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A CERTAIN SUBSPACE OF CHARACTERISTIC ZERO OF $(l^1)^*$

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Abstract

We construct an example of a subspace [1] V of the conjugate $E^* = l^\infty$ of $E = l^1$ with characteristic $r(V) = 0$ and satisfying the following two conditions:
(K_1) if $x_n \to x_0$ for $\sigma(E, V)$, then $\lim ||x_n|| \geqq ||x_0||$,
(K_2) If $x_n \to x_0$ for $\sigma(E, V)$ and

$$\lim_{n\to\infty} ||x_n|| = ||x_0||, \text{ then } \lim_{n\to\infty} ||x_n - x_0|| = 0.$$

Introduction

Let E be a Banach space, E^* its conjugate and V a subspace of E^*. The unit ball of $E(E^*, V$ respectively) we denote by $S_E(S_{E^*}, S_V$ respectively). Dixmier ([2]) defined the characteristic $r(V)$ of V as follows:

$$r(V) = \sup\{\alpha : \alpha \geqq 0 \text{ and } \alpha S_{E^*} \subset \overline{S_V}^{\sigma(E^*, E)}\}.$$

Clearly $r(V) > 0$ implies that V is $\sigma(E^*, E)$-dense in E^*, but the converse is not true (see [2] for an example).

The following two results involve characteristics.

PROPOSITION 1: ([6, *proposition* 4.1]). *Let E be a Banach space and let V be a separable subspace of E^*. Then (K_1) is equivalent to $r(V) = 1$.*

PROPOSITION 2: ([3], *see also* [9, *p.* 486]) *Let E be a separable Banach space and let V be a subspace of E^* with $r(V) > 0$. Then there exists an equivalent norm $|||\cdot|||$ on E for which (K_1) and (K_2) hold.*

Our example shows that in proposition 1 the separability of V is essential and also that in proposition 2 the condition $r(V) > 0$ is not necessary.

First we prove, setting $E = l^1$, $E^* = l^\infty$, that for each $k \in N$ there exists a (non-separable) subspace V_k of E^* such that (K_1) and (K_2) hold whereas

$$r(V_k) \leqq \frac{1}{k}.$$

[1] Apparently the problem of the existence of such a subspace was raised by Kadec. We thank Prof. Singer for communicating it to us and for some discussions resulting in the proof of proposition 1.

This V_k will be a suitable quasi-complement of c_0 in E^*, which we define by modifying a construction of Rosenthal ([8]). This leads, by a procedure of taking l^1-sums, to a subspace V of E^* satisfying both (K_1) and (K_2) and with $r(V) = 0$.

We begin by sketching a proof of proposition 1 which differs from the one suggested by Mil'man.

PROOF OF PROPOSITION 1: We first observe that (K_1) is equivalent to the sequential $\sigma(E, V)$-closedness of S_E. Since V is separable, the topology $\sigma(E, V)$ is metrizable when restricted to bounded subsets of E. Hence the sequential $\sigma(E, V)$-closure and the $\sigma(E, V)$-closure of S_E coincide. Thus (K_1) means that S_E is $\sigma(E, V)$-closed and this in turn is equivalent, by [2, Théorème 8], to $r(V) = 1$.

Observe that $r(V) = 1$ implies (K_1) also for non-separable V, by [2, Théorème 8]. The separability of V is needed only for the proof of the converse implication.

One should also note that (K_1) implies that V is $\sigma(E^*, E)$-dense, whether V is separable or not.

Our example will be based on the following

LEMMA: *Let* $E = l^1$, $E^* = l^\infty$ *and let* V *be a* $\sigma(l^\infty, l^1)$*-dense quasi-complement of* c_0 *in* l^∞ *(We assume* c_0 *to be imbedded in* l^∞ *in the canonical way). Then we have: If* $x_n \to x_0$ *for* $\sigma(l^1, V)$ *and* $\{x_n\}$ *is norm-bounded, then* $\|x_n - x_0\| \to 0$. *In particular,* (K_1) *and* (K_2) *are satisfied.*

Proof: Let $\{x_{n'}\}$ be any subsequence of $\{x_n\}$. Since l^1 is the dual of the separable space c_0, $\{x_{n'}\}$ contains (see [1]) a $\sigma(l^1, c_0)$-convergent subsequence $\{x_{n''}\}$. Thus $\{x_{n''}\}$ is $\sigma(l^1, c_0)$-Cauchy as well as $\sigma(l^1, V)$-Cauchy and therefore $\sigma(l^1, c_0 + V)$-Cauchy. Since $c_0 + V$ is norm-dense in l^∞, the boundedness of $\{x_{n''}\}$ now implies that $\{x_{n''}\}$ is $\sigma(l^1, l^\infty)$-Cauchy and therefore norm-convergent (see [4, p. 281]), say to x. V being $\sigma(l^\infty, l^1)$-dense in l^∞, $\sigma(l^1, V)$-limits are unique. This evidently implies that $x = x_0$. We have now shown that any subsequence of $\{x_n\}$ contains a subsequence converging to x_0 in norm. Hence $\|x_n - x_0\| \to 0$.

The statement proved clearly implies (K_2), and also (K_1), since (K_1) is equivalent to the sequential $\sigma(l^1, V)$-closedness of S_{l^1}.

In order to understand our example it is necessary to recall briefly Rosenthal's construction of a quasi-complement of c_0 in l^∞ (cf. [8]). This construction is based on the following observations, the complete proofs of which can be found in [8].

(i) A subspace X of a Banach space E is quasi-complemented in E if and only if there exists a $\sigma(E^*, E)$-closed subspace Y of E^* such that $Y \cap X^\perp = \{0\}$ and $Y_\perp \cap X = \{0\}$. Indeed, if Y has these properties, then $Y_\perp$ is a quasi-complement of X in E.

(ii) If Y is a reflexive subspace of E^*, then Y is $\sigma(E^*, E)$-closed. This follows from the Krein-Šmulian theorem.

(iii) If an infinite compact topological space S contains an infinite perfect subset, then $C(S)^*$ contains a subspace isomorphic to l^2.

Rosenthal's construction ([8]) of a quasi-complement of c_0 now proceeds as follows. We may identify l^∞ with $C(\beta N)$, where βN denotes the Stone-Cech compactifation of N. Then $c_0^\perp$ can be identified with $C(\beta N/N)^*$. Since $\beta N \setminus N$ is an infinite perfect compact Hausdorff space, (iii) implies that $c_0^\perp$ contains l^2 isomorphically. Let $H \subset c_0^\perp$ be isomorphic to l^2 and let $\{\mu_1, \cdots, \mu_n, \cdots\}$ be a basis of H equivalent to the orthonormal basis of l^2. We assume that $\|\mu_n\| = 1$ $(n = 1, 2, \cdots)$. For each $n \in N$ let δ_n be the Dirac measure on N concentrated at n. Then the closed linear span of $\{\delta_n : n \in N\}$ in $(l^\infty)^*$ can be identified with l^1, by the canonical map. Now let G be the closed linear span of

$$\left\{\frac{\delta_n}{n} + \mu_n : n \in N\right\}.$$

It is easily verified that G is isomorphic to H and therefore $\sigma((l^\infty)^*, l^\infty)$-closed, by (ii). Finally, $G \cap c_0^\perp = G_\perp \cap c_0 = \{0\}$, so $V = G_\perp$ is a quasi-complement of c_0 by (i).

Since, in this construction, $V^\perp \cap l^1 = G \cap l^1 = \{0\}$, V is $\sigma(l^\infty, l^1)$-dense in l^∞, so the lemma applies.

EXAMPLE: We now show that by a slight modification of the construction described above we can obtain for each $k \in N$ a $\sigma(l^\infty, l^1)$-dense quasi-complement V_k of c_0 with $r(V_k) \leqq 1/k$.

Let $k \in N$ be arbitrary and let G_k be the closed linear span of $k\delta_1 + \mu_1$ and

$$\left\{\frac{\delta_n}{n} + \mu_n : n = 2, 3, \cdots\right\}.$$

Clearly G_k is isomorphic to H and therefore $\sigma((l^\infty)^*, l^\infty)$-closed, by (ii). Again, as before it is easily verified that

$$G_k \cap c_0^\perp = (G_k)_\perp \cap c_0 = \{0\}.$$

Therefore $V_k = (G_k)_\perp$ is a quasi-complement of c_0 in l^∞, by (i). Also

$$V_k^\perp \cap l^1 = G_k \cap l^1 = \{0\},$$

so that V_k is $\sigma(l^\infty, l^1)$-dense in l^∞.

Next we show that

$$r(V_k) \leqq \frac{1}{k}.$$

By [2, Théorème 9] it suffices to prove that

$$\overline{(l^1, V_k^\perp)} \leqq \frac{1}{k}$$

(Here $\overline{(X, Y)}$, for arbitrary subspaces X and Y of a Banach space E, denotes the inclination of X to Y, i.e. the distance of the unit sphere of X to Y (cf. [9]). Clearly, since $\delta_1 \in S_{l^1}$ and

$$\delta_1 + \frac{1}{k}\mu_1 \in G_k,$$

we have

$$\overline{(l^1, G_k)} \leqq \left\| \delta_1 - \left(\delta_1 + \frac{1}{k}\mu_1\right) \right\| = \frac{1}{k},$$

which proves our claim, since $G_k = V_k^\perp$.

Now, for each $k \in N$, let $E_k = l^1$, $E_k^* = l^\infty$ and let V_k be the $\sigma(E_k^*, E_k)$-dense quasi-complement of c_0 in E_k^* with

$$r(V_k) \leqq \frac{1}{k}$$

that was constructed above. Then, putting

$$E = (E_1 \oplus E_2 \oplus \cdots \oplus E_k \oplus \cdots)_{l^1},$$

we have

$$E^* \equiv (E_1^* \oplus E_2^* \oplus \cdots \oplus E_k^* \oplus \cdots)_{l^\infty}.$$

We will show that

$$V = (V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus \cdots)_{l^\infty} \subset E^*$$

satisfies (K_1) and (K_2) whereas $r(V) = 0$.

To prove (K_1), it suffices to show that S_E is sequentially $\sigma(E, V)$-closed. Let $\{x^{(n)}\}_{n=1}^\infty$, with $x^{(n)} = (x_1^{(n)}, x_2^{(n)} \cdots) \in E (n \in N)$, be a sequence in S_E which converges for $\sigma(E, V)$ to $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots) \in E$. We must show that $\|x^{(0)}\| \leqq 1$. For this it is enough to prove that for an arbitrary $k \in N$

$$\|\pi_k(x^{(0)})\| = \sum_{n=1}^{k} \|x_n^{(0)}\| \leqq 1,$$

where π_k is the natural projection of E onto $(E_1 \oplus \cdots \oplus E_k \oplus \{0\} \oplus \cdots)_{l^1}$, which we identify with $(E_1 \oplus E_2 \oplus \cdots \oplus E_k)_{l^1}$. Clearly the sequence

$$\{\pi_k(x^{(n)})\}_{n=1}^\infty$$

converges to $\pi_k(x^{(0)})$ for $\sigma(\pi_k(E), \pi_k^*(V)) = \sigma((E_1 \oplus \cdots \oplus E_k)_{l^1},$

$(V_1 \oplus \cdots \oplus V_k)_{l^\infty})$. Since $\|\pi_k(x^{(n)})\| \leqq 1$ for all $n \in N$, $(E_1 \oplus \cdots \oplus E_k)_{l^1}$ (which is isometric to l^1) is isometric to the dual of the separable space

$$\underbrace{(c_0 \oplus \cdots \oplus c_0)}_{k\ factors}{}_{l^\infty},$$

and $(V_1 \oplus \cdots \oplus V_k)_{l^\infty}$ is a

$$\sigma((E_1^* \oplus \cdots \oplus E_k^*)_{l^\infty}, (E_1 \oplus \cdots \oplus E_k)_{l^1})\text{-dense}$$

quasi-complement of $(c_0 \oplus \cdots \oplus c_0)_{l^\infty}$ in

$$(E_1^* \oplus \cdots \oplus E_k^*)_{l^\infty},$$

the Lemma applies here and yields that $\|\pi_k(x^{(0)})\| \leqq 1$. Hence $\|x^{(0)}\| \leqq 1$, since $k \in N$ was arbitrary.

To show that (K_2) holds, let us assume that $x^{(n)} \to x^{(0)}$ for $\sigma(E, V)$ and that $\|x^{(n)}\| \to \|x^{(0)}\|$. We may also assume that $\|x^{(0)}\| = 1$. Let $\varepsilon > 0$ be arbitrary and let $k \in N$ be such that

$$(1) \qquad 1-\varepsilon < \|\pi_k(x^{(0)})\| \leqq 1$$

As in the proof of (K_1) it follows from the Lemma that

$$\|\pi_k(x^{(n)}) - \pi_k(x^{(0)})\| \to 0$$

$(n \to \infty)$. Hence there exists an $n_0 \in N$ such that

$$(2) \qquad \|\pi_k(x^{(n)}) - \pi_k(x^{(0)})\| < \varepsilon \ (n \geqq n_0),$$

and therefore, by (1),

$$(3) \qquad \|\pi_k(x^{(n)})\| > \|\pi_k(x^{(0)})\| - \varepsilon > 1-2\varepsilon \ (n \geqq n_0)$$

We may also assume that

$$(4) \qquad \|x^{(n)}\| < 1+\varepsilon \ (n \geqq n_0)$$

Thus

$$(5) \quad \|x^{(n)} - \pi_k(x^{(n)})\| = \|x^{(n)}\| - \|\pi_k(x^{(n)})\| < 1+\varepsilon-(1-2\varepsilon) = 3\varepsilon \quad (n \geqq n_0)$$

It follows now from (1), (2), (3), (4) and (5) that

$$\|x^{(n)} - x^{(0)}\| \leqq \|x^{(n)} - \pi_k(x^{(n)})\| + \|\pi_k(x^{(n)}) - \pi_k(x^{(0)})\| + \|\pi_k(x^{(0)}) - x^{(0)}\| < 3\varepsilon+\varepsilon+\varepsilon = 5\varepsilon \ (n \geqq n_0)$$

This proves (K_2).

Finally, let us show that $r(V) = 0$. We have

$$S_{E^*} = \prod_{k=1}^{\infty} S_{E^*_k}$$

and it is easily seen that

$$\overline{S_V}^{\sigma(E^*, E)} = \prod_{k=1}^{\infty} \overline{S_{V_k}}^{\sigma(E_k^*, E_k)}.$$

By the definition of $r(V_k)$

$$\alpha S_{E_k^*} \not\subset \overline{S_{V_k}}^{\sigma(E^*_k, E_k)} \text{ for all } \alpha > \frac{1}{k}\ (k \in N).$$

It follows that

$$\alpha S_{E^*} \not\subset \overline{S_V}^{\sigma(E^*, P)} \text{ for all } \alpha > 0.$$

Thus $r(V) = 0$. This completes the example.

We conclude with a general result on quasi-complements of c_0 in l^∞. All such quasi-complements obtained by Rosenthal's construction are $\sigma(l^\infty, l^1)$-dense in l^∞. This may not be the case in general. However, all quasi-complements of c_0 are 'almost' $\sigma(l^\infty, l^1)$-dense in l^∞, as we show in the following

PROPOSITION 3: *Let V be a quasi-complement of c_0 in l^∞. Then the $\sigma(l^\infty, l^1)$- closure V' of V in l^∞ has finite codimension in l^∞.*

PROOF: Suppose that $\dim l^\infty/V' = \infty$. Then we have, since $V_\perp = V'_\perp$, that $\dim V_\perp = \infty$ and, of course, $\dim l^1/V_\perp = \infty$. By [7, Lemma 2]$V_\perp$ contains a subspace L with $\dim L = \infty$ which is complemented in l^1. Let M be a complement of L in l^1. Then $l^\infty = L^\perp \oplus M^\perp$. By [5] both $L^\perp$ and $M^\perp$ are isomorphic to l^∞. In particular $M^\perp$ is non-separable. Since $L \subset V_\perp$ we have $V \subset L^\perp$. Furthermore, l^∞/V is separable, by the definition of V, whereas $l^\infty/_{L^\perp} \cong M^\perp$ is not. This is a contradiction, since the canonical map $l^\infty/V \to l^\infty/_{L^\perp}$ is a continuous surjection.

REFERENCES

[1] S. BANACH: *Théorie des opérations linéaires*. Warszawa (1932).

[2] J. DIXMIER: Sur un théorème de Banach. *Duke Math. J. 15* (1948) 1057–1071.

[3] M. I. KADEC and A. PELCZYNSKI: Basic sequences, biorthogonal systems and norming sets in Banach and Fréchet spaces. *Studia Math. 25* (1965) 297–323 (Russian).

[4] G. KÖTHE: *Topological vector spaces I*. Springer Verlag (1969).

[5] J. LINDENSTRAUSS: On complemented subspaces of *m*. *Israel J. Math. 5* (1967) 153–156.

[6] V. D. MIL'MAN: The geometric theory of Banach spaces, Part II. *Uspehi Math. Nauk, 26*, (1971) 73–149 (Russian). English translation in *Russian Math. Surveys 26* 79–163 (1971).

[7] A. PELCZYNSKI: Projections in certain Banach spaces. *Studia Math. 19* (1960) 209–228.
[8] H. P. ROSENTHAL: On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\mu)$. *J. Funct. Anal.* 4 (1969) 176–214.
[9] I. SINGER: *Bases in Banach spaces I.* Springer Verlag (1970).

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