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CONVOLUTIONS AND FACTORIZATION THEOREMS

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Introduction

It is our purpose in this paper to formulate a general notion of convolution and present three factorization theorems in terms of the generalized convolution operation. In 1957, W. Rudin [5] proved that every function in $L_1(R)$ can be expressed as the convolution product of two suitable functions in $L_1(R)$. Since then several results of this type, usually called factorization theorems, are proved by various authors. We mention the papers of P. J. Cohen [1] and of P. C. Curtis and A. Figa-Talamanca [2], since our theorems are generalizations of some of their results. Cohen [1] proved that if one considers $L_1(G)$, for a compact group G, as an algebra of operators under convolution acting on the continuous functions on G, then each continuous function could be written as the convolution product of another continuous function with an integrable function. Curtis and Figa-Talamanca [2] obtain, among other things, generalizations of Cohen's results. Our theorems are inspired by these results and further generalize them in terms of the convolution operation we formulate below. The paper is organized in the following way. In Section 1, we formulate our version of the convolution operation and in Section 2 we present some examples and state our theorems. Section 3 contains all proofs.

1. Convolutions

The convolution of two Lebesgue integrable functions f, g on the real line R is a well known concept, resulting in the (integrable) function $f * g$ given by the formula:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y.$$

Generalization of this concept to locally compact groups is also well known and is extensively used in abstract harmonic analysis. In this setting the above formula takes the form:

* For the most part the contents of this paper are included in a doctoral thesis written by R. K. Heckman under the direction of R. R. Chivukula and presented to the University of Nebraska.

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\,\mathrm{d}y$$

where G is a locally compact group, $\mathrm{d}y$ is a (fixed) Haar measure on G and f, g are integrable with respect to $\mathrm{d}y$. Also one talks about the convolution of two (finite, regular) Borel measures on G and the above formulas become special cases of the convolutions of measures (namely, of two measures each of which is absolutely continuous with respect to Haar measure).

As indicated above, the classical convolution operation is customarily defined by means of integrals. But from a functional analysis point of view, it is best to think of convolution as an operation between linear functionals. We adapt this view and our generalization will appear(in the same way) as an operation between linear functionals. Before we write our definition, it may be helpful to describe how the (classical) convolution of measures is defined in terms of linear functionals. This is as follows. Let G be a locally compact group and let $C_0(G)$ be the Banach space (with supremum norm) of all real or complex valued continuous functions on G which vanish at infinity. Let $M(G)$ denote the Banach space of all bounded regular Borel measures on G with the total variation norm. As usual $M(G)$ is (identified with) the normed dual of $C_0(G)$. For any function $f \in C_0(G)$ and any element $x \in G$, let ${}_xf$ denote the left translate of f by x; that is ${}_xf$ is the function on G defined by $({}_xf)(y) = f(xy)$ for all $y \in G$. Clearly ${}_xf \in C_0(G)$. Now for any measure $\mu \in M(G) = C_0^*(G)$ and any function $f \in C_0(G)$, let $\bar{\mu}f$ denote the function defined as $(\bar{\mu}f)(x) = \mu({}_xf)$ for all $x \in G$. It is easy to prove that $\bar{\mu}f \in C_0(G)$. Finally, for any λ, $\mu \in M(G)$ one defines $\lambda * \mu$, called the convolution of the measures λ and μ, by the equation $(\lambda * \mu)(f) = \lambda(\bar{\mu}f)$ for all $f \in C_0(G)$ It is not hard to prove that $\lambda * \mu$ is a bounded linear functional on the Banach space $C_0(G)$. Hence, $\lambda * \mu \in C_0^*(G) = M(G)$; and it is straightforward to check that $\lambda * \mu$ as defined above coincides with the classical convolution of the measures λ and μ.

The advantage in considering convolution as an operation between linear functionals arises from the fact that as such it can be extended to more general situations. For example one could start, in great generality, with a semigroup S and a linear space $\mathscr{F}$ of functions on S. Assuming that $\mathscr{F}$ is closed under left translations and that L is a linear functional on $\mathscr{F}$ with the property that $\bar{L}f \in \mathscr{F}$ for all $f \in \mathscr{F}$, where $(\bar{L}f)(x) = L({}_xf)$, one could then define $M * L$, the convolution of M and L, by $(M * L)(f) = M(\bar{L}f)$ where M is any linear functional on $\mathscr{F}$. Clearly $M * L$ is a linear functional on the linear space $\mathscr{F}$ and this construction reduces to the classical convolution of measures if one takes $S = G$

and $\mathscr{F} = C_0(G)$. This is exactly the point of view taken by Hewitt and Ross [3]. We generalize the above construction as follows.

DEFINITION 1: Let S and T be sets and let $\mathscr{F}_S$ and $\mathscr{F}_T$ be linear spaces of real or complex valued functions on S and T respectively. Suppose in addition that S is a collection of transformations of T. For $f \in \mathscr{F}_T$ and $s \in S$, define ${}_sf$ by $({}_sf)(t) = f(s(t))$ for all $t \in T$; and suppose that for each $f \in \mathscr{F}_T$ and $s \in S$, ${}_sf \in \mathscr{F}_T$ (in otherwords, $\mathscr{F}_T$ is S-invariant). If M is a linear functional on $\mathscr{F}_T$ and if $f \in \mathscr{F}_T$, we denote by $\overline{M}f$ the function on S defined as $(\overline{M}f)(s) = M({}_sf)$for all $s \in S$. Suppose further that M is such that $\overline{M}f \in \mathscr{F}_S$ for all $f \in \mathscr{F}_T$. Now if L is any linear functional on $\mathscr{F}_S$, then the function whose value at any $f \in \mathscr{F}_T$ is $L(\overline{M}f) = (L \circ \overline{M})(f)$ is well defined. We call this function the convolution of the functionals L and M and denote it by $L * M$.

It is easy to verify that the function $L * M$, as defined above, is a linear functional on the linear space $\mathscr{F}_T$. Thus our convolution is an operation between (certain) linear functionals and results in a linear functional. Moreover if we take $S = T$ to be a semigroup and take the action of $s \in S$ to be left translation we obtain, as a special case, the convolution of Hewitt and Ross mentioned earlier. Thus Definition 1 is a generalization of def. 19.1, p. 262, Hewitt and Ross [3].

2. Results and Examples

We are now in a position to state our theorems. As indicated in the introduction, these theorems are inspired by some of the results in Curtis and Figa-Talamanca [2]. Specifically, we adopt their Theorem 2.3 and Corollary 2.4 by reformulating them in terms of our convolution. To this end, let the sets S and T and linear spaces $\mathscr{F}_S$ and $\mathscr{F}_T$ be as in Definition 1. Further let $\mathscr{L}(\mathscr{F}_S)$ and $\mathscr{L}(\mathscr{F}_T)$ be (certain, prechosen) linear spaces of linear functionals on $\mathscr{F}_S$ and $\mathscr{F}_T$ respectively. Suppose that for each $M \in \mathscr{L}(\mathscr{F}_T)$ and each $f \in \mathscr{F}_T$, the function $\overline{M}f \in \mathscr{F}_S$. Then, as in Definition 1, we can construct the convolution product $L * M$ for all $L \in \mathscr{L}(\mathscr{F}_S)$ and $M \in \mathscr{L}(\mathscr{F}_T)$. Now suppose further that $L * M \in \mathscr{L}(\mathscr{F}_T)$. [Note: Of course $L * M$ is a linear functional on $\mathscr{F}_T$, as always. We are here assuming that $L * M$ belongs to the already chosen linear space $\mathscr{L}(\mathscr{F}_T)$ for all L and M.] Under these assumptions, we may regard each L as generating a linear mapping $\tilde{L}$ of the linear space $\mathscr{L}(\mathscr{F}_T)$ into itself, defined by $\tilde{L}(M) = L * M$ for all $M \in \mathscr{L}(\mathscr{F}_T)$. Further we can also define a mapping I by putting $I(L) = \tilde{L}$. It is easy to see that the mapping I is a linear mapping of $\mathscr{L}(\mathscr{F}_S)$ into the space of all linear mappings of $\mathscr{L}(\mathscr{F}_T)$ into itself. The stage is now set for Theorems A and B, stated below, which are respectively reformulations of Theorem 2.3 and Corollary 2.4 of Curtis and Figa-Talamanca [2].

THEOREM A: *Let* S, T, $\mathscr{F}_S$, $\mathscr{F}_T$, $\mathscr{L}(\mathscr{F}_S)$, $\mathscr{L}(\mathscr{F}_T)$ *and* $\tilde{L}$ *be as described above. Suppose that* $\mathscr{L}(\mathscr{F}_S)$ *and* $\mathscr{L}(\mathscr{F}_T)$ *are Banach spaces and that* $L * M \in \mathscr{L}(\mathscr{F}_T)$ *for all* $L \in \mathscr{L}(\mathscr{F}_S)$ *and* $M \in \mathscr{L}(\mathscr{F}_T)$. *Suppose further that* $I : L \to \tilde{L}$ *is a continuous one-one linear mapping of* $\mathscr{L}(\mathscr{F}_S)$ *into* $B(\mathscr{L}(\mathscr{F}_T))$, *the algebra of all bounded linear operators on the Banach space* $\mathscr{L}(\mathscr{F}_T)$, *with* $||I|| \leqq c$, *a positive real number. Suppose that* $\mathscr{L}(\mathscr{F}_S)$ *is the closed linear span of a bounded set* $\mathscr{E}$, $||\mathscr{E}|| \leqq N$, *with the property that if* $\tilde{\mathscr{E}} = \{\tilde{E} | E \in \mathscr{E}\}$, *then for each finite subset* $\{\tilde{E}_1, \cdots, \tilde{E}_n\} \subseteq \tilde{\mathscr{E}}$ *and* $\varepsilon > 0$ *there exists* $\tilde{E} \in \tilde{\mathscr{E}}$ *satisfying* $||\tilde{E} \circ \tilde{E}_i - \tilde{E}_i|| < \varepsilon$, $i = 1, \cdots, n$. *Then* $\mathscr{L}(\mathscr{F}_S) * \mathscr{L}(\mathscr{F}_T) = \{L * M | L \in \mathscr{L}(\mathscr{F}_S), M \in \mathscr{L}(\mathscr{F}_T)\}$ *is a closed subspace* Y *of* $\mathscr{L}(\mathscr{F}_T)$. *Furthermore, for each* $M' \in Y$ *and* $\delta > 0$ *there exist* $L \in \mathscr{L}(\mathscr{F}_S)$, $||L|| \leqq N$, *and* $M'' \in Y$ *satisfying* $||M' - M''|| < \delta$ *and* $L * M'' = M'$.

THEOREM B: *Let* S, T, $\mathscr{F}_S$, $\mathscr{F}_T$, $\mathscr{L}(\mathscr{F}_S)$, $\mathscr{L}(\mathscr{F}_T)$ *and* $\tilde{L}$ *be as above. Suppose that* $\mathscr{L}(\mathscr{F}_T)$ *is a Banach space and that* $\mathscr{L}(\mathscr{F}_S)$ *is a Banach algebra with a bounded left approximate identity* $\mathscr{E} = \{E_\alpha\}$. [*That is,* $\mathscr{E}$ *is directed and* $\lim E_\alpha L = L$ *for each* $L \in \mathscr{L}(\mathscr{F}_S)$.] *Let* $I : L \to \tilde{L}$ *be a bounded faithful representation of* $\mathscr{L}(\mathscr{F}_S)$ *into* $B(\mathscr{L}(\mathscr{F}_T))$, *the algebra of all bounded linear operators on the Banach space* $\mathscr{L}(\mathscr{F}_T)$. *Let* $Y = \{M \in \mathscr{L}(\mathscr{F}_T) | \lim_\alpha \tilde{E}_\alpha(M) = \lim_\alpha E_\alpha * M = M, \tilde{E}_\alpha \in I(\mathscr{E})\}$. *Then* Y *is a closed subspace of* $\mathscr{L}(\mathscr{F}_T)$ *and* $Y = \mathscr{L}(\mathscr{F}_S) * \mathscr{L}(\mathscr{F}_T)$. *Moreover, for each* $M' \in Y$ *and* $\delta > 0$ *there exist* $L \in \mathscr{L}(\mathscr{F}_S)$ *and* $M'' \in Y$ *such that* $||M' - M''|| < \delta$ *and* $L * M'' = M'$. *Furthermore,* $||L||$ *can be taken to be bounded by a fixed constant.*

We now proceed to describe the setting for our first factorization theorem. This description provides an example of an instance where the full force of our Definition 1 comes into play. Let X be a locally compact Hausdorff space and let G be a locally compact Hausdorff topological group. Suppose that each $g \in G$ is a homeomorphism of X onto itself. Now, in the notation of Definition 1, we take $S = G$ and $T = X$. Further, we take $\mathscr{F}_S = \mathscr{F}_G = L_\infty(G)$ the Banach space of all essentially bounded complex valued Borel measurable functions on G; and we take $\mathscr{F}_T = \mathscr{F}_X = C_0(X)$ the Banach space of all continuous complex valued func tions on X which vanish at infinity. Since these spaces $L_\infty(G)$ and $C_0(X)$ are well known (for example, see [3]) we shall feel free to use their properties as needed. Next we need to specify spaces of linear functionals $\mathscr{L}(\mathscr{F}_G)$ and $\mathscr{L}(\mathscr{F}_X)$. We simply let $\mathscr{L}(\mathscr{F}_X) = C_0^*(X)$ the normed dual of $C_0(X)$. For $\mathscr{L}(\mathscr{F}_G)$ we take the space $\hat{L}_1(G)$ defined as follows.

DEFINITION 2: Given $f \in L_1(G)$ and $H \in L_\infty(G)$, define

$$\hat{f}(H) = \int_G f(g)H(g^{-1})\,dg$$

where dg denotes a (fixed) Haar measure on G. Let $\hat{L}_1(G) = \{\hat{f} | f \in L_1(G)\}$. Clearly this space $\hat{L}_1(G)$ is a linear space of linear functionals on $L_\infty(G)\}$, our choice for $\mathscr{F}_G$. [In fact each $\hat{f}$ is a continuous (even $w*$-continuous) linear functional on $L_\infty(G)$]. Now it remains to show that $\hat{f} * M$ is properly defined for all $\hat{f} \in \hat{L}_1(G)$ and $M \in C_0^*(X)$, in accord with Definition 1. The following three propositions will do this.

(i) If $F \in C_0(X)$ and $g \in G$, then ${}_gF \in C_0(X)$.

(ii) If $M \in C_0^*(X)$ and $F \in C_0(X)$, then $\overline{M}F$ is a bounded right uniformly continuous complex valued function on G and hence $\overline{M}F \in C(G) \subseteq L_\infty(G)$ where $C(G)$ denotes the Banach space (in supremum norm) of all bounded continuous complex valued functions on G.

(iii) If $\hat{f} \in \hat{L}_1(G)$ and $M \in C_0^*(X)$, then $\hat{f} * M \in C_0^*(X)$. Thus we have $\hat{L}_1(G) * C_0^*(X) = \{\hat{f} * M | \hat{f} \in \hat{L}_1(G), M \in C_0^*(X)\} \subseteq C_0^*(X)$.

Each of these propositions can be proved by standard methods of functional analysis and so we omit their proofs. For a discussion of uniform continuity on topological groups we refer the reader to Kelley [4]. We are now ready for our first factorization theorem.

THEOREM 1: *Let G be a locally compact group of homeomorphisms of the locally compact topological space X. Assume that the mapping $g \to {}_gF$ from G into $C_0(X)$ is uniformly continuous for each $F \in C_0(X)$. Suppose also that $\hat{f}_1 * M = \hat{f}_2 * M$ for all $M \in C_0^*(X)$ implies $f_1 = f_2$ in $L_1(G)$. Then $\hat{L}_1(G) * C_0^*(X)$ is a closed subspace Y of $C_0^*(X)$ and furthermore given $M' \in Y$ and $\delta > 0$, there exist $\hat{f} \in \hat{L}_1(G)$ and $M'' \in Y$ satisfying $M' = \hat{f} * M''$ and $||M' - M''|| < \delta$. Moreover $||\hat{f}||$ can be taken to be bounded by a fixed constant.*

Our next result (Theorem 2, below) is an extension of Cor. 2.4 [2], in that the need for the representation to be faithful is partially eliminated. We use Theorem 2 to obtain two more factorization theorems (namely, Theorems 3 and 4). It may be noted that since Theorem B (stated earlier) is an adaptation of Cor. 2.4 [2], our extension (namely, Theorem 2) applies to Theorem B as well. This is how Theorem 2 will be used in proving Theorems 3 and 4.

THEOREM 2: *Let $\mathscr{A}$ be a Banach algebra with a bounded left approximate identity $\mathscr{E}$. [That is, $\mathscr{E} = \{e_\alpha\}$ is directed, and $\lim e_\alpha a = a$ for each $a \in \mathscr{A}$; and $||e_\alpha|| < c$ for some positive c.] Let $\sigma : \mathscr{A} \to B(X)$ be a bounded representation of $\mathscr{A}$ in the algebra $B(X)$ of bounded operators on a Banach space X satisfying the property that if $e_\alpha, e_\beta \in \mathscr{E}$ and $\sigma(e_\alpha) = \sigma(e_\beta)$, then*

$e_\alpha = e_\beta$. Let $Y = \{x \in X | \lim_\alpha E_\alpha x = x,\ E_\alpha = \sigma(e_\alpha),\ e_\alpha \in \mathscr{E}\}$. Then Y is a closed subspace of X and $\mathscr{A}X = \{\sigma(a)x | a \in \mathscr{A},\ x \in X\} = Y$. Moreover, for each $y \in Y$ and $\delta > 0$ there exist $a \in \mathscr{A}$ and $z \in Y$ such that $||z-y|| < \delta$ and $\sigma(a)z = y$. Furthermore, a can be taken to be bounded by a fixed constant.

In our next factorization theorem (Theorem 3, below) the setting is slightly different from that of Theorem 1. Instead of G being a group of homeomorphisms on X, we now assume that G acts topologically to the left on X. [This means that there is given a continuous mapping from $G \times X$ into X, denoted by $(g, x) \to gx$, such that $ex = x$ and $g_1(g_2 x) = (g_1 g_2)x$ where e is the identity of G and $g_1,\ g_2 \in G$ and $x \in X$.] The choices for the various spaces remain the same and it follows that $\hat{f} * M$ is properly defined, in accord with Definition 1. One difference may be noted here. The condition that the mapping $g \to {}_gF$ is uniformly continuous was part of the hypothesis in Theorem 1. We are able to prove this property in the present context.

THEOREM 3: *Let G be a locally compact topological group acting topologically to the left on the locally compact topological space X. Assume that G is first countable and that the map $g \to gx$ is one-one and open for some (one element) $x \in X$. Let $Y = \hat{L}_1(G) * C_0^*(X)$. Then Y is a closed subspace of $C_0^*(X)$. Moreover, there exists a left approximate identity $\mathscr{E} = \{e_\alpha\}$ in $L_1(G)$ such that $Y = \{M \in C_0^*(X) | \lim \hat{e}_\alpha * M = M\}$. Also, given $M' \in Y$ and $\delta > 0$ there exist $M'' \in Y$ and $\hat{f} \in \hat{L}_1(G)$ satisfying $M' = \hat{f} * M''$ and $||M' - M''|| < \delta$. Moreover, $||\hat{f}||$ can be taken to be bounded by a fixed constant. Furthermore for each $M \in C_0^*(X)$ and $F \in C_0(X)$, we have* $\lim_\alpha(\hat{e}_\alpha * M)(F) = M(F)$.

In our next (and final) theorem the setting is the same as in Theorem 3, except that it is assumed, in addition, that X carries a uniform structure compatible with its topology. We denote by $C_u(X)$ the Banach space (with supremum norm) of all uniformly continuous complex valued functions on X; and use $C_u(X)$ in place of $C_0(X)$. The choices for the other spaces remain the same and it follows that $\hat{f} * M$ is properly defined in accord with Definition 1, and that $\hat{f} * M \in C_u^*(X)$ for all $\hat{f} \in \hat{L}_1(G)$ and $M \in C_u^*(X)$. At this point it may be interesting to note that the elements in $C_u^*(X)$ are not, in general, countably additive measures; whereas the elements of $C_0^*(X)$ are such measures. However, it will be seen in the next section that Theorem 4 is proved in much the same way as Theorem 3.

THEOREM 4: *Let G be a locally compact topological group which operates topologically to the left on the locally compact topological space X and further let X be a uniform space with uniformity $\mathscr{U}$ compatible with the topology of X. Assume that each $g \in G$ (that is, the mapping $x \to gx$) is* *uniformly continuous from X to X with respect to the uniformity $\mathscr{U}$; and that if $V \in \mathscr{U}$, there exists an open set U in G containing the identity of G such that if $g \in U$, then the ordered pair $(x, gx) \in V$ for all $x \in X$. Further assume that G is first countable and that the mapping $g \to gx$ from G to X is one-one and open for some (one element) $x \in X$. Let $Y = \hat{L}_1(G) * C_u^*(X)$. Then Y is a closed subspace of $C_u^*(X)$. Moreover, there exists an approximate identity $\mathscr{E} = \{e_\alpha\}$ in $L_1(G)$ such that $Y = \{M \in C_u^*(X) | \lim_\alpha \hat{e}_\alpha * M = M\}$. Also, given $M' \in Y$ and $\delta > 0$ there exist $\hat{f} \in \hat{L}_1(G)$ and $M'' \in Y$ satisfying $M'' = \hat{f} * M'$ and $||M' - M''|| < \delta$. Moreover $||\hat{f}||$ can be taken to be bounded by a fixed constant. Furthermore, for each $M \in C_u^*(X)$ and $F \in C_u(X)$, we have $\lim_\alpha (\hat{e}_\alpha * M)(F) = M(F)$.*

This completes the statements of all our theorems. The proofs are given in the next section. We note that each of the convolutions described above illustrate our Definition 1. In a sense, these are natural generalizations of well known constructions involving convolutions. We close this section by giving an example which seems to be entirely new. We have some results in this context and we plan to publish them (and others that we are working on) in a separate communication.

EXAMPLE: Let X be a Banach space and $B = B(X)$ be the Banach algebra of all bounded linear operators from X into X. Now, in the notation of Definition 1, we take $T = X$ and $S = B = B(X)$. Further we let $\mathscr{F}_S = \mathscr{F}_B = B^*$ and $\mathscr{F}_T = \mathscr{F}_X = X^*$. Now the following three propositions can be proved in a straight-forward manner: (i) given $F \in X^*$ and $g \in B$, ${}_gF \in X^*$; (ii) If $M \in X^{**}$ and $F \in X^*$, then $\overline{M}F \in B^*$; and (iii) If $L \in B^{**}$ then $L * M$, given by $(L * M)(F) = L(\overline{M}F)$ for all $F \in X^*$ and $M \in X^{**}$, is well defined and $L * M \in X^{**}$. Thus elements of $B(X)^{**}$ may be convolved with elements of X^{**} for any Banach space X; or, alternatively, elements of $B(X)^{**}$ can be considered as convolution operators on the Banach space X^{**}.

3. Proofs

As noted in Section 2, Theorems A and B are reformulations of Theorem 2.3 and Corollary 2.4 of [2] in terms of our convolution given in Definition 1; and we therefore do not give their proofs here.

PROOF OF THEOREM 1: For convenience, we break the proof into the following three propositions:

(a) $\hat{L}_1(G)$ is a Banach algebra.

(b) $\hat{L}_1(G)$ has a bounded left approximate identity $\mathscr{E} = \{E_\alpha\}$.

(c) The mapping $I: \hat{L}_1(G) \to \tilde{\hat{L}}_1(G) = \{\tilde{\hat{f}} | \hat{f} \in \hat{L}_1(G)\}$ is a bounded faithful representation of $\hat{L}_1(G)$ in the algebra of all bounded linear operators on the Banach space $C_0^*(X)$.

Once these three propositions are proved, Theorem 1 will follow as a direct consequence of Theorem B. We now proceed to prove the propositions (a), (b) and (c).

(a) It is easy to verify that $\hat{L}_1(G)$ is a linear space and that if we define

$$||\hat{f}|| = \sup\left\{|\hat{f}(h)| \,\middle|\, ||h||_\infty \leqq 1,\, h \in L_\infty(G)\right\}$$
$$= \sup\left\{\left|\int_G f(g)h(g^{-1})\,\mathrm{d}g\right| \,\middle|\, ||h||_\infty \leqq 1,\, h \in L_\infty(G)\right\},$$

then $||\hat{f}|| = ||f||_1$. Hence the mapping $J : L_1(G) \to \hat{L}_1(G)$ taking f to $\hat{f}$ is a linear isometry and thus $\hat{L}_1(G)$ is a Banach space. Next we define $\hat{f}_1 \Delta \hat{f}_2 = \widehat{f_2 * f_1}$ for all $f_1, f_2 \in L_1(G)$; where $f_2 * f_1$ denotes the standard convolution in the group algebra $L_1(G)$ given by

$$(f_2 * f_1)(x) = \int_G f_2(xy^{-1})f_1(y)\,\mathrm{d}y$$

and $\mathrm{d}y$ stands for a (fixed) right Haar measure on G. Now it is clear that the mapping J taking $f \in L_1(G)$ to $\hat{f} \in \hat{L}_1(G)$ is an anti-isomorphism and linear isometry and hence $\hat{L}_1(G)$ is a Banach algebra.

(b) As is well known (see [3], p. 303), the group algebra $L_1(G)$ admits a bounded right approximate identity, say $\{e_\alpha\}_{\alpha \varepsilon A}$ where A is an appropriate directed set. This means that $\lim_\alpha (f * e_\alpha) = f$ for every $f \in L_1(G)$. We let $\mathscr{E} = \{E_\alpha | E_\alpha = \hat{e}_\alpha,\, \alpha \in A\}$. It is now easy to see that $\mathscr{E}$ is a bounded left approximate identity for $\hat{L}_1(G)$.

(c) First, we recall that the mapping I has been introduced in Section 2, immediately preceding Theorem A. For the convenience of the reader we show how it works in the present context. We have already seen that $\hat{f} * M \in C_0^*(X)$ for every $\hat{f} \in \hat{L}_1(G)$ and $M \in C_0^*(X)$. The mapping $I : \hat{L}_1(G) \to \tilde{\hat{L}}_1(G)$ is defined by $I(\hat{f}) = \tilde{\hat{f}}$ where $\tilde{\hat{f}}(M) = \hat{f} * M$ for every $M \in C_0^*(X)$. It is clear that $\tilde{\hat{f}}$ is indeed a bounded linear operator on the Banach space $C_0^*(X)$. We also have for every $F \in C_0(X)$,

$$(\tilde{\hat{f}}(M))(F) = (\hat{f} * M)(F) = \int_G f(g)(\overline{M}F)(g^{-1})\,\mathrm{d}g,$$

where as usual, $\mathrm{d}g$ denotes right Haar measure and $(\overline{M}F)(g^{-1}) = M({}_{g^{-1}}F)$. It was noted earlier that $\overline{M}F \in C(G) \subset L_\infty(G)$. We use the above integral representation to show that the mapping I is a bounded faithful representation of $\hat{L}_1(G)$ in the algebra of bounded linear operators on $C_0^*(X)$.

We need to show the following: (i) $||I||$ is finite; (ii) I is linear; (iii) I is multiplicative, that is $I(\hat{f}_1 \Delta \hat{f}_2) = I(\hat{f}_1) \circ I(\hat{f}_2)$ for all $\hat{f}_1, \hat{f}_2 \in \hat{L}_1(G)$ and $\circ$ denotes composition of operators; and finally (iv) I is one-one, that is if $\hat{f}_1, \hat{f}_2 \in \hat{L}_1(G)$ and $I(\hat{f}_1) = I(\hat{f}_2)$ then $\hat{f}_1 = \hat{f}_2$. The assertions (i) and (ii) are immediate consequences of the properties of the mapping I and (iv) is part of the hypothesis of Theorem 1. We prove (iii) as follows:

Let $\hat{f}_1, \hat{f}_2 \in \hat{L}_1(G)$. Then $I(\hat{f}_1 \Delta \hat{f}_2) = I(\widehat{f_2 * f_1}) = \widetilde{\widehat{f_2 * f_1}}$. Thus if $M \in C_0^*(X)$ and $F \in C_0(X)$, then by the integral representation, we have

$$(\widetilde{\widehat{f_2 * f_1}}(M))(F) = ((\widehat{f_2 * f_1}) * M)(F) = \int_G (f_2 * f_1)(g)(\overline{M}F)(g^{-1})\,\mathrm{d}g$$
$$= \int_G \int_G f_2(gh^{-1}) f_1(h)(\overline{M}F)(g^{-1})\,\mathrm{d}h\,\mathrm{d}g.$$

On the otherhand we have

$$[I(\hat{f}_1) \circ I(\hat{f}_2)](M)(F) = [(\tilde{\hat{f}}_1 \circ \tilde{\hat{f}}_2)(M)](F) = [\hat{f}_1 * (\tilde{\hat{f}}_2(M))](F)$$
$$= \int_G f_1(h)[\overline{(\hat{f}_2 * M)}F](h^{-1})\,\mathrm{d}h$$
$$\overset{v}{=} \int_G f_1(h)(\hat{f}_2 * M)({}_{h^{-1}}F)\,\mathrm{d}h$$
$$= \int_G \int_G f_1(h) f_2(g)(\overline{M}_{h^{-1}}F)(g^{-1})\,\mathrm{d}g\,\mathrm{d}h$$
$$= \int_G \int_G f_1(h) f_2(g) M({}_{g^{-1}h^{-1}}F)\,\mathrm{d}g\,\mathrm{d}h$$
$$= \int_G \int_G f_1(h) f_2(g)(\overline{M}F)(g^{-1}h^{-1})\,\mathrm{d}g\,\mathrm{d}h.$$

Now by a careful use of Fubini's theorem (see [6], p. 269 and [3], p. 153) and the right invariance of the chosen Haar measure, we conclude that the two integrals giving the values of $[I(\hat{f}_1 \Delta \hat{f}_2)(M)](F)$ and $[(I(\hat{f}_1) \circ I(\hat{f}_2))(M)](F)$ are equal for every $M \in C_0^*(X)$ and $F \in C_0(X)$. Thus we have proved (iii) and completed the proof of Theorem 1.

COROLLARY 1: *If the right approximate identity* $\{e_\alpha\}_{\alpha \in A}$ *in* $L_1(G)$ *is appropriately chosen, then* $\lim_\alpha (\hat{e}_\alpha * M)(F) = M(F)$ *for each* $M \in C_0^*(X)$ *and* $F \in C_0(X)$.

PROOF: Let $A = \{\alpha\}$ denote the collection of all open subsets of G with compact closure and containing the identity of G. Then A is partially ordered and directed by set inclusion. For each $\alpha \in A$, define e_α on G by $e_\alpha(g) = 1/\lambda(\alpha)$ if $g \in \alpha$ and $e_\alpha(g) = 0$ otherwise, where λ denotes the chosen right Haar measure on G. We now have $\{e_\alpha\}_{\alpha \varepsilon A}$ is a right approximate identity for $L_1(G)$ and also

$$\int_G e_\alpha(g)\,\mathrm{d}g = 1$$

for every α.

Now let $M \in C_0^*(X)$ and let μ denote the finite regular Borel measure which corresponds to M by the Riesz representation theorem. Then we have for every $F \in C_0(X)$,

$$\begin{aligned}
|(E_\alpha * M)(F) - M(F)| &= \left| \int_G \int_X e_\alpha(g) F(g^{-1}x)\,\mathrm{d}\mu(x)\,\mathrm{d}g - \int_X F(x)\,\mathrm{d}\mu(x) \right| \\
&= \left| \int_G \int_X e_\alpha(g) F(g^{-1}x)\,\mathrm{d}\mu(x)\,\mathrm{d}g \right. \\
&\quad \left. - \int_G \int_X e_\alpha(g) F(x)\,\mathrm{d}\mu(x)\,\mathrm{d}g \right| \\
&= \left| \int_G \int_X e_\alpha(g) [{}_{g^{-1}}F(x) - F(x)]\,\mathrm{d}\mu(x)\,\mathrm{d}g \right| \\
&\leqq \int_G \int_X e_\alpha(g) |{}_{g^{-1}}F(x) - F(x)| \mathrm{d}|\mu|(x)\,\mathrm{d}g \\
&\leqq \int_G e_\alpha(g) ||{}_{g^{-1}}F - F||\, |\mu|(X)\,\mathrm{d}g \to 0,
\end{aligned}$$

since the e_α are supported on 'small neighborhoods' of the identity of G and the mapping $g \to {}_gF$ is right uniformly continuous. This proves the corollary.

Proof of Theorem 2: Let $\mathscr{E}' = \sigma(\mathscr{E}) = \{E_\alpha | \sigma(e_\alpha),\ e_\alpha \in \mathscr{E}\}$. Then $||E_\alpha|| = ||\sigma(e_\alpha)|| \leqq ||\sigma||\, ||e_\alpha|| \leqq ||\sigma||\ C$ which means $||\mathscr{E}'|| \leqq ||\sigma||\, C$. Since $\mathscr{E}$ is a left approximate identity, given $\varepsilon > 0$ and a finite subset $\{e_1, \cdots, e_n\}$ of $\mathscr{E}$ there exists $e \in \mathscr{E}$ such that $||ee_i - e_i|| < \varepsilon/||\sigma||$, $i = 1, \cdots, n$. (See [2], p. 170.) But then

$$||EE_i - E_i|| = ||\sigma(e)\sigma(e_i) - \sigma(e_i)|| = ||\sigma(ee_i - e_i)|| \leqq ||\sigma||\, ||ee_i - e_i|| < \varepsilon.$$

Hence $\mathscr{E}'$ satisfies the hypothesis of Theorem 2.2 in [2] and if $\mathscr{A}'$ denotes the Banach algebra generated by $\mathscr{E}'$, then we can conclude that

$$Y = \mathscr{A}'X = \{Tx | T \in \mathscr{A}', x \in X\} = \{x \in X | \lim_\alpha E_\alpha x = x, E_\alpha \in \mathscr{E}'\}$$

is a closed subspace of X.

It only remains to show that $\mathscr{A}X = \{\sigma(a)x | a \in \mathscr{A}\} = \mathscr{A}'X = Y$. Since $\lim_\alpha e_\alpha a = a$ for each $a \in \mathscr{A}$, it follows that $\lim_\alpha E_\alpha A = \lim_\alpha$

$\sigma(e_\alpha a) = \sigma(a) = A$, so that $\mathscr{A}X \subseteq Y$. Equality of these last two spaces will follow if we prove $Y \subseteq \mathscr{A}X$. But this is an immediate consequence of the last assertion of the theorem, which we prove now. Let $0 < b < 1/(4\|\sigma\|C)$. By Theorem 2.2 [2], if $y \in Y$ and $\delta > 0$ there exist $z \in Y$ such that $\|z-y\| < \delta$ and a sequence $\{E_k\} \subset \mathscr{E}'$ such that if

$$A = \sum_{k=1}^{\infty} b(1-b)^{k-1}E_k$$

then $Az = y$. Now let $e_k = \sigma^{-1}(E_k)$, which is meaningful since σ is faithful on $\mathscr{E}$. Clearly

$$\sum_{k=1}^{\infty} b(1-b)^{k-1}e_k$$

converges absolutely in $\mathscr{A}$ if $0 < b < 1$, since $\|\mathscr{E}\|$ is bounded. Thus if $0 < b < 1/(4\|\sigma\|C)$ and $0 < b < 1$, then

$$a = \sum_{k=1}^{\infty} b(1-b)^{k-1}e_k \in \mathscr{A}$$

and $\sigma(a) = A$. Now $\sigma(a)z = Az = y$. Also

$$\|a\| \leqq C\sum_{k=1}^{\infty} b(1-b)^{k-1}.$$

Proof of Theorem 3: It is easy to verify that if $F \in C_0(X)$, then ${}_gF \in C_0(X)$ for each $g \in G$. Next we prove a lemma.

Lemma 1: *If $F \in C_0(X)$, then the mapping $g \to {}_gF$ from G into $C_0(X)$ is (right) uniformly continuous.*

Proof of Lemma 1: We have to prove that given $\varepsilon > 0$, there exists open subset U of G containing the identity such that if $g_1 g_2^{-1} \in U$, then $\|{}_{g_1}F - {}_{g_2}F\| = \sup\{|{}_{g_1}F(x) - {}_{g_2}F(x)| \mid x \in X\} < \varepsilon$. Since $\|{}_gF\| = \|F\|$ for any $F \in C_0(X)$ and $g \in G$, it is enough to show $\sup_x |F(x) - {}_{g_1 g_2^{-1}}F(x)| < \varepsilon$. Also, since the space $C_{00}(X)$(continuous functions with compact supports) is norm dense in $C_0(X)$, we may assume that F has compact support, say S. Now let $x \in S$ and let V_x be an open neighborhood of x such that $|F(x) - F(y)| < \varepsilon/2$ for all $y \in V_x$. By the continuity of the mapping $(g, x) \to g(x)$, there exists an open neighborhood U_x of the identity in G and an open neighborhood $V_x' \subseteq V_x$ of x such that $\{g(x) \mid (g, x) \in U_x \times V_x'\} \subseteq V_x$. Also there exist $U_x' \subseteq U_x$, an open neighborhood of the identity, and $V_x'' \subseteq V_x'$ an open neighborhood of x such that $\{g(x) \mid (g, x) \in U_x' \times V_x''\} \subseteq V_x'$. Now if $g \in U_x'$ and $y \in V_x'' \subseteq V_x$, then $g(y) \in V_x' \subseteq V_x$, which implies that $|F(y) - {}_gF(y)| \leqq |F(y) - F(x)| + |F(x) - F(gy)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. The family $\{V_x'' \mid x \in S\}$ is an open covering of the compact S

and hence there exists a finite subcovering, $\{V''_{x_1}, \cdots, V''_{x_n}\}$. Let

$$U' = \bigcap_{i=1}^{n} U'_{x_i}$$

and let $U = U' \cap U'^{-1}$. Then U is nonempty (contains the identity) and open in G.

We claim that for each $x \in X$ and $g \in U$, $|F(x) - {}_gF(x)| < \varepsilon$ which implies $||F - {}_gF|| < \varepsilon$ and proves the lemma. To this end, consider $|F(x) - {}_gF(x)|$, where $g \in U$. If

$$x \in \bigcup_{i=1}^{n} V'_{x_i},$$

then $x \in V'_{x_k} \subseteq V_{x_k}$ for some $1 \leqq k \leqq n$ and $g \in U \subseteq U' \subseteq U'_{x_k} \subseteq U_{x_k}$ and thus $g(x) \in V_{x_k}$ and we have $|F(x) - {}_gF(x)| \leqq |F(x) - F(x_k)| + |F(x_k) - F(g(x))| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. On the otherhand, if

$$x \in X \setminus \bigcup_{i=1}^{n} V'_{x_i},$$

then $x \notin S$ and hence $F(x) = 0$. Consider $g(x)$. There are three mutually exclusive possibilities for the location of $g(x)$:

(i) $g(x) \in X \setminus \bigcup_{i=1}^{n} V'_{x_i}$;

(ii) $g(x) \in \bigcup_{i=1}^{n} V'_{x_i} \setminus \bigcup_{i=1}^{n} V''_{x_i}$;

and

(iii) $g(x) \in \bigcup_{i=1}^{n} V''_{x_i}$.

In case (i) $F(g(x)) = 0$, so $|F(x) - {}_gF(x)| = 0 < \varepsilon$. In case (ii) again $F(g(x)) = 0$ and $|F(x) - {}_gF(x)| = 0 < \varepsilon$. Finally in case

(iii), $g(x) \in \bigcup_{i=1}^{n} V''_{x_i}$

and so $g(x) \in V''_{x_k}$ for some $1 \leqq k \leqq n$. Since $g \in U$ and U is a symmetric neighborhood, $g^{-1} \in U$ and so

$$g^{-1}(g(x)) = x \in V'_{x_k} \subseteq \bigcup_{i=1}^{n} V'_{x_i}$$

which contradicts the fact that

$$x \in X \setminus \bigcup_{i=1}^{n} V'_{x_i}.$$

Thus for all $x \in X$ and $g \in U$ we have $|F(x) - {}_gF(x)| < \varepsilon$ and the lemma is proved.

Now we are ready to proceed with the proof of Theorem 3. First we observe that as in the proof of Theorem 1, we have: if $M \in C_0^*(X)$ and $F \in C_0(X)$, then $\overline{M}F \in C(G) \subseteq L_\infty(G)$; and if $\hat{f} \in \hat{L}_1(G)$ and $M \in C_0^*(X)$, then $\hat{f} * M \in C_0^*(X)$. Thus we have $\hat{L}_1(G) * C_0^*(X) \subseteq C_0^*(X)$. Now the rest of Theorem 3 will follow from Theorem 2 if we carry out the following five steps: (a) $\hat{L}_1(G)$ is a Banach algebra, (b) $\hat{L}_1(G)$ has a bounded left approximate identity, (c) $I : \hat{f} \to \tilde{\hat{f}}$ is a bounded representation of $\hat{L}_1(G)$ in the algebra of bounded linear operators on the Banach space $C_0^*(X)$, (d) the left approximate identity stated in (b) is faithfully represented by I, and finally (e) that if $\{\hat{e}_\alpha\}$ is the left approximate identity given in (b), then $\lim_\alpha (\hat{e}_\alpha * M)(F) = M(F)$ for each $M \in C_0^*(X)$ and $F \in C_0(X)$.

(a) This was done in part (a) of the proof of Theorem 1.

(b) This was also done in part (b) of the proof of Theorem 1, but we construct a special left approximate identity as follows, which will aid in proving the remaining parts. Since G is first countable and locally compact, there exists a sequence of open sets $\{U_n\}_1^\infty$ in G each containing the identity and having compact closure and such that $U_n \subseteq U_m$ if $n \geqq m$

$$\text{and} \bigcap_{n=1}^{\infty} U_n = \{i\}$$

where i denotes the identity of G. Let λ denote the right Haar measure on G and define a sequence $\{e_n\}_1^\infty$ of functions on G as follows: For each n, let $e_n(g) = 1/\lambda(U_n)$ if $g \in U_n$ and let $e_n(g) = 0$ otherwise. It is now easily verified that $\{e_n\}_1^\infty$ is a right approximate identity for the group algebra $L_1(G)$ and hence $\{\hat{e}_n\}_1^\infty$ is a left approximate identity for $\hat{L}_1(G)$. Finally, we remark that $\{\hat{e}_n\}_1^\infty$ will be kept fixed in the remainder of the proof (of Theorem 3).

(c) Same as part (c) of the proof of Theorem 1.

(d) We need to show that if $\hat{e}_m * M = \hat{e}_p * M$ for all $M \in C_0^*(X)$, then $e_m = e_p$. Let $\hat{e}_m * M = \hat{e}_p * M$. Then for $F \in C_0(X)$, we have

$$(\hat{e}_m * M)(F) = \int_G e_m(g) M({}_{g^{-1}}F)\,\mathrm{d}g$$
$$= \int_G \int_X e_m(g) F(g^{-1}x)\,\mathrm{d}\mu(x)\,\mathrm{d}g$$

where μ is the finite regular Borel measure that corresponds to the functional M by Riesz theorem and $\mathrm{d}g$, as usual, denotes the right Haar measure on G. Similarly, we get

$$(\hat{e}_p * M)(F) = \int_G \int_X e_p(g) F(g^{-1}x)\,\mathrm{d}\mu(x)\,\mathrm{d}g.$$

We may suppose $m \geqq p$, without loss of generality. Now $(\hat{e}_m * M)(F) = (\hat{e}_p * M)(F)$ implies that

$$\int_G \int_X [e_m(g) - e_p(g)] F(g^{-1}x)\,\mathrm{d}\mu(x)\,\mathrm{d}g = 0$$

for all $F \in C_0(X)$ and for all bounded regular Borel measures μ on X. Applying Fubini's theorem, as in part (c)(iii) of the proof of Theorem 1, we get

$$\int_X \int_G [e_m(g) - e_p(g)] F(g^{-1}x)\mathrm{d}g\mathrm{d}\mu(x) = 0.$$

Now let $$H(x) = \int_G [e_m(g) - e_p(g)] F(g^{-1}x)\,\mathrm{d}g.$$

Then we have $$\int_X H(x)\,\mathrm{d}\mu(x) = 0$$

for every μ. It is easy to see that if $F \in C_{00}(X)$ then the corresponding $H \in C_{00}(X)$, that is H is continuous with compact support if F is continuous with compact support. Thus if we start with a function $F \in C_{00}(X)$ then it follows that $H(x) = 0$ identically for all $x \in X$.

By hypothesis there exists an $x \in X$ such that the mapping $g \to g(x)$ is one-one and open. Also $m \geqq p$ implies $U_m \subseteq U_p$ and hence $e_m - e_p \geqq 0$ on U_m. Now the set $U_m^{-1}x$ is open in X, since U_m^{-1} is open in G, and $x \in U_m^{-1}x$. Since the space x is locally compact (and Hausdorff), it is completely regular and there exists (by Urysohn's Lemma) a continuous function F on X such that $F(X) \subseteq [0, 1]$, $F(x) = 1$, $F(y) = 0$ for all $y \in X \setminus U_m^{-1}x$ (a closed set not containing x) and finally F has compact support since F vanishes outside the compact set $\overline{U_m^{-1}x} \supseteq U_m^{-1}x$. Thus $F \in C_{00}(X)$. Now the only way the corresponding

$$H(x) = \int_G [e_m(g) - e_p(g)] F(g^{-1}x)\,\mathrm{d}g$$

can equal to zero (as established above) is if $e_m = e_p$ (as desired).

(e) Let $M \in C_0^*(X)$ and $F \in C_0(X)$. Then

$$\begin{aligned} |(\hat{e}_n * M)(F) - M(F)| &= \left| \int_G e_n(g) M({}_{g^{-1}}F)\,\mathrm{d}g - M(F) \right| \\ &= \left| \int_G e_n(g) M({}_{g^{-1}}F - F)\,\mathrm{d}g \right| \\ &\leqq \int_G |e_n(g)|\, \|M\|\, \|{}_{g^{-1}}F - F\|\,\mathrm{d}g \to 0 \end{aligned}$$

since the map $g \to {}_gF$ is uniformly continuous and the weights e_n are supported on 'small' neighborhoods of the identity of G. Thus $\lim_n (\hat{e}_n * M)(F) = M(F)$ for each $M \in C_0^*(X)$ and $F \in C_0(X)$ which proves (e) and completes the proof of Theorem 3.

PROOF OF THEOREM 4: First we note that as in the proofs of theorems 1 and 3, we have: if $g \in G$ and $F \in C_u(X)$, then ${}_gF \in C_u(X)$; if $M \in C_u^*(X)$ and $F \in C_u(X)$, then $\overline{M}F \in C(G) \subseteq L_\infty(G)$; and $\hat{f} * M \in C_u^*(X)$ for each $\hat{f} \in \hat{L}_1(G)$ and $M \in C_u^*(X)$. Further, like Theorem 3, our present theorem will follow from Theorem 2 if we prove the following five steps: (a) $\hat{L}_1(G)$ is a Banach algebra, (b) $\hat{L}_1(G)$ has a bounded left approximate identity, (c) $I : \hat{f} \to \tilde{\hat{f}}$ is a bounded representation of $\hat{L}_1(G)$ in the algebra of all bounded linear operators on the Banach space $C_u^*(X)$, (d) the left approximate identity mentioned in (b) is faithfully represented by I, and finally (e) if $\{\hat{e}_\alpha\}$ denotes the left approximate identity mentioned in (b) then $\lim_\alpha (\hat{e}_\alpha * M)(F) = M(F)$ for every $M \in C_u^*(X)$ and $F \in C_u(X)$.

Now (a), (b) and (c) are proved exactly as in the proof of Theorem 3. Since $C_0(X) \subseteq C_u(X)$, we can again prove that the left approximate identity is faithfully represented by I, in the same way as we did in part (d) of the proof of Theorem 3. Finally, the hypothesis that given $V \in \mathscr{U}$ there exists an open neighborhood U of the identity in G such that $(x, g(x)) \in V$ for each $g \in U$ and $x \in X$, clearly implies that the mapping $g \to {}_gF$ from G to $C_u(X)$ is right uniformly continuous. Using these facts we can prove part (e) just as we did part (e) of Theorem 3.

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