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Introduction

In this paper we discuss closures of open analytic polyhedra. The main problem is to find conditions on the defining functions for the polyhedron under which the closure is a holomorphic set.

A complex analytic manifold of dimension n is called a Stein manifold if three conditions are satisfied: $\mathrm{Hol}(M)$, the collection of holomorphic functions on M, separates the points of M; for every $x \in M$ there exist n functions in $\mathrm{Hol}(M)$ which provide local coordinates at x; finally for every compact subset K of M, $\mathrm{hull}_{\mathrm{Hol}(M)}(K) = \{x \in M : |f(x)| \leqq \|f\|_K$ for all $f \in \mathrm{Hol}(M)\}$ is a compact subset of M. Here $\|\,.\,\|_K$ denotes the supremum norm on K. A compact set K in M is called $\mathrm{Hol}(M)$ – convex if $K = \mathrm{hull}_{\mathrm{Hol}(M)}(K)$. Note that an open subset M of $\mathbb{C}^n$ is a Stein manifold if the last condition in the above definition is satisfied. In this paper we only consider Stein manifolds which are open subsets of $\mathbb{C}^n$.

A compact subset K of $\mathbb{C}^n$ is called a holomorphic set if K is the intersection of Stein manifolds in $\mathbb{C}^n$. K is said to be holomorphically convex if K is the continuous homomorphism space of the function algebra $H(K)$ consisting of uniform limits on K of restrictions to K of functions holomorphic in a neighborhood of K. From a characterization by Birtel, [3], of holomorphically convex subsets of $\mathbb{C}^n$ it is evident that holomorphic sets are also holomorphically convex. The converse is not true, see [4], [12].

One can show that the interior of a holomorphically convex set in $\mathbb{C}^n$ is a Stein manifold, [2]. We are interested in the converse statement: is the closure of a relatively compact Stein manifold in $\mathbb{C}^n$ always a holomorphically convex set? Or perhaps a holomorphic set? The answer is no; take for M

$$\{(z, w) \in \mathbb{C}^2 : |z| < |w|, \tfrac{1}{2} < |w| < 1\} \cup \{(z, w) \in \mathbb{C}^2 : |w| < |z|, \tfrac{1}{2} < |z| < 1\}.$$

Now the closure $\overline{M}$ of M contains the topological boundary of the closed unit polydisc in $\mathbb{C}^2$ and the origin in $\mathbb{C}^2$ is not contained in $\overline{M}$. So the smallest holomorphic set containing $\overline{M}$ is the closed unit polydisc in $\mathbb{C}^2$ and hence does not coincide with $\overline{M}$.

We here try to solve this problem for special types of Stein manifolds: open analytic polyhedra. An open analytic polyhedron in a Stein manifold M is a relatively compact subset of M of the form

$$\{x \in M: |f_i(x)| < 1, i = 1, \cdots, N\},$$

where $f_1, \cdots, f_N \in \mathrm{Hol}(M)$ and $N \geqq n$. In a similar way one defines a compact analytic polyhedron in M. This is a compact subset of M of the form

$$\{x \in M: |f_i(x)| \leqq 1, i = 1, \cdots, N\},$$

where $f_1, \cdots, f_N \in \mathrm{Hol}(M)$ and $N \geqq n$. In case $N = n$, the polyhedron is called a special analytic polyhedron. For convenience we will always assume that none of the defining functions are constant on any of the components of M. Note that an open analytic polyhedron is a Stein manifold and that a $\mathrm{Hol}(M)$-convex subset K of a Stein manifold M is the intersection of open analytic polyhedra, so a holomorphic set. For information on (special) analytic polyhedra, see [1].

It is easy to see that the closure of an open analytic polyhedron

$$P = \{x \in M: |f_i(x)| < 1, i = 1, \cdots, N\}$$

is not always a compact analytic polyhedron defined by the same functions as the open one. Just take $M = \mathbb{C}$, $N = 2$, $f_1 = (2z+1)/3$, $f_2 = (z^2-3)/2$. Here $Q = \{z \in \mathbb{C}: |f_1(z)| \leqq 1, |f_2(z)| \leqq 1\}$ is not connected and the closure of P consists of one of the components of Q.

It can also happen that $Q = \{x \in M: |f_i(x)| \leqq 1, i = 1, \cdots, N\}$ is not a compact subset of M: $M = \mathbb{C}$, $N = 3$, $f_1 = e^{z^2}$, $f_2 = e^{iz-1}$, $f_3 = e^{cz}$, $c = e^{-\frac{1}{4}\pi i}$. Note that Q contains the half line

$$\{z \in \mathbb{C}: z = x+iy, y+x = 0, y \geqq -1\}.$$

In these examples every irreducible component of a variety of the form

$$\{x \in M: f_j(x) = e^{i\theta_j}, j \in J\},$$

where $J \subset \{1, \cdots, N\}$, is either entirely contained in the closure of $\{x \in M: |f_j(x)| < 1, j \in J\}$ or else does not meet this set. This also need not always be the case: let $M = \mathbb{C}^2$, $N = 3$, $f_1(z, w) = (w(z-1)-1)(z-1)+1$, $f_2(z, w) = z$, $f_3(z, w) = \frac{1}{4}w$. Let $V = \{(z, w) \in \mathbb{C}^2: f_1(z, w) = f_2(z, w) = 1\}$, so $V = \{(z, w) \in \mathbb{C}^2: z = 1\}$. Then $(1, w)$, $|w| < \frac{1}{4}$, is not in the closure of $P' = \{(z, w) \in \mathbb{C}^2: |f_1(z, w)| < 1, \ |f_2(z, w)| < 1\}$, since points (z, w) with $|z| < 1$ and $|w| < \frac{1}{4}$ are not in P'. Indeed, let (z, w) as above, hence $\alpha = z-1 = x+iy \in D = \{t \in \mathbb{C}: |t+1| < 1\} = \{x^2+y^2+2x < 0\}$. So $|w(z-1)^2| < \frac{1}{4}|\alpha|^2 = \frac{1}{4}(x^2+y^2) < -\frac{1}{2}x$. Hence $\mathrm{Re}(f_1(z, w)-1) = \mathrm{Re}(-\alpha+w\alpha^2) \geqq -x +\frac{1}{2}x = -\frac{1}{2}x > 0$. So $-\alpha+w\alpha^2 \notin D$, therefore $|f_1(z, w)| \geqq 1$. But $(1, 2)$ is in the closure of P' since the points $(x+1+i\sqrt{-3x/2}, 2)$ are in the closure of P' if $x < 0$, x sufficiently close to 0.

In the next we will show that if Q is a compact special analytic polyhedron then Q is the closure of its interior. We also show that under certain conditions on the set of points where the defining functions for an open analytic polyhedron P in $M \subset \mathbb{C}^n$ do not provide local coordinates, the closure $\bar{P}$ of P (this notation will be used throughout the following) is $\mathrm{Hol}(M)$-convex, hence a holomorphic set. Note that this condition is of the type Hoffman needs in some of his results ([9]). Finally in the case $\dim M = 2$ we completely solve our problem by proving $\bar{P}$ is a holomorphic set.

We will make use freely of results concerning function algebras, complex analytic manifolds and analytic varieties, stated in [6], [7], [8].

The simplest type of an open analytic polyhedron is the interior of a compact special polyhedron. The next result is well-known but the proof gives an indication of the techniques we want to use.

THEOREM 1: *Let* $Q = \{x \in M : |f_i(x)| \leqq 1, i = 1, \cdots, n\}$ *be a compact special analytic polyhedron in the Stein manifold* $M \subset \mathbb{C}^n$. *Then the open analytic polyhedron* $P = \{x \in M : |f_i(x)| < 1,\ i = 1, \cdots, n\}$ *is dense in* Q.

This result follows immediately from a theorem in [10], p. 132:

Let N be an n-dimensional complex analytic manifold and $f: N \to \mathbb{C}^m$ holomorphic. Then f is open iff $\dim_x f^{-1}f(x) = n - m$ for all $x \in N$.

Now in a neighborhood $N \subset M$ of Q, the varieties $\{z \in N : (f_1, \cdots, f_n)(z) = (f_1, \cdots, f_n)(x)\}$ for $x \in N$ are compact hence consist of a finite number of points. The above result shows that $f = (f_1, \cdots, f_n)$ is open hence P is dense in Q.

Another way to see this is the following: the varieties $\{z \in M : f_j(z) = e^{i\theta_j}, j = 1, \cdots, n\}$ consist of a finite number of points. So every irreducible component B of $\{z \in M : f_{j(s)}(z) = e^{i\theta_{j(s)}}, s = 1, \cdots, m\}$, $m \leqq n$, which meets Q is $n-m$ dimensional. If $x \in B$ is a regular point of $\{z \in M : f_{j(s)}(z) = e^{i\theta_{j(s)}}, s = 1, \cdots, m\}$ there exists a neighborhood U of x such that $\{z \in U : f_{j(s)}(z) = e^{i\theta_{j(s)}}, s = 1, \cdots, l\}$ is purely $n-l$ dimensional for $l \leqq m$. It then follows easily that x is in the closure of $\{z \in M : |f_{j(s)}(z)| < 1, s = 1, \cdots, m\}$. So B is in this closure. Hence P is dense in Q.

Let P be an open analytic polyhedron, defined by $f_1, \cdots, f_N$. If a variety of the form

$$\{x \in M : f_j(x) = e^{i\theta_j}, j \in J\},$$

with $J \subset \{1, \cdots, N\}$ consisting of m elements, is under consideration we always will assume $\theta_j = 0$, $j \in J$ and $J = \{1, \cdots, m\}$. This causes no loss of generality since we always may multiply f_j by unimodular constants and reorder the functions $f_1, \cdots, f_N$.

We now prove several lemmas to be used later on. We denote the Silov boundary of a function algebra A on a compact space X by ∂A. It is well-known that the peak points for A are dense in ∂A if X is metrizable. If F is a collection of functions on a topological space X, separating the points of X and containing the constant functions on X, and if Y is a compact sub set of X, then $[F|Y]$ will be the function algebra on Y generated by the restrictions to Y of the elements in F.

LEMMA 1: *Let* $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$, $x \in \{y \in M : f_i(y) = 1, i = 1, \cdots, m\}$. *Then* $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P} \cap \{y \in M : f_i(y) = 1, i = 1, \cdots, m\})$.

PROOF: Suppose the lemma were not true, then there is $g \in \mathrm{Hol}(M)$ such that $g(x) = 1 > \|g\|_{\bar{P} \cap \{y \in M : f_i(y) = 1, i = 1, \cdots, m\}}$.

Putting $h = 2^{-m}(f_1+1) \cdots (f_m+1)$ we have

$$(gh^k)(x) = 1 > \|gh^k\|_{\bar{P}}$$

for sufficiently large k, in contradiction with $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$.

Now $\{y \in M : f_i(y) = 1, i = 1, \cdots, m\}$ is a subvariety of M, so $\{y \in M : f_i(y) = 1, i = 1, \cdots, m\} = V \cup W$, where V is the union of the branches of the variety which do not meet x and W is the union of those branches which do meet x. Both V and W are subvarieties of M.

LEMMA 2: *With notation as above*

$$\textit{if } x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P}),\ x \in \{y \in M : f_i(y) = 1, i = 1, \cdots, m\},$$

then $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P} \cap (W \backslash V))^-$.

PROOF: Suppose the lemma is false. Since M is Stein, there exists $G \in \mathrm{Hol}(M)$ with $G(x) = 1$, $G = 0$ on V. Let $a = \|G\|_{\bar{P}}$. Note that $a > 0$. We can find $H \in \mathrm{Hol}(M)$ such that $H(x) = 1$ and $\|H\|_{(\bar{P} \cap (W \backslash V))^-} \leqq 1/(2a)$. So $GH(x) = 1$, $\|GH\|_{(\bar{P} \cap (W \backslash V))^-} \leqq \frac{1}{2}$, $GH = 0$ on $\bar{P} \cap V$. Hence the supremum norm of GH on $\bar{P} \cap \{y \in M : f_i(y) = 1, i = 1, \cdots, m\}$ is $\leqq \frac{1}{2}$ while $GH(x) = 1$, in contradiction with the initial assumption.

LEMMA 3: $\partial[\mathrm{Hol}(M)|(\bar{P} \cap (W \backslash V))^-] \subset \partial[\mathrm{Hol}(M)|\bar{P}]$.

PROOF: We abbreviate $A = [\mathrm{Hol}(M)|(\bar{P} \cap (W \backslash V))^-]$. Let z be a peak point for A which is contained in V. Let X be the closure of the set of peak points for A which are not contained in V. Suppose $z \notin X$. Now the function $f \in A$, peaking at z is such that $f(z) = 1$, $\|f\|_X < a$, $0 < a < 1$. Now there is a point $w \in \bar{P} \cap (W \backslash V)$ such that $|f(w)| = b > a$. Since $w \notin V$ there exists $g \in \mathrm{Hol}(M)$ such that $g(w) = 1$ and $g = 0$ on V. Now $|(gf^k)(w)| = b^k > a^k \ \|g\|_{(\bar{P} \cap (W \backslash V))^-} \geqq \|gf^k\|_{\partial A}$ for sufficiently large k, a contradiction. Hence z is a limit point of peak points for A which are contained in $\bar{P} \cap (W \backslash V)$.

Let y be a peak point of A which is not contained in V. Let f be a peak function for y. So there are $f_n \in \mathrm{Hol}(M)$ such that $\lim f_n = f$ on $(\bar{P} \cap (W \backslash V))^-$ and $\|f_n\|_{(\bar{P} \cap (W \backslash V))^-} = 1$. Let U, $U \subset\subset M$, be a neighborhood of y in $\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$ which has a positive distance to V. We may assume that $|f_n(x)| < \varepsilon$ for all $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P} \cap (W \backslash V))^-$, $x \notin U$ if n sufficiently large, where ε is a positive number smaller than 1.

Now let h be the peak function of lemma 1. For k sufficiently large, a component of

$$\{y \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P}) \colon |(h^k f_n)(y)| \geqq 1\}$$

is contained in U.

Now we use the following form of Rossi's local maximum modulus principle [13, theorem 5.3]: Let A be a function algebra on ΔA, $f \in A$ and $\alpha \in \mathbb{R}$, $0 \leqq \alpha \leqq \|f\|_{\Delta A}$. Then every component of $\{y \in \Delta A \colon |f(y)| \geqq \alpha\}$ meets ∂A. So U contains a point of $\partial[\mathrm{Hol}(M)|\bar{P}]$. We can do this for any neighborhood U of y. Hence $y \in \partial[\mathrm{Hol}(M)|\bar{P}]$. So $\partial A \subset \partial[\mathrm{Hol}(M)|\bar{P}]$.

Let $B(P)$ denote the set of all points of M at which at least $n = \dim M$ of the defining functions for P are 1 in absolute value.

LEMMA 4: *$\partial H(\bar{P})$ and $\partial[\mathrm{Hol}(M)|\bar{P}]$ are contained in $B(P)$.*

Comment: In [5] Bremermann states the same result in section 5.2, page 253. It seems to us that the argument produced in his proof is incomplete. However using similar arguments as in section 6.5, page 258, the result can be proved, using the analogue result for compact analytic polyhedra, [9].

LEMMA 5: $\partial[\mathrm{Hol}(M)|(\bar{P} \cap (W \backslash V))^-] \subset B(P)$.

The statement follows immediately from the previous two lemmas.

Let $f_1, \cdots, f_n$ be elements of $\mathrm{Hol}(M)$. We define the Jacobian of $f_1, \cdots, f_n$ as follows.

$$J(f_1, \cdots, f_n)(y) = \det\left(\frac{\partial f_i}{\partial z_j}\right)(y), \qquad y \in M.$$

It is clear that $J(f_1, \cdots, f_n)(y) \neq 0$ if and only if $f_1, \cdots, f_n$ provide local coordinates at y.

Let S be the set of all $y \in B(P)$ such that $J(f_I)(y) = 0$ for all $I \subset \{1, \cdots. N\}$ consisting of n elements such that $|f_i(y)| = 1$ for all $i \in I$; Here f_I stands for $(f_{i(1)}, \cdots, f_{i(n)})$ where $\{i(1), \cdots, i(n)\} = I$.

THEOREM 2: *Suppose there is an open neighborhood U of $\bar{P}$ such that $U \cap S = \emptyset$. Then $\bar{P}$ is* $\mathrm{Hol}(M)$*-convex.*

PROOF: Let $x \in \text{hull}_{\text{Hol}(M)}(\bar{P})$, $x \notin \bar{P}$; then $f_1(x) = \cdots = f_m(x) = 1$, $|f_{m+1}(x)| < 1, \cdots, |f_N(x)| < 1$ for some $m \leqq N$ (see the remark before lemma 1). As in lemma 2 we consider the varieties V and W.

Suppose $m \geqq n$. Now dim $W > 0$, otherwise $W = \{x\}$, in contradiction with $x \notin \bar{P}$. By lemma 5, $W \cap \bar{P} \cap B(P) \neq \emptyset$, say y is contained in this intersection. There are k functions f_i such that $|f_i(y)| = 1, k \geqq n$. By assumption there is a subset I of $\{1, \cdots, N\}$ consisting of n elements such that $J(f_I)(y) \neq 0$ and $|f_i(y)| = 1, i \in I$. If $I \subset \{1, \cdots, m\}$ then y is an isolated point of the variety $\{z \in M : f_i(z) = 1, i \in I\}$, hence of W, in contradiction with the fact that W is connected. Suppose $I \not\subset \{1, \cdots, m\}$. By the Jacobian condition not all f_i, $i \in I$, are constant (near y) on W. So close to y, i.e. in $U \cap W$, we can find a point y' where at most $k-1 \geqq n$ functions are 1 in absolute value. Repeating this process, we end op with $y_0 \in W$, $I \subset \{1, \cdots, m\}$ consisting of n elements and f_I such that $|f_i(y_0)| = 1$ for $i \in I$, $J(f_I)(y_0) \neq 0$, as above a contradiction.

Now suppose $m < n$, hence dim $W > 0$. First, since $x \notin \bar{P}$, the dimension of every branch of W is $> n-m$ (see proof of theorem 1). Also since $x \notin \bar{P}$, for any $g_{m+1}, \cdots, g_n \in \text{Hol}(M)$, there is a neighborhood of x in W where $J(f_1, \cdots, f_m, g_{m+1}, \cdots, g_n) = 0$. Hence by the identity principle for analytic functions on irreducible varieties, $J(f_1, \cdots, f_m, g_{m+1}, \cdots, g_n)$ vanishes identically on W for all choices of $g_{m+1}, \cdots, g_n \in \text{Hol}(M)$. Again, let $y \in W \cap \bar{P} \cap B(P)$. As before there is $I \subset \{1, \cdots, N\}$ consisting of n elements such that $|f_i(y)| = 1, i \in I$, and $J(f_I)(y) \neq 0$. If $I \supset \{1, \cdots, m\}$ we have a contradiction with the above observation about Jacobians. If $\{1, \cdots, m\} \not\subset I$, we find $y' \in W$ close to y such that at least n functions are 1 in absolute value at y'. Moreover the number of functions f_i which are 1 in absolute value at y' is at least one less than this number at y. This is possible since the dimension of every branch of W is $> n-m$. Continuing the above process we end up with n functions f_i, $i \in I$, $I \supset \{1, \cdots, m\}$, and a point $y_0 \in W$ such that $|f_i(y_0)| = 1$, $i \in I$ and $J(f_I)(y_0) \neq 0$. A contradiction. Hence $\text{hull}_{\text{Hol}(M)}(\bar{P}) = \bar{P}$, so $\bar{P}$ is a holomorphic set.

REMARK: One can prove a slightly stronger result:

Let T be the set of all $y \in B(P)$ such that y is not an isolated point of $\{z \in M : f_I(z) = f_I(y)\}$ for all $I \subset \{1, \cdots, N\}$ consisting of n elements such that $|f_i(y)| = 1$ for all $i \in I$.

If there is a neighborhood U of $\bar{P}$ such that $U \cap T = \emptyset$, then $\bar{P}$ is Hol(M)-convex.

The proof of theorem 2 depends on the fact that for an n-tuple f_I with $J(f_I)(x) \neq 0$, $|f_i(x)| = 1$, $i \in I$, and a positive-dimensional variety W through x, not all f_i are constantly 1 in absolute value on W. The same is true replacing $J(f_I)(x) \neq 0$ by: x is isolated point of $\{z \in M : f_I(z) = f_I(x)\}$ (cf. theorem 11, p. 108 of [8]).

For our main result we need the following theorem, due to Stein [14, proposition 2].

Let X be a connected complex analytic manifold, Y a complex space, $\tau: X \to Y$ a holomorphic mapping such that $\dim_x \{\tau^{-1}(\tau(x))\}$ does not depend on x. Let L, L_n $(n = 1, 2, \cdots)$ be connected components of level sets of τ. Assume that there is a point $x_0 \in L$ such that every neighborhood of x_0 is met by almost all L_n. Then every point of L has the same property.

THEOREM 3: *Let P be an open analytic polyhedron in a Stein manifold M which is a subset of $\mathbb{C}^2$. Then $\bar{P}$ is a holomorphic set.*

PROOF: Assume M is connected. This causes no loss of generality because we may restrict our attention to components of M.

Let $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$, $x \notin \bar{P}$. Let P be defined by $f_1, \cdots, f_N$ and suppose $f_1(x) = \cdots = f_m(x) = 1$, $|f_{m+1}(x)| < 1, \cdots, |f_N(x)| < 1$. Let W as before be the union of the branches of $\{y \in M: f_i(y) = 1,\ i = 1, \cdots, m\}$ through x. Now $\dim W = 1$, otherwise x would be in $\bar{P}$ since $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P} \cap W)$.

Let I be the subvariety of M defined as

$$I = \left\{ z \in M: \frac{\partial f_i}{\partial z_1}(z) = \frac{\partial f_i}{\partial z_2}(z) = 0 \text{ for some } i,\ 1 \leqq i \leqq N \right\},$$

and let J be the variety defined by $J = \{z \in M: J(f_i, f_j)(z) = 0$ for some i, j for which $J(f_i, f_j)$ is not identically zero$\}$. Then I and J are of dimension $\leqq 1$. We will show $x \in I \cup J$. Suppose $x \notin I \cup J$ then

$$\frac{\partial f_1}{\partial z_1}(x) \neq 0 \text{ or } \frac{\partial f_1}{\partial z_2}(x) \neq 0, \text{ say } \frac{\partial f_1}{\partial z_1}(x) \neq 0.$$

Let

$$A = \left\{ z \in M: \frac{\partial f_1}{\partial z_1}(z) = 0 \right\}$$

and suppose $y \notin A$. In a neighborhood U of y f_1 and z_2 provide local coordinates. Since $J(f_1, f_i)(x) = 0$ for all i, $2 \leqq i \leqq m$, because $\dim W = 1$, $J(f_1, f_i)$ is identically zero by the assumption $x \notin J$. So the value of f_i on U depends only on the value of f_1, i.e. $\mathit{dim}_y \{z \in M: (f_1, \cdots, f_m)(z) = (f_1, \cdots, f_m)(y)\} = 1$.

Now there is a point $y \in \bar{P} \cap W$ which is not contained in A. Take a neighborhood U of y as above and choose a sequence $\{y_n\}$ of points in $P \cap U$ converging to y. Let L_n be the connected component of the level set $\{z \in M \backslash A: (f_1, \cdots, f_m)(z) = (f_1, \cdots, f_m)(y_n)\}$ through y_n. These sets are 1-dimensional.

Now apply Stein's theorem to $M\backslash A$ and the map $(f_1, \cdots, f_m)$. This shows that every point of $W\backslash A$, in particular x, is met by almost all L_n, i.e. $x \in \bar{P}$, a contradiction.

REMARK: To prove that $\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})\backslash\bar{P}$ is contained in a one-dimensional analytic variety one also could use the remark on convergence of analytic varieties on page 5 of [14].

The above shows that for every $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$, $x \notin \bar{P}$, there is a neighborhood U of x in $\mathbb{C}^2\backslash\bar{P}$ and a one-dimensional subvariety V of U such that $U \cap \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P}) \subset V$. Let f be a polynomial such that $f(x) = 0$ and f has no other zeroes on V (after shrinking U if necessary). Let U_1 be an open neighborhood of x, relatively compact in U. We may assume that if $Z = \{z \in U : f(z) = 0\}$, $Z\backslash U_1$ has a positive distance to $\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$ (by shrinking U if necessary). Since $\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$ is a holomorphic set there is a Stein manifold $S \supset \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$ such that $S \cap Z \subset U_1$. Consider the following data for a Cousin I problem: $(S \cap U;\ 1/f)$, $(S\backslash\bar{U}_1;\ 0)$. On the Stein manifold S this problem is solvable, so there is a meromorphic function m on S such that $m - 1/f \in \mathrm{Hol}(S \cap U)$ and m is holomorphic on $S\backslash\bar{U}_1$. Therefore the Stein manifold $\{z \in S : |m(z)| < C\}$ for C sufficiently large contains $\bar{P}$, but does not contain x. In other words $\bar{P}$ is a holomorphic set.

REMARK: All results stated above can be proved for the more general situation where M is a Stein manifold which is a Riemann domain.

COROLLARY: *Let P be an open rational polyhedron in a Stein manifold M in $\mathbb{C}^2$, i.e. P is defined by rational functions with pole sets which do not meet M. Suppose that every function in* $\mathrm{Hol}(M)$ *can be approximated on K by rational functions with poles off K for every compact subset K of M. If $H^2(\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P}); \mathbb{Z}) = 0$, then $\bar{P}$ is rationally convex.*

PROOF: The condition on M means that $\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$ contains the rationally convex hull of $\bar{P}$. As in the proof of theorem 3 every point $x \in \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P})$, $x \notin \bar{P}$, is contained in a one-dimensional variety V such that for small enough neighborhoods U of x

$$U \cap \mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P}) \subset U \cap V.$$

As above choose Z and a Stein manifold S. Since $H^2(\mathrm{hull}_{\mathrm{Hol}(M)}(\bar{P}); \mathbb{Z}) = 0$, we may assume, replacing S by a smaller Stein manifold, there is an $f \in \mathrm{Hol}(S)$ such that $Z \cap S = \{f = 0\}$ (see [15], p. 286). Now f has no zeroes on $\bar{P}$ and f is holomorphic on the rationally convex hull of $\bar{P}$, so can be approximated on the rationally convex hull of $\bar{P}$ by rational functions. This shows that x is not contained in the rationally convex hull of $\bar{P}$, hence $\bar{P}$ is rationally convex.

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