.

LEMMA 1: *Let X be a complex space, $f \in H^0(X, \mathcal{O})$; let $\mathscr{C} = \{C_i\}_{i \in I}$ ($I = \{1, \cdots, m\}$ $m \leqq \infty$) be the family of the analytic irreducible components of $Z(X, f)$ and suppose that $\mathscr{C}$ does not contain any irreducible*

* While preparing this work the author had a fellowship of C.N.R. and was part of G.N.S.A.G.A. of C.N.R.

component of X. *For every* $i \in \{1, \cdots, d\} \subset I$ *let a function* g_i *be given associated, in* X, *to* C_i.

Then g_i *is irreducible in the ring* $H^0(X, \mathcal{O})$ *for* $1 \leqq i \leqq d$. *Furthermore there are d natural numbers* $\lambda_1, \cdots, \lambda_d$ *such that the following factorization holds*

$$f = k_d(g_1)^{\lambda_1} \cdots (g_d)^{\lambda_d}$$

and $k_d(x) \neq 0$ *for every* $x \in X - \bigcup_{j \geqq d+1} C_j$.

Proof: Every g_i is irreducible in $H^0(X, \mathcal{O})$. Indeed, if $g_i = ab$ with $a, b \in H^0(X, \mathcal{O})$ then we may suppose without loss of generality $Z(X, a) = C_i$, because C_i is irreducible. Let $z \in C_i$; we call $\mathcal{M}_z$ the maximal ideal of $\mathcal{O}_z$; there is a suitable natural number s such that $(g_i)_z \in \mathcal{M}_z^{s-1}$, $(g_i)_z \notin \mathcal{M}_z^s$; by the definition of g_i there exists a germ $c_z \in \mathcal{O}_z$ for which $a_z = (g_i)_z c_z$; it follows that $(g_i)_z = (g_i)_z c_z b_z$ and hence $b_z \notin \mathcal{M}_z$; therefore b is a unit in $H^0(X, \mathcal{O})$.

We will prove the assumptions by induction on d. Let $d = 1$. Since f vanishes on C_1, it follows from the definition of g_1 that there exists $k \in H^0(X, \mathcal{O})$ such that

$$f = kg_1.$$

We observe that if k vanishes in a point $z' \in C_1 - \bigcup_{j \geqq 2} C_j$, then k vanishes on the whole of C_1. Indeed, if U is a neighborhood of z' in X with $U \cap (\bigcup_{j \geqq 2} C_j) = \emptyset$, then we have $U \cap Z(k) = U \cap Z(k) \cap C_1$; hence from the equality of analytic set germs

$$Z(k)_{z'} = (Z(k) \cap C_1)_{z'}$$

it follows $\operatorname{codim}_{\mathbb{C}}(Z(k) \cap C_1)_{z'} = 1 = \operatorname{codim}_{\mathbb{C}}(C_1)_{z'}$; as C_1 is irreducible, it follows $Z(k) \cap C_1 = C_1$.

Then there is $k' \in H^0(X, \mathcal{O})$ with $k = k'g_1$, $f = (k')g_1^2$. If k' vanishes in $z'' \in C_1 - \bigcup_{j \geqq 2} C_j$, then $k'_{|C_1} = 0$; but since f has a zero of finite order in every point of C_1, a number $\lambda_1 \geqq 1$ must exist such that

$$f = k_1(g_1)^{\lambda_1}$$

with $k_1 \in H^0(X, \mathcal{O})$ and $k_1(z'') \neq 0$; it follows that $k_1(x) \neq 0$ for every $x \in C_1 - \bigcup_{j \geqq 2} C_j$, hence $k_1(x) \neq 0$ for $x \in X - \bigcup_{j \geqq 2} C_j$.

Suppose that the thesis holds for $d-1$. We have

$$f = k_{d-1}(g_1)^{\lambda_1} \cdots (g_{d-1})^{\lambda_{d-1}}$$

with $k_{d-1}(x) \neq 0$ for every $x \in X - \bigcup_{j \geqq d} C_j$.

One has $k_{d-1}(x) = 0$ for every $x \in C_d$; therefore there is $h \in H^0(X, \mathcal{O})$ such that $k_{d-1} = hg_d$, $f = h(g_1)^{\lambda_1} \cdots (g_{d-1})^{\lambda_{d-1}} g_d$. If $h(w) = 0$ at a point

$w \in C_d - \bigcup_{j \geqq d+1} C_j$, then h vanishes on C_d; indeed, $Z(h)_w = (Z(h) \cap C_d)_w$. As in the case $d = 1$, we deduce that there exists $\lambda_d \geqq 1$ with

$$f = k_d(g_1)^{\lambda_1} \cdots (g_{d-1})^{\lambda_{d-1}}(g_d)^{\lambda^d}$$

where $k_d \in H^0(X, \mathcal{O})$ and $k_d(x) \neq 0$ for every $x \in C_d - \bigcup_{j \geqq d+1} C_j$, as required.

Let f be a holomorphic function on a complex space X; let $K \subset X$; f_K denotes the 'holomorphic function' induced by f on K; i.e. f_K is the class of the functions g which are holomorphic in a neighborhood of K and such that $g_{|U} = f_{|U}$ in some neighborhood $U = U_g$ of K in X.

The ring of all the holomorphic functions on K is denoted by $H(K)$.

We say that the *property* (C) holds on X when, for every holomorphic function f on X, every irreducible analytic component D of $Z(X, f)$ has, in X, an associated function; it is said that the property (C) holds on K when K has an open neighborhood base $\mathscr{B}$ such that the property (C) holds on every $B \in \mathscr{B}$.

LEMMA 2: *Let X be a complex space; assume that $f \in H^0(X, \mathcal{O})$ does not vanish identically on any irreducible analytic component of X. Let K be a compact semianalytic subset of K and let the property (C) hold on K.*

Then f_K is uniquely expressible as finite product of irreducible elements of $H(K)$, except for units and for the order of the factors.

PROOF:

(I) *Existence of the factorization.* Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open neighborhood base of K in X such that on every $U \in \mathscr{U}$ the property (C) holds.

Let $\mathscr{C}_1$ be the family of the irreducible components of $Z(U_1, f_{|U_1})$ which contain points of K. As K is compact, $\mathscr{C}_1$ has only a finite number of elements. Lemma 1 proves that, without loss of generality, we may suppose that $f_{|U_1}$ is irreducible in $H^0(U_1, \mathcal{O})$; indeed, there is a factorization $f = uq_1 \cdots q_r$ where $q_1, \cdots, q_r$ are irreducible elements of $H^0(U_1, \mathcal{O})$ and $u \in H^0(U_1, \mathcal{O})$ induces on K a function u_K that is a unit in $H(K)$.

Assume that there is no factorization of f_K as required by the thesis.

Under this hypothesis we shall prove, in the following step (a), that there is a suitable sequence $\{U_h, x_h\}_{h \geqq 1}$ with $U_h \in \mathscr{U}$ and $x_h \in U_h$ satisfying certain properties specified below and, later in (b), we shall prove that the existence of this sequence, implies a contradiction.

(a) We define a sequence $\{U_h, x_h\}_{h \geqq 1}$ such that $U_h \in \mathscr{U}$ and $x_h \in U_h \cap K$ for every $h \geqq 1$; furthermore we impose that the four following properties hold for every $h \geqq 1$:

(1) there are $m_h \geqq 1$.functions $g_1^{(h)}, \cdots, g_{m_h}^{(h)}$ holomorphic on U_h and irreducible in $H^0(U_h, \mathcal{O})$ such that on U_h

$$f = k^{(h)}(g_1^{(h)})^{\lambda_{h,1}} \cdots (g_{m_h}^{(h)})^{\lambda_{h,m}}$$

where $\lambda_{h,1}, \cdots, \lambda_{h,m}$ are non zero natural numbers and $k^{(h)} \in H^0(U_h, \mathcal{O})$ is such that $k^{(h)}(x) \neq 0$ for every $x \in K$;

(2) if $\mathscr{C}_h$ is the family of the irreducible components C of $Z(U_h, f_{|U_h})$ such that $C \cap K \neq \emptyset$, then $\mathscr{C}_h = \{Z(U_h, g_1^{(h)}), \cdots, Z(U_h, g_{m_h}^{(h)})\}$ and $\mathscr{C}_h$ has exactly m_h elements;

(3) $m_h \geqq h$;

(4) there are h different elements $C_1, \cdots, C_h$ of $\mathscr{C}_h$ with $x_1 \in C_1, \cdots, x_h \in C_h$.

To define (U_1, x_1) it suffices to call x_1 an arbitrary point of $Z(U_1, f_{|U_1}) \cap K$.

Now, we assume $(U_1, x_1), \cdots, (U_{h-1}, x_{h-1})$ already defined and we define (U_h, x_h).

First of all we choose U_h. Since f_K has no finite decomposition as product of irreducible elements of $H(K)$, there is, at least, a function $g_j^{(h-1)}$, say $g_1^{(h-1)}$, which satisfies the following properties. On an open set $V \in \mathscr{U}$, $V \subset U_{h-1}$ two holomorphic functions a, b are given such that $g_1^{(h-1)} = ab$, $Z(V, a) \cap K \neq \emptyset$, $Z(V, b) \cap K \neq \emptyset$; furthermore there is a point $x \in K$ such that $a(x) \neq 0$, $b(x) = 0$. We set $U_h := V$. Certainly $\mathscr{C}_h$ is a finite set; let m_h be the cardinality of $\mathscr{C}_h$; for every $G_j \in \mathscr{C}_h$ $(j = 1, \cdots, m_h)$, we call $g_j^{(h)}$ a holomorphic function on U_h, associated to G_j. Property (2) is true. Lemma 1 implies that also property (1) holds. For every $j \in \{1, \cdots, m_{h-1}\}$ we write briefly C_j to indicate $Z(U_{h-1}, g_j^{(h-1)})$; let $D_j^1, \cdots, D_j^{i_j}$ be the different irreducible components of $C_j \cap U_h$, such that $D_j^i \cap K \neq \emptyset$ for every $i = 1, \cdots, i_j$. Certainly $i_j \geqq 1$ for every such j; furthermore the definition of U_h implies $i_1 > 1$.

The analytic sets D_j^i are 1-codimensional in U_h (we consider the complex dimension); hence they are irreducible components of $Z(U_h, f_{|U_h})$ i.e. they are elements of $\mathscr{C}_h$. If $D_j^i = D_t^s$ then $(i, j) = (s, t)$; indeed, there exist points which are both in C_i and in C_t and which are regular in $Z(U_{h-1}, f_{|U_{h-1}})$. It follows $m_h \geqq m_{h-1} + 1$, that is condition (3) is fulfilled.

Let $C_1, \cdots, C_{h-1}$ be different elements of $\mathscr{C}_{h-1}$ such that

$$x_1 \in C_1, \cdots, x_{h-1} \in C_{h-1};$$

for each $j = 1, \cdots, h-1$ the analytic subset $C_j \cap U_h$ of U_h contains at least one element D_j of $\mathscr{C}_h$ such that $x_j \in D_j$; it was previously observed that $D_1, \cdots, D_{h-1}$ are different one from the other. Let $D \in \mathscr{C}_h$ such that $D \neq D_j$ for every $j \in \{1, \cdots, h-1\}$; pick a point x_h in $D \cap K$.

With this definition $\{(U_1, x_1), \cdots, (U_h, x_h)\}$ satisfies (1), (2), (3) and also (4).

In step (b) we shall find a contradiction.

(b) Let x be a cluster point of the sequence $\{x_h\}_{h \geqq 1}$; since K is compact,

$x \in K$. We call $\mathcal{M}_x$ the maximal ideal of $\mathcal{O}_x$; $f_x \in \mathcal{M}_x$; let $s \in \mathbb{N}$ be such that $f_x \in \mathcal{M}_x^{s-1}$, $f_x \notin \mathcal{M}_x^s$.

Let $\mathcal{T}$ be the coherent ideal sheaf on U_1 associated to the analytic subset $Z(U_1, f_{|U_1})$ of U_1 and let $\mathcal{F} = \mathcal{O}/\mathcal{T}$.

With regard to $\mathcal{F}$ there is an open neighborhood base $\mathcal{B}$ of x in K such that every $V \in \mathcal{B}$ has an open neighborhood base $\mathcal{X}_V$ in X so that every $W \in \mathcal{X}_V$ is an $\mathcal{F}$-privileged neighborhood of x. The existence of $\mathcal{B}$ and of the families $\mathcal{X}_V$ is proved in ([1], théor. (1.4), (1.5)).

Let $V \in \mathcal{B}$; V contains s terms $x_{i_1}, \cdots, x_{i_s}$ with $i_1 < \cdots < i_s$.

Let us consider U_{i_s}; because of property (4), there are s irreducible components $C_1, \cdots, C_s$ of $Z(U_{i_s}, f_{|U_{i_s}})$ different one from the other such that $x_{i_j} \in C_j$ for every $j = 1, \cdots, s$. Let S be the union of the irreducible components of $Z(U_{i_s}, f_{|U_{i_s}})$ which contain x; because of (2) for everyone of these components C there is a suitable t, $1 \leqq t \leqq s$, such that $C = Z(U_{i_s}, g_t^{(i_s)})$. But the functions $g_j^{(i_s)}$ vanishing in x may not be more than $s-1$, because of (1). Hence we may suppose, without loss of generality, that C_1 is not contained in S. Let $W \in \mathcal{X}_V$ be an $\mathcal{F}$-privileged neighborhood of V in X such that $W \subset U_{i_s}$ and let $y \in W \cap C_1$, $y \notin S$. Property (C) implies that a function $F \in H^0(U_{i_s}, \mathcal{O})$ exists that vanishes in the points of S and only in these points. The function F induces a section $F' \in H^0(W, \mathcal{F})$; F' is not the zero section of $\mathcal{F}$ on W because $F(y) \neq 0$; but $F'_x \in \mathcal{F}_x$ is the zero germ. This is a contradiction.

(II) *Uniqueness of the decomposition* in irreducible factors. Let

$$f_K = \delta\alpha_1^{m_1} \cdots \alpha_t^{m_t}$$

$$f_K = \lambda\beta_1^{p_1} \cdots \beta_s^{p_s}$$

where δ, λ are units in $H(K)$ and $\alpha_1, \cdots, \alpha_t$, $\beta_1, \cdots, \beta_s$ are irreducible elements of $H(K)$. Furthermore assume that, for $i \neq j$ and $1 \leqq i, j \leqq t$, α_i and α_j, are not associated elements in $H(K)$ and that, for $i \neq j$ and $1 \leqq i, j \leqq s$, β_i and β_j are not associated elements in $H(K)$.

In a suitable neighborhood $U \in \mathcal{U}$ there are holomorphic functions $d, a_1, \cdots, a_t, l, b_1, \cdots, b_s$ such that they induce respectively $\delta, \alpha_1, \cdots, \alpha_t$, $\lambda, \beta_1, \cdots, \beta_s$ and on U

$$f = da_1^{m_1} \cdots a_t^{m_t} = lb_1^{p_1} \cdots b_s^{p_s}.$$

Each a_i, for $i = 1, \cdots, t$, defines an irreducible component V_i of $Z(U, a_i)$ such that $V_i \cap K \neq \emptyset$. Such a V_i is unique; indeed, if $Z_i \neq V_i$ is an irreducible component of $Z(U, a_i)$ and $Z_i \cap K \neq \emptyset$, then, by lemma 1 and by recalling that property (C) holds on U, we can write $a_i = vgh$ where v, g, h are holomorphic functions on U and g, h are respectively associated in U to V_i and to Z_i; it follows that $\alpha_i = v_K g_K h_K$; this is a contradiction, because g_K and h_K are not units in $H(K)$. Let $i, j \in \{1, \cdots, t\}$, $i \neq j$, then $V_i \neq V_j$. Assume, in fact, $V_i = V_j$; let g_i be a holomorphic function associated in U to V_j; since g_i divides a_i and a_j in $H(U, \mathcal{O})$, it follows that α_i, α_j are associated elements in $H(K)$. Every V_i is an irreducible component of $Z := Z(U, f_{|U})$; indeed, if $x \in V_i$, then $\dim_{\mathbb{C}}(V_i)_x = \dim_{\mathbb{C}} X_x - 1 = \dim_{\mathbb{C}} Z_x$; vice versa for every irreducible component C of Z such that $C \cap K \neq \emptyset$, there is one $i \in \{1, \cdots, t\}$ such that $C = V_i$. Every b_j defines, for $j = 1, \cdots, s$, one and only one irreducible component W_j of $Z(U, b_j)$ such that $W_j \cap K \neq \emptyset$; by what we have just proved it follows that $t = s$ and, eventually changing the indicization of $b_1, \cdots, b_s$, $V_i = W_i$ for every $i \in \{1, \cdots, t\}$. Then we can deduce that for each i the elements α_i, β_i are associated in the ring $H(K)$.

Pick, now, $i \in \{1, \cdots, t\}$ let $x \in V_i$ be a regular point of Z with $d(x) \neq 0$, $l(x) \neq 0$ and let g_i be a function associated to V_i in U. Recall that V_i is an irreducible component of Z; by lemma 1, $a_i = s_i g_i^r$ and $s_i(x) \neq 0$; by the irreducibility of α_i it follows $r = 1$. Furthermore $a_j(x) \neq 0$, $b_j(x) \neq 0$ for $j \neq i$. Then on U we can write

$$f = f_1 g_i^{m_i} = f_2 g_i^{p_i}$$

where $f_1, f_2 \in H^0\ (U, \mathcal{O})$, $f_1(x) \neq 0$, $f_2(x) \neq 0$; it follows that $m_i = p_i$. The lemma is proved.

Note that property (C) holds on each Cousin-II space U in which the local rings $\mathcal{O}_{U,y}$ are U.F.D. (= unique factorization domains) for every $y \in U$. Thus, we have proved that if K is a semianalytic, connected, compact Cousin-II subset of a complex space X and if K has an open neighborhood U in X so that the rings $\mathcal{O}_y$ are U.F.D. for every $y \in U$, then $H(K)$ is an U.F.D.

By a different way, in [2] a criterion is given for the factoriality of $H(K)$ in the case that K is a compact Stein set and that every element of $H(K)$ is expressible as product of irreducible elements.

REFERENCES

[1] J. Frisch: Points de platitude d'un morphisme d'espaces analytiques complexes. *Inv. Math. 4* (1967) 118–138.

[2] K. Langmann: Ringe holomorpher Funktionen und endliche Idealverteilungen. *Schriftenreihe des Math. Inst. Münster* (1971) 1–125.

(Oblatum 8–V–1974)

Università di Pisa
Istituto di Matematica
Via Derna, 1 – Pisa (Italy)