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THE USE OF INJECTIVE-LIKE STRUCTURES IN MODEL THEORY

H. Simmons

The use of homogeneous-universal and saturated structures in model theory is now well established. Recently Mycielski et al have introduced into model theory compact structures and Fisher has introduced existentially complete structures. Here we attempt a uniform approach to these and related concepts. Our central theme is that of injective-like structures, i.e. structures which have properties resembling the properties of injective objects in categories.

This paper is complementary to [5] and although a detailed knowledge of that paper is not required to understand this paper, the reader will find that a familiarity with the ideas of [5] (especially the latter part of Section 5) will help him.

The background facts required are given in Section 1. In Section 2 we define and construct several kinds of injective-like structures. These constructions are carried out in a category-theoretic setting. In Section 3 we characterize these injective-like structures in model-theoretic terms, and consider several related kinds of structures. These we call compact-like structures, (they are generalizations of saturated structures).

In Section 4 we (essentially) consider the relationship between these injective-like structures and model complete theories or model companions. In Section 5 we indicate how the back and forth argument can be used in conjunction with injective-like structures. Finally in Section 6 we consider the connection between homogeneous-universal structures, saturated structures and injectivelike structures.

There have been several factors which have influenced me while discovering the results presented here. Several of the ideas came naturally while writing [5]. After reading a first draft of [5] Paul Bacsich suggested several improvements and drew my attention to the work of Mycielski et al. After ruminating on these results and those contained in [1] I began to develop the results given here.

Finally both I and Angus Macintyre had been attempting to eliminate the forcing technique from the construction of Robinson's generic structures (using infinite forcing). Together we eventually realized that the way to do this was to use injective-like structures (see theorem 5.8).

Bacsich has independently obtained some of the results presented here, and [1] suggests that Fisher knows some of these results also.

I thank both Paul Bacsich and Angus Macintyre for their comments and suggestions, without which I could not have written this paper.

1. Introduction

Throughout the paper L is some fixed first order language of cardinality λ, κ is some fixed cardinal such that $\lambda < \kappa$, and r is some fixed positive integer. We are concerned with the model theory of L so, unless we state otherwise, all theories T are L-theories, all structures $\mathfrak{A}$ are L-structures, all formulas ϕ are L-formulas, etc. Initially the reader may find it convenient to put $r = 0$, in which case several of the results proved here will already be familiar to him.

We use standard notation and terminology together with some that we used in [5]. In several places we use [5, theorem 1.1] without explicit reference.

Throughout the paper we are concerned with injections.

$$\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{B}$$

between structures $\mathfrak{A}$, $\mathfrak{B}$. For any point

$$\boldsymbol{a} = (a_1, \cdots, a_k)$$

of $\mathfrak{A}$ we let

$$f(\boldsymbol{a}) = (f(a_1), \cdots, f(a_k))$$

so that $f(\boldsymbol{a})$ is a point of $\mathfrak{B}$. We use a similar notation with infinite sequences $\vec{a}$ taken from $\mathfrak{A}$.

We say f is elementary if for each formula $\phi(\boldsymbol{v})$ and point $\boldsymbol{a}$ of $\mathfrak{A}$,

$$\mathfrak{A} \models \phi(\boldsymbol{a}) \Rightarrow \mathfrak{B} \models \phi(f(\boldsymbol{a})).$$

Notice that if f is an insertion (the injection associated with an inclusion) then f is elementary if and only if $\mathfrak{A} \prec \mathfrak{B}$.

We say f is $\prec_r$-like if for each formula $\phi(\boldsymbol{v}) \in \forall_r$ and point $\boldsymbol{a}$ of $\mathfrak{A}$,

$$\mathfrak{A} \models \phi(\boldsymbol{a}) \Rightarrow \mathfrak{B} \models \phi(f(\boldsymbol{a})).$$

Thus every injection is $\prec_0$-like and an insertion is $\prec_r$-like if and only if $\mathfrak{A} \prec_r \mathfrak{B}$.

Given any theory T we can form several categories based on the models of T.

Definition: For any theory T, $\boldsymbol{M}_r(T)$ is the category whose objects are the models of T and whose morphisms are the $\prec_r$-like injections between these models. Also $\boldsymbol{M}_\infty(T)$ is the category whose objects are the models of T and whose morphisms are the elementary injections between these models.

In what follows $\boldsymbol{C}$ is one of these categories. Of course these categories may not be distinct. For instance if T is model complete then $\boldsymbol{M}_0(T) = \boldsymbol{M}_\infty(T)$.

In Section 2 we will construct several objects inside these categories. To do this we will require two methods of construction, namely union of chains (*UC*), and the amalgamation property (*AP*).

DEFINITION: Let $\boldsymbol{C}$ be any of the above categories. $\boldsymbol{C}$ has *UC* if for each ascending chain

$$\mathfrak{A}_0 \rightarrowtail \mathfrak{A}_1 \rightarrowtail \cdots \rightarrowtail \mathfrak{A}_i \rightarrowtail \cdots i < \mu$$

of insertions of $\boldsymbol{C}$ the union

$$\cup \{\mathfrak{A}_i : i < \mu\}$$

is an object of $\boldsymbol{C}$. A theory T has UC_r if $\boldsymbol{M}_r(T)$ has *UC* and has UC_∞ if $\boldsymbol{M}_\infty(T)$ has *UC*.

A theorem of Tarski shows that every theory has UC_∞. The following theorem is a generalization of theorems of Tarski and Chang-Los-Suszko. It gives a simple characterization of the theories with UC_r.

THEOREM (1.1): *For any theory T the following are equivalent.*

(i) *T is $\forall_{r+2}$-axiomatizable.*

(ii) *For any two structures $\mathfrak{A}, \mathfrak{B}$.*

$$\mathfrak{A} \prec_{r+1} \mathfrak{B} \models T \Rightarrow \mathfrak{A} \models T.$$

(iii) *T has UC_r.*

Notice that *UC* is not a category-theoretic property (i.e. not definable in category-theoretic terms) since it refers to insertions. However *AP* is a category-theoretic property.

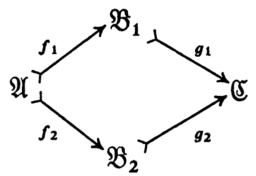
DEFINITION: Let $\boldsymbol{C}$ be any of the above categories. $\boldsymbol{C}$ has *AP* if for each pair of injections

$$\mathfrak{A} \xrightarrow{f_1} \mathfrak{B}_1, \mathfrak{A} \xrightarrow{f_2} \mathfrak{B}_2$$

of $\boldsymbol{C}$ there are injections

$$\mathfrak{B}_1 \xrightarrow{g_1} \mathfrak{C}, \mathfrak{B}_2 \xrightarrow{g_2} \mathfrak{C}$$

of $\boldsymbol{C}$ such that



commutes, i.e. $g_1 \circ f_1 = g_2 \circ f_2$. A theory T has AP_r if $\boldsymbol{M}_r(T)$ has AP, and has AP_∞ if $\boldsymbol{M}_\infty(T)$ has AP.

Any two elementary equivalent structures have isomorphic elementary extensions and so any theory T has AP_∞. To see this we argue as follows.

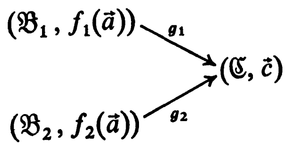
Consider any two elementary injections

$$\mathfrak{A} \overset{f_1}{\rightarrowtail} \mathfrak{B}_1,\ \mathfrak{A} \overset{f_2}{\rightarrowtail} \mathfrak{B}_2$$

and let $\vec{a}$ be an enumeration of $\mathfrak{A}$. Thus we have

$$(\mathfrak{B}_1, f_1(\vec{a})) \equiv (\mathfrak{B}_2, f_2(\vec{a}))$$

and so we obtain



for some structure $(\mathfrak{C}, \vec{c})$ and elementary injections g_1, g_2. This gives $g_1 \circ f_1 = g_2 \circ f_2$, as required.

There is no simple characterization of theories with AP_r, however many of the applications of AP_r can be avoided by using the following theorem.

THEOREM (1.2): *Let T be any theory. For any pair of injections*

$$\mathfrak{A} \overset{f_1}{\rightarrowtail} \mathfrak{B}_1,\ \mathfrak{A} \overset{f_2}{\rightarrowtail} \mathfrak{B}_2$$

where $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ are models of T, f_1 is $\prec_{r+1}$-like, and f_2 is $\prec_r$-like, there are injections

$$\mathfrak{B}_1 \overset{g_1}{\rightarrowtail} \mathfrak{C},\ \mathfrak{B}_2 \overset{g_2}{\rightarrowtail} \mathfrak{C}$$

where $\mathfrak{C}$ is a model of T, g_1 is $\prec_r$-like, g_2 is elementary and $g_1 \circ f_1 = g_2 \circ f_2$.

PROOF: Use the above technique with [5, lemma 3.3].

In Section 6 we will require the joint embedding property.

DEFINITION: The category $\boldsymbol{C}$ has *JEP* if for any two objects $\mathfrak{A}_1$, $\mathfrak{A}_2$ of $\boldsymbol{C}$ there are injections

$$\mathfrak{A}_1 \rightarrowtail \mathfrak{B},\ \mathfrak{A}_2 \rightarrowtail \mathfrak{B}$$

of C. A theory T has JEP_r if $M_r(T)$ has JEP, and has JEP_∞ if $M_\infty(T)$ has JEP.

The following characterization of theories with JEP_r is easily proved.

THEOREM (1.3): *For any theory T the following are equivalent.*

(i) *T has JEP_r.*

(ii) *For any two sentences σ_1, σ_2 in $\forall_{r+1}$,*

$$T \vdash \sigma_1 \vee \sigma_2 \Rightarrow T \vdash \sigma_1 \text{ or } T \vdash \sigma_2 .$$

2. Injective-like structures

Recall that an object $\mathfrak{M}$ of a category C is injective (in C) if for each morphism $\mathfrak{A} \longrightarrow \mathfrak{M}$ and injection $\mathfrak{A} \rightarrowtail \mathfrak{B}$ there is a morphism $\mathfrak{B} \longrightarrow \mathfrak{M}$ such that

$$\begin{array}{ccc} \mathfrak{A} & \longrightarrow & \mathfrak{M} \\ & \searrow \quad \nearrow & \\ & \mathfrak{B} & \end{array}$$

commutes. In any of the categories C we are considering all morphisms are injections hence (on cardinality grounds) C has no injective objects. Thus we are free to introduce into C injective-like objects by suitably modifying the definition of injective. We do this in several ways, these ways being variations on the following theme.

DEFINITION: The object $\mathfrak{M}$ of C is κ-*WI* (κ-weakly injective) in C if for each pair of injections $\mathfrak{A} \rightarrowtail \mathfrak{M}$, $\mathfrak{A} \rightarrowtail \mathfrak{B}$ of C with card $(\mathfrak{B}) < \kappa$, there is an injection $\mathfrak{B} \rightarrowtail \mathfrak{M}$ of C such that the above diagram commutes.

At this stage the reader may like to convince himself that the $\kappa - WI$ of $M_\infty(T)$ are exactly the κ-saturated models of T, and the κ-homogeneous-universal models of T are all κ-*WI* in $M_0(T)$.

To obtain the several kinds of injective-like structures we modify the above definition by putting further restrictions on the injections $\mathfrak{A} \rightarrowtail \mathfrak{M}$, $\mathfrak{A} \rightarrowtail \mathfrak{B}$, $\mathfrak{B} \rightarrowtail \mathfrak{M}$. To define and discuss these structures we repeatedly consider the following situation.

$$\left.\begin{array}{l} \begin{array}{ccc} \mathfrak{A} & \overset{f}{\rightarrowtail} & \mathfrak{M} \\ & {\scriptstyle g}\searrow & \\ & & \mathfrak{B} \end{array} \\ \text{where } \mathfrak{A}, \mathfrak{B}, \mathfrak{M} \text{ are models of } T \text{ and the cardinals} \\ \qquad \mathfrak{a} = \text{card}\,(\mathfrak{A}),\ \mathfrak{b} = \text{card}\,(\mathfrak{B}),\ \mathfrak{m} = \text{card}\,(\mathfrak{M}) \\ \text{satisfy } \mathfrak{a} \leqq \mathfrak{b} < \kappa. \end{array}\right\} \quad (S_1)$$

In what follows p, q, r are positive integers or ∞, and '$\prec_\infty$' means '$\prec$'.

Definition: The model $\mathfrak{M}$ of T is κ-(p, q, r)-I over T if for each situation (S_1) where f is $\prec_p$-like and g is $\prec_q$-like, there is a $\prec_r$-like injection $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ such that $f = h \circ g$.

We are mainly interested in three special kinds of the κ-$(p, q, r)-I$. These classes are

$$\begin{aligned} \kappa\text{-}WI_r &= \kappa\text{-}(r, r, r)-I, \\ \kappa\text{-}WJ_r &= \kappa\text{-}(\infty, r, r)-I, \\ \kappa\text{-}WK_r &= \kappa\text{-}(\infty, \infty, r)-I. \end{aligned}$$

The most important of these three classes is $\kappa - WJ_r$.

Notice that the $\kappa - WI_r$ are the $\kappa - WI$ of $M_r(T)$. Also $\kappa - WI_\infty = \kappa - WJ_\infty = \kappa - WK_\infty$.

The following theorem follows immediately from the definitions involved.

Theorem (2.1): (a) *If* $p_1 \leqq p_2, q_1 \leqq q_2, r_2 \leqq r_1$ *then*

$$\kappa-(p_1, q_1, r_1)-I \subseteq \kappa-(p_2, q_2, r_2)-I.$$

(b) *If* $\kappa_2 \leqq \kappa_1$ *then*

$$\kappa_1-(p, q, r)-I \subseteq \kappa_2(p, q, r)-I.$$

From part (a) of this theorem we have the following inclusions (where '$\subseteq$' has been replaced by '$\longrightarrow$').

$$\begin{array}{ccccc} & & & & \kappa - WI_\infty \\ & & & & \downarrow \\ \kappa - WI_r & \longrightarrow & \kappa - WJ_r & \longrightarrow & \kappa - WK_r \end{array}$$

Before we consider the existence of the $\kappa - WX$ for $X \in \{I_r, J_r, K_r, I_\infty\}$ we will show that restricting our attention to these four cases is not much of a restriction. There are many equalities between the various classes of injectivelike structures. Roughly speaking if either p or q is too large then it might as well be ∞. This is shown in the next three theorems.

Theorem (2.2): $\kappa-(q+1, q, r)-I = \kappa-(\infty, q, r)-I$

Proof: From theorem 2.1 it is sufficient to show that the right hand side is included in the left hand side.

Suppose $\mathfrak{M}$ is $\kappa-(\infty, q, r)-I$ over T and consider any situation (S_1) where f is $\prec_{q+1}$-like and g is $\prec_q$-like. We must produce a suitable $\prec_r$-like injection $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$.

We have

$$\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{M} = \mathfrak{A} \overset{f'}{\rightarrowtail} \mathfrak{A}' \prec_r \mathfrak{M}$$

for some $\prec_{q+1}$-like injection f' and model $\mathfrak{A}'$ of cardinality $\mathfrak{a}+\lambda < \kappa$. Theorem 1.2 gives us a commuting diagram

$$\begin{array}{ccc} \mathfrak{A} & \overset{f'}{\rightarrowtail} & A' \prec \mathfrak{M} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathfrak{B} & \underset{k}{\rightarrowtail} & \mathfrak{B}' \end{array}$$

where k is elementary and g' is $\prec_q$-like. We may assume that card $(\mathfrak{B}) = \mathfrak{b}+\mathfrak{a}+\lambda < \kappa$. Now $\mathfrak{M}$ is $\kappa-(\infty, q, r)-I$ and so we have a commuting diagram

$$\begin{array}{ccc} \mathfrak{A}' & \prec & \mathfrak{M} \\ {\scriptstyle g'}\searrow & & \nearrow{\scriptstyle h'} \\ & \mathfrak{B}' & \end{array}$$

for some $\prec_r$-like injection h'. The required injection is obtained by putting $h = h' \circ k$.

THEOREM (2.3): $\kappa-(p, r+1, r)-I = \kappa-(p, \infty, r)-I.$

PROOF: From theorem 2.1 it is sufficient to show that the right hand side is included in the left hand side.

Suppose $\mathfrak{M}$ is $\kappa-(p, \infty, r)-I$ over T and consider any situation (S_1) where f is $\prec_p$-like and g is $\prec_{r+1}$-like. We must produce a suitable $\prec_r$-like injection $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$.

We have

$$\mathfrak{A} \overset{g'}{\rightarrowtail} \mathfrak{B}' = \mathfrak{A} \overset{g}{\rightarrowtail} \mathfrak{B} \prec_r \mathfrak{B}'$$

for some elementary injection g' and model $\mathfrak{B}'$ of cardinality $\mathfrak{b}+\lambda < \kappa$. But $\mathfrak{M}$ is $\kappa-(p, \infty, r)-I$ so we have a commuting diagram

$$\begin{array}{ccc} \mathfrak{A} & \overset{f}{\rightarrowtail} & \mathfrak{M} \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle h'} \\ \mathfrak{B} & \prec_r & \mathfrak{B}' \end{array}$$

for some $\prec_r$-like injection h'. Thus putting

$$\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M} = \mathfrak{B} \prec_r \mathfrak{B}' \overset{h'}{\rightarrowtail} \mathfrak{M}$$

we obtain the required injection.

Theorem (2.4): $\kappa-(r, \infty, r)-I = \kappa-(\infty, \infty, r)-I$.

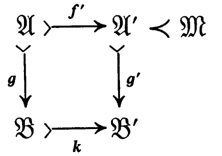
Proof: From theorem 2.1 it is sufficient to show that the right hand side is included in the left hand side.

Suppose $\mathfrak{M}$ is $\kappa-(\infty, \infty, r)-I$ over T and consider any situation (S_1) where f is $\prec_r$-like and g is elementary. We must produce a suitable $\prec_r$-like injection $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$.

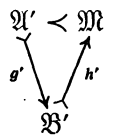
We have

$$\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{M} = \mathfrak{A} \overset{f'}{\rightarrowtail} \mathfrak{A}' \prec \mathfrak{M}$$

for some $\prec_r$-like injection f' and model $\mathfrak{A}'$ of cardinality $\mathfrak{a}+\lambda < \kappa$. Theorem 1.2 gives us a commuting diagram



where k is $\prec_r$-like and g' is elementary. We may assume that card $(\mathfrak{B}) = \mathfrak{b}+\mathfrak{a}+\lambda < \kappa$. Now $\mathfrak{M}$ is $\kappa-(\infty, \infty, r)-I$ and so we have a commuting diagram



for some $\prec_r$-like injection h'. The required injection is obtained by putting $h = h' \circ k$.

These three theorems show that we may restrict out attention to the following four cases.

(1) $p \leqq q \leqq r$.
(2) $\infty, q \leqq r$.
(3) ∞, ∞, r.
(4) p, ∞, r with $p < r$.

However it is reasonable to consider only the cases $p \geqq \min(q, r)$, hence (4) can be neglected. Thus, for any fixed r, the three largest classes of $\kappa-(p, q, r)-I$ structures to be considered are

(1) $\kappa-(r, r, r)-I$ i.e. $\kappa-WI_r$,
(2) $\kappa-(\infty, r, r)-I$ i.e. $\kappa-WJ_r$,
(3) $\kappa-(\infty, \infty, r)-I$ i.e. $\kappa-WK_r$.

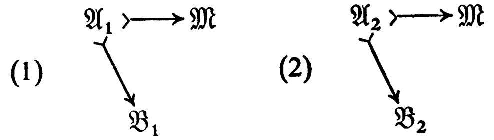
Of course, these three classes coincide for $r = \infty$.

In the remainder of this section we consider the question of existence of these injective-like objects. For each $X \in \{I_r, J_r, K_r, I_\infty\}$ we show that if T satisfies certain conditions then each model of T is embeddable in some member of $\kappa - WX$. For each X the proof follows the same pattern along the following lines. Consider any $\mathfrak{M} \models T$. First we calculate the number of essentially different situations (S_1). This is done in the counting lemma. Secondly we construct an insertion $\mathfrak{M} \rightarrowtail \mathfrak{M}'$ (of the appropriate kind) where $\mathfrak{M}'$ is a first approximation to a member of $\kappa - WX$. This is done in the one step lemma. Finally we escalate this construction to obtain $\mathfrak{M} \rightarrowtail \mathfrak{M}^* \in \kappa - WX$.

The most delicate of these constructions, that for $X = J_r$, will be done in detail. At the same time we will state the appropriate lemmas and theorems for the cases $X = I_r$ and $X = I_\infty$. The case $X = K_r$ is a corollary of the case $X = I_\infty$.

First we must formalize the notion 'essentially different situations (S_1)'.

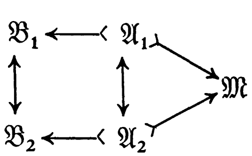
Consider some fixed $\mathfrak{M} \models T$ of cardinality $\mathfrak{m}$. We say that two situations (S_1),



are essentially the same if there are isomorphisms

$$\mathfrak{A}_1 \longleftrightarrow \mathfrak{A}_2, \quad \mathfrak{B}_1 \longleftrightarrow \mathfrak{B}_2$$

such that



commutes. Clearly if (1), (2) are essentially the same then there is an injection $\mathfrak{B}_1 \rightarrowtail \mathfrak{M}$ (with the required commuting property) if and only if there is an injection $\mathfrak{B}_2 \rightarrowtail \mathfrak{M}$ (with the required commuting property). Thus we need consider only essentially distinct situations (S_1). In particular we can suppose that f is an insertion and that $\mathfrak{B}$ is a subset of some fixed set of cardinality κ, (say κ itself).

Counting Lemma (2.5): *Suppose $\kappa \leqq \mathfrak{m}$. For any given $\mathfrak{M} \models T$ of cardinality $\mathfrak{m}$ there are at most $\mathfrak{m}^{\kappa^*}$ essentially distinct situations (S_1).*

Proof: First fix $\mathfrak{a}$ and $\mathfrak{b}$. Then, remembering the above remarks, we have

(i) at most $\mathfrak{m}^{\mathfrak{a}}$ different $\mathfrak{A}$,
(ii) at most $2^{\lambda\mathfrak{b}}$ different $\mathfrak{B}$,
and for fixed $\mathfrak{A}, \mathfrak{B}$ we have
(iii) at most $\mathfrak{b}^{\mathfrak{a}}$ different g.
Thus there are
(iv) at most $\mathfrak{m}^{\mathfrak{a}} \cdot 2^{\lambda\mathfrak{b}} \cdot \mathfrak{b}^{\mathfrak{a}}$ essentially different situations (S_1).
But $\mathfrak{a} \leqq \mathfrak{b} < \kappa \leqq \mathfrak{m}$ so that

$$\mathfrak{m}^{\mathfrak{a}} \cdot 2^{\lambda\mathfrak{b}} \cdot \mathfrak{b}^{\mathfrak{a}} \leqq \mathfrak{m}^{\lambda+\mathfrak{b}}$$

Thus, freeing $\mathfrak{a}$ and $\mathfrak{b}$, there are at most

$$\sum_{\mathfrak{a} \leqq \mathfrak{b} < \kappa} \mathfrak{m}^{\lambda+\mathfrak{b}}$$

essentially different situations (S_1). Now

$$\begin{aligned}\sum_{\mathfrak{a} \leqq \mathfrak{b} < \kappa} \mathfrak{m}^{\lambda+\mathfrak{b}} &= \mathfrak{m}^{\lambda} \cdot \sum_{\mathfrak{a} \leqq \mathfrak{b} < \kappa} \mathfrak{m}^{\mathfrak{b}} \\ &= \mathfrak{m}^{\lambda} \cdot \mathfrak{m}^{\kappa^*} \\ &= \mathfrak{m}^{\kappa^*}\end{aligned}$$

since $\lambda < \kappa$, which gives the required result.

The following lemma is the crucial step in the construction of the $\kappa - WJ_r$.

ONE STEP LEMMA (2.6): *Suppose T is $\forall_{r+2}$-axiomatizable, and that $\kappa \leqq \mathfrak{m}$ where $\mathfrak{m}^{\kappa^*} = \mathfrak{m}$. For each model $\mathfrak{M}$ of T of cardinality $\mathfrak{m}$ there is a $\prec_r$-like insertion $\mathfrak{M} \overset{k}{\rightarrowtail} \mathfrak{M}'$ into a model $\mathfrak{M}'$ of T of cardinality $\mathfrak{m}$, such that for any situation (S_1) where $k \circ f$ is $\prec_{r+1}$-like and g is $\prec_r$-like, there is a $\prec_r$-like injection $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ such that $h \circ g = k \circ f$.*

PROOF: Let

$$\{\mathfrak{A}_i \overset{g_i}{\rightarrowtail} \mathfrak{B}_i : i < \mathfrak{m}\}$$

be an enumeration of all essentially distinct situations (S_1). (Here we are using lemma 2.5 and assuming that each f_i is an insertion.) We construct an ascending chain of $\prec_r$-like insertions

$$\mathfrak{M} \rightarrowtail \mathfrak{M}_0 \rightarrowtail \cdots \longrightarrow \mathfrak{M}_i \longrightarrow \cdots \; i < \mathfrak{m}$$

such that for each $i < \mathfrak{m}$,
(a_i) $\mathfrak{M}_i$ is a model of T of cardinality $\mathfrak{m}$,
(b_i) if

$$\mathfrak{A}_i \overset{f_i}{\rightarrowtail} \mathfrak{M} \overset{k_i}{\rightarrowtail} \mathfrak{M}_i$$

is $\prec_{r+1}$-like (where k_i is the obvious insertion) and g_i is $\prec_r$-like then there is an elementary insertion $\mathfrak{B}_i \overset{h_i}{\rightarrowtail} \mathfrak{M}_i$ such that $k_i \circ f_i = h_i \circ g_i$.

Suppose we have $\mathfrak{M}_j$ for all $j < i$. Let

$$\mathfrak{N}_i = \cup \{\mathfrak{M}_j : j < i\}$$

(with $\mathfrak{N}_0 = \mathfrak{M}$) so that $\mathfrak{N}_i$ is a model of T of cardinality $\mathfrak{m}$. If

$$\mathfrak{A}_i \overset{f_i}{\rightarrowtail} \mathfrak{M} \rightarrowtail \mathfrak{N}_i$$

is not $\prec_{r+1}$-like or g_i is not $\prec_r$-like put $\mathfrak{M}_i = \mathfrak{N}_i$. Otherwise, using theorem 1.2 we have a commuting diagram

$$\begin{array}{ccccc} \mathfrak{M} & \rightarrowtail & \mathfrak{N}_i & \rightarrowtail & \mathfrak{M}_i \\ {\scriptstyle f_i}\uparrow & & & & \uparrow{\scriptstyle h_i} \\ \mathfrak{A}_i & & \overset{}{\underset{g_i}{\rightarrowtail}} & & \mathfrak{B}_i \end{array}$$

where $\mathfrak{N}_i \rightarrowtail \mathfrak{M}_i$ is $\prec_r$-like and h_i is elementary. Now $\lambda < \kappa \leqq \mathfrak{m}$ so we can assume $\mathfrak{M}_i$ has cardinality $\mathfrak{m}$, and replacing $\mathfrak{M}_i$ by a suitable isomorphic copy we can assume that $\mathfrak{N}_i \rightarrowtail \mathfrak{M}_i$ is an insertion. Thus we get (a_i, b_i).

Finally we put

$$\mathfrak{M}' = \cup \{\mathfrak{M}_i : i < \mathfrak{m}\}$$

so that $\mathfrak{M}'$ is a model of T of cardinality $\mathfrak{m}$ and the insertion $\mathfrak{M} \overset{k}{\rightarrowtail} \mathfrak{M}'$ is $\prec_r$-like.

Consider any situation

$$\begin{array}{ccc} \mathfrak{M} & \overset{k}{\rightarrowtail} & \mathfrak{M}' \\ {\scriptstyle f}\uparrow & & \\ \mathfrak{A} & \underset{g}{\rightarrowtail} & \mathfrak{B} \end{array}$$

where card $(\mathfrak{B}) < \kappa$, $k \circ f$ is $\prec_{r+1}$-like and g is $\prec_r$-like. We may suppose that

$$(f, \mathfrak{A}, g, \mathfrak{B}) = (f_i, \mathfrak{A}_i, g_i, \mathfrak{B}_i)$$

for some $i < \mathfrak{m}$. Then (considering stage i of the construction of $\mathfrak{M}'$) we see that

$$\mathfrak{A}_i \overset{f_i}{\rightarrowtail} \mathfrak{M} \rightarrowtail \mathfrak{N}_i$$

is $\prec_{r+1}$-like and so we get a commuting diagram

$$\begin{array}{ccccc} \mathfrak{M} & \rightarrowtail & \mathfrak{M}_i & \rightarrowtail & \mathfrak{M}' \\ {\scriptstyle f_i}\uparrow & & \uparrow{\scriptstyle h_i} & & \\ \mathfrak{A}_i & \underset{g_i}{\rightarrowtail} & \mathfrak{B}_i & & \end{array}$$

for some elementary h_i. But $\mathfrak{M}_i \rightarrowtail \mathfrak{M}'$ is $\prec_r$-like, hence

$$\mathfrak{B}_i \overset{h_i}{\rightarrowtail} \mathfrak{M}_i \rightarrowtail \mathfrak{M}'$$

is $\prec_r$-like, as required.

To construct the κ-*WI* we use the following lemma in place of lemma 2.6.

LEMMA (2.7): *Suppose* **C** *has UC and AP, and that* $\kappa \leqq \mathfrak{m}$ *where* $\mathfrak{m}^{\kappa^*} = \mathfrak{m}$. *For each object* $\mathfrak{M}$ *of* **C** *of cardinality* $\mathfrak{m}$ *there is an insertion* $\mathfrak{M} \overset{k}{\rightarrowtail} \mathfrak{M}'$ *of* **C** *with* $\mathfrak{M}'$ *of cardinality* $\mathfrak{m}$, *such that for any situation* (S_1) *there is an injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}'$ *such that* $h \circ g = k \circ f$.

Let

$$\kappa' = \begin{cases} \kappa \text{ if } \kappa \text{ is regular} \\ \kappa^+ \text{ if } \kappa \text{ is singular} \end{cases}$$

and for each cardinal $\mathfrak{m}$ let

$$\mathfrak{m}^* = \begin{cases} \mathfrak{m}^{\kappa^*} \text{ if } \kappa \text{ is regular} \\ \mathfrak{m}^{\kappa^*} \text{ if } \kappa \text{ is singular.} \end{cases}$$

Notice that $\kappa' \leqq \mathfrak{m}^*$ and $\mathfrak{m}^{*\kappa^*} = \mathfrak{m}^*$.

THEOREM (2.8): *Suppose T is* $\forall_{r+2}$*-axiomatizable. Then for each model* $\mathfrak{M}$ *of T of cardinality* $\mathfrak{m}$ *there is a model* $\mathfrak{M}^*$ *of T of cardinality* $\mathfrak{m}^*$ *such that* $\mathfrak{M} \prec_r \mathfrak{M}^*$ *and* $\mathfrak{M}^*$ *is* $\kappa - WJ_r$ *over T.*

PROOF: Consider any $\prec_r$-like insertion $\mathfrak{M} \rightarrowtail \mathfrak{M}_0$ into a model $\mathfrak{M}_0$ of cardinality $\mathfrak{m}^*$. We now repeatedly use lemma 2.6 to construct an ascending chain of $\prec_r$-like insertions

$$\mathfrak{M}_0 \rightarrowtail \mathfrak{M}_1 \rightarrowtail \cdots \rightarrowtail \mathfrak{M}_i \rightarrowtail \cdots \quad i < \kappa'$$

by

$$\mathfrak{M}_i = (\cup \{\mathfrak{M}_j : j < i\})'$$

for $0 < i < \kappa'$. Then we put

$$\mathfrak{M}^* = \cup \{\mathfrak{M}_i : i < \kappa'\}.$$

Clearly $\mathfrak{M}^*$ has cardinality $\mathfrak{m}^*$, thus we must show that $\mathfrak{M}^*$ is $\kappa - WJ_r$.

Consider any situation

$$\begin{array}{ccc} \mathfrak{A} & \overset{f^*}{\rightarrowtail} & \mathfrak{M}^* \\ {\scriptstyle g}\downarrow & & \\ \mathfrak{B} & & \end{array}$$

where card $(\mathfrak{B}) < \kappa, f^*$ is elementary and g is $\prec_r$-like. Then, by the choice of κ', we have

$$\mathfrak{A} \overset{f^*}{\rightarrowtail} \mathfrak{M} = \mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{M}_i \prec_r \mathfrak{M}_{i+1} \prec_r \mathfrak{M}^*$$

for some $i < \kappa'$. Notice that, since f^* is elementary, the injection

$$\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{M}_i \prec_r \mathfrak{M}_{i+1}$$

is $\prec_{r+1}$-like. Thus, using lemma 2.6, we have a commuting diagram

$$\begin{array}{ccccc} \mathfrak{M}_i & \prec_r & \mathfrak{M}_{i+1} & \prec_r & \mathfrak{M}^* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle h} & & \\ \mathfrak{A} & \underset{g}{\rightarrowtail} & \mathfrak{B} & & \end{array}$$

for some $\prec_r$-like injection h. Hence we obtain the required injection of $\mathfrak{B}$ into $\mathfrak{M}^*$.

The κ-*WI* are obtained by iterating lemma 2.7. We have the following.

THEOREM (2.9): *Suppose **C** has UC and AP. Then each object $\mathfrak{M}$ of **C** of cardinality $\mathfrak{m}$ is embeddable in some object $\mathfrak{M}^*$ of **C** of cardinality $\mathfrak{m}^*$, where $\mathfrak{M}^*$ is $\kappa - WI$ over **C**.*

COROLLARY (2.10): *Suppose T is $\forall_{r+2}$-axiomatizable and has AP_r. Then for each model $\mathfrak{M}$ of T of cardinality $\mathfrak{m}$ there is a model $\mathfrak{M}^*$ of T of cardinality $\mathfrak{m}^*$ such that $\mathfrak{M} \prec_r \mathfrak{M}^*$ and $\mathfrak{M}^*$ is $\kappa - WI_r$ over T.*

COROLLARY (2.11): *For each model $\mathfrak{M}$ of T of cardinality $\mathfrak{m}$ there is a model $\mathfrak{M}^*$ of T of cardinality $\mathfrak{m}^*$ such that $\mathfrak{M} \prec \mathfrak{M}^*$ and $\mathfrak{M}^*$ is $\kappa - WI_\infty$ over T.*

COROLLARY (2.12): *For each model $\mathfrak{M}$ of T of cardinality $\mathfrak{m}$ there is a model $\mathfrak{M}^*$ of T of cardinality $\mathfrak{m}^*$ such that $\mathfrak{M} \prec \mathfrak{M}^*$ and $\mathfrak{M}^*$ is $\kappa - WK_r$ over T.*

3. Compact-like structures

We wish to relate the injective-like properties of the last section to more standard model-theoretic properties. This we do via the realization of various kinds of types.

Let T be some fixed theory and $\mathfrak{M}$ some fixed model of T. We want to consider various kinds of types over $\mathfrak{M}$. To do this we enrich our language L in two ways. First we extend the sequence $\{v_i : i < \omega\}$ of object variables to a sequence $\{v_i : i < \kappa\}$. This does not essentially alter the powers of expression of L. Secondly we add, for each member of $\mathfrak{M}$, a new object constant denoting that member of $\mathfrak{M}$. (We will use the same letter for the member and its name.) Let $L(\mathfrak{M})$ be the resulting language, so $L(\mathfrak{M})$ is designed to talk about $\mathfrak{M}$ and its superstructures. We refer to the new object constants of $L(\mathfrak{M})$ as parameters.

In the obvious way we have the set $\exists_{r+1}(\mathfrak{M})$ of $L(\mathfrak{M})$-formulas.

The following definition is four definitions in one and should be read as such.

DEFINITION: A

(i) strict $-\kappa-\exists_{r+1}$-type
(ii) strict-κ-type
(iii) $\kappa-\exists_{r+1}$-type
(iv) κ-type

over $\mathfrak{M}$ is a set, $\Sigma(\vec{m}, \vec{v})$, of formulas

(i) in $\exists_{r+1}(\mathfrak{M})$
(ii) [no restriction]
(iii) in $\exists_{r+1}(\mathfrak{M})$
(iv) [no restriction]

containing a sequence, $\vec{m}$, of parameters of length $< \kappa$, and a sequence, $\vec{v}$, of free variables

(i) of finite length.
(ii) of finite length.
(iii) of length $< \kappa$.
(iv) of length $< \kappa$.

It is clear what is mean by a type $\Sigma(\vec{m}, \vec{v})$ (of a certain kind) over $\mathfrak{M}$ being satisfied or finitely satisfied in $\mathfrak{M}$ or a superstructure $\mathfrak{N}$ of $\mathfrak{M}$. Notice that

$$\mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

means that $\Sigma(\vec{m}, \vec{v})$ is satisfied in $\mathfrak{N}$. The following lemma deals with finite satisfiability in $\mathfrak{M}$.

LEMMA (3.1): *For any type $\Sigma(\vec{m}, \vec{v})$ over $\mathfrak{M}$ the following are equivalent.*

(i) *There is a structure $\mathfrak{N}$ such that*

$$\mathfrak{M} \prec \mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v}).$$

(ii) *$\Sigma(\vec{m}, \vec{v})$ is finitely satisfied in M.*

PROOF: Obvious.

We now come to the definitions of the six kinds of compact-like structures. Again we give these six definitions in one. In these definitions '$\mathfrak{M} \prec_\infty \mathfrak{N}$' means '$\mathfrak{M} \prec \mathfrak{N}$'.

DEFINITION: A model $\mathfrak{M}$ of T is

(i) $\kappa-\exists_{r+1}$-replete
(ii) $\kappa-\exists_{r+1}$-saturated
(iii) κ-saturated
(iv) $\kappa-\exists_{r+1}$-complete
(v) $\kappa-\exists_{r+1}$-compact
(vi) κ-compact

over T if for each

(i) strict-$\kappa-\exists_{r+1}$-type
(ii) strict-$\kappa-\exists_{r+1}$-type
(iii) strict-κ-type
(iv) $\kappa-\exists_{r+1}$-type
(v) $\kappa-\exists_{r+1}$-type
(vi) κ-type

$\Sigma(\vec{m}, \vec{v})$ over $\mathfrak{M}$, if there is a model $\mathfrak{N}$ of T such that

$$\mathfrak{M} \prec_q \mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

where

(i) $q = r$
(ii) $q = \infty$
(iii) $q = \infty$
(iv) $q = r$
(v) $q = \infty$
(vi) $q = \infty$

then

$$\mathfrak{M} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v}).$$

Several of these kinds of compact-like structures have already appeared in the literature. I have used the names by which these kinds are already known; hence the haphazard nomenclature.

(a) The κ-saturated structures are now well established. The $\kappa-\exists_{r+1}$-saturated structures are named by analogy with these.

(b) The κ-compact and $\kappa-\exists_{r+1}$-compact structures are related to the compact structures of Mycielski. See [2], [6], [3], and [7].

(c) E. Fisher has introduced the $\kappa-\exists_1$-complete structures (see [1]), and the $\kappa-\exists_{r+1}$-complete structures are named by analogy with these.

(d) The $\kappa-\exists_{r+1}$-replete structures are not the same as the replete structures of Keisler. (These are now called saturated structures).

The following inclusions are easy to check. (Again we have replaced '$\subseteq$' by '$\longrightarrow$'.)

$$\begin{array}{ccccc}
\kappa-\exists_{r+1}\text{-complete} & \longrightarrow & \kappa-\exists_{r+1}\text{-compact} & \longleftarrow & \kappa\text{-compact} \\
\downarrow & & \downarrow & & \downarrow \\
\kappa-\exists_{r+1}\text{-replete} & \longrightarrow & \kappa-\exists_{r+1}\text{-saturated} & \longleftarrow & \kappa\text{-saturated}
\end{array}$$

The compact-like structures in the top line of this diagram are in fact injective-like. To show this we will consider several times the following situation.

$$\left.\begin{array}{l}
\begin{array}{ccc}
\mathfrak{A} & \overset{f}{\rightarrowtail} & \mathfrak{M} \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
\mathfrak{B} & \underset{f'}{\rightarrowtail} & \mathfrak{C}
\end{array} \\
\text{where } \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{M} \text{ are models} \\
\text{of } T, \text{ card } (\mathfrak{B}) < \kappa \text{ and } g' \circ f = f' \circ g.
\end{array}\right\} (S_2)$$

In each of the following three theorems parts (iii), (iv) are derived from a suggestion of Angus Macintyre.

THEOREM (3.2): *For any model* $\mathfrak{M}$ *of* T *the following are equivalent.*

(i) $\mathfrak{M}$ is $\kappa - WJ_r$ *over* T.

(ii) $\mathfrak{M}$ *is* $\kappa-\exists_{r+1}$*-complete over* T.

(iii) *For each situation* (S_2) *where* f', g' *are both* $\prec_r$*-like there is a* $\prec_r$*-like injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ *such that* $f = h \circ g$.

(iv) *For each situation* (S_2) *where* f, f' *are both elementary and* g, g' *are both* $\prec_r$*-like there is a* $\prec_r$*-like injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ *such that* $f = h \circ g$.

PROOF: (i) $\Rightarrow$ (ii). Suppose (i) and that

$$\mathfrak{M} \prec_r \mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

where $\mathfrak{N} \models T$ and $\Sigma(\vec{m}, \vec{v})$ is a $\kappa-\exists_{r+1}$-type over $\mathfrak{M}$.

First consider any $\mathfrak{A} \prec \mathfrak{M}$ such that card $(\mathfrak{A}) < \kappa$ and each of the parameters $\vec{m}$ are taken from $\mathfrak{A}$. Then consider any $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B} \prec \mathfrak{N}$, card $(\mathfrak{B}) < \kappa$, and

$$\mathfrak{B} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v}).$$

Notice that $\mathfrak{A} \prec_r \mathfrak{B}$, so we have (by (i)) an $\mathfrak{A}$-preserving, $\prec_r$-like injection $\mathfrak{B} \rightarrowtail \mathfrak{M}$. But $\Sigma(\vec{m}, \vec{v}) \subseteq \exists_{r+1}(\mathfrak{M})$, and so

$$\mathfrak{M} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

as required.

(ii) ⇒ (iii). Suppose (ii) and that we have a situation (S_2) where f', g' are both $\prec_r$-like. Let $\vec{a} = \{a_i : i < \mathfrak{a}\}$ be an enumeration of $\mathfrak{A}$, where $\mathfrak{a} = \text{card}(\mathfrak{A}) < \kappa$. Let

$$g(\vec{a}) = \{g(a_i) : i < \mathfrak{a}\}$$

with a similar notation when the other injections are involved. Thus $g(\vec{a})$ is a partial enumeration of $\mathfrak{B}$. Let $\vec{b}$ be an enumeration of the remaining part of $\mathfrak{B}$ (so $\vec{b}$ has length $< \kappa$). Let $\Sigma(\vec{u}, \vec{v})$ be the set of parameter free formulas such that

$$\Sigma(g(\vec{a}), \vec{b}) = \text{Th}(\mathfrak{B}, g(\vec{a}), \vec{b}) \cap \forall_r.$$

Thus $\Sigma(f(\vec{a}), \vec{v})$ is a $\kappa - \exists_{r+1}$-type over $\mathfrak{M}$.

To construct the required injection it is sufficient to show that

$$\mathfrak{M} \models (\exists \vec{v}) \wedge \Sigma(f(\vec{a}), \vec{v}).$$

Now we have

$$\mathfrak{B} \models (\exists \vec{v}) \wedge \Sigma(g(\vec{a}), \vec{v})$$

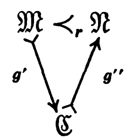
hence, since f' is $\prec_r$-like we have

$$\mathfrak{C} \models (\exists \vec{v}) \wedge \Sigma(f' \circ g(\vec{a}), \vec{v}).$$

But $f' \circ g = g' \circ f$ so that

$$\mathfrak{C} \models (\exists \vec{v}) \wedge \Sigma(g' \circ f(\vec{a}), \vec{v}).$$

However, g' is $\prec_r$-like, so we have a commuting diagram



for some suitable structure $\mathfrak{N}$ and isomorphism g''. Thus

$$\mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(g'' \circ g' \circ f(\vec{a}), \vec{v})$$

i.e.

$$\mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(f(\vec{a}), \vec{v}).$$

The required result now follows from (ii).

(iii) ⇒ (iv) is trivial.

(iv) ⇒ (i). Suppose (iv) and that we have a situation (S_1) where f is elementary and g is $\prec_r$-like. Using theorem 1.2 we obtain a situation (S_2) where f, f' are both elementary and g, g' are both $\prec_r$-like. Thus (iv) gives us the required $\prec_r$-like injection.

This completes the proof of the theorem.

The following two theorems are proved in exactly the same way.

THEOREM (3.3): *For any model* $\mathfrak{M}$ *of* T *the following are equivalent.*

(i) $\mathfrak{M}$ *is* $\kappa - WI_\infty$ *over* T.

(ii) $\mathfrak{M}$ *is* κ*-compact over* T.

(iii) *For each situation* (S_2) *where* f', g' *are both elementary there is an elementary injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ *such that* $f = h \circ g$.

(iv) *For each situation* (S_2) *where* f, f', g, g' *are all elementary there is an elementary injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ *such that* $f = h \circ g$.

THEOREM (3.4): *For any model* $\mathfrak{M}$ *of* T *the following are equivalent.*

(i) $\mathfrak{M}$ *is* $\kappa - WK_r$ *over* T.

(ii) $\mathfrak{M}$ *is* $\kappa - \exists_{r+1}$*-compact over* T.

(iii) *For each situation* (S_2) *where* f' *is* $\prec_r$*-like and* g' *is elementary there is an injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ *such that* $f = h \circ g$.

(iv) *For each situation* (S_2) *where* f, f', g, g' *are all elementary there is a* $\langle_r$*-like injection* $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$ *such that* $f = h \circ g$.

Finally we point out that there is no need to use unstrict types. We state the following theorem without proof.

THEOREM (3.5): *For any model* $\mathfrak{M}$ *of* T *we have* (i) $\Leftrightarrow$ (ii), $(j) \Leftrightarrow (jj)$, $(k) \Leftrightarrow (kk)$ *where*

(i) $\mathfrak{M}$ *is* $\kappa - \exists_{r+1}$*-complete,*

(ii) $\mathfrak{M}$ *is* $\kappa - \exists_{r+1}$*-replete,*

(j) $\mathfrak{M}$ *is* $\kappa - \exists_{r+1}$*-compact,*

(jj) $\mathfrak{M}$ *is* $\kappa - \exists_{r+1}$*-saturated,*

(k) $\mathfrak{M}$ *is* κ*-compact,*

(kk) $\mathfrak{M}$ *is* κ*-saturated.*

The proof of this theorem shows, in fact, that in all cases the types used can be assumed to contain no more than one free variable.

4. *r*-complete theories and *r*-companions

Theorem 1.1 is a generalization of a well known theorem concerning $\forall_2$-axiomatizable theories. One particular class of such theories are the model complete theories, and there are several characterizations of these. All of these characterizations can be generalized. For instance we have the following theorem.

THEOREM (4.1): *For any theory* T *the following are equivalent.*

(i) *For each formula* $\phi(\mathbf{v})$ *there is a formula* $\psi(\mathbf{v}) \in \exists_{r+1}$ *such that*

$$T \vdash (\forall \mathbf{v})[\phi(\mathbf{v}) \longleftrightarrow \psi(\mathbf{v})]. \qquad (e)$$

(ii) *For each formula* $\phi(\mathbf{v})$ *there is a formula* $\psi(\mathbf{v}) \in \forall_{r+1}$ *such that* (e) *holds.*

(iii) *For any two models* $\mathfrak{A}$, $\mathfrak{B}$ *of* T,

$$\mathfrak{A} \prec_r \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$$

(iv) *For any two models* $\mathfrak{A}$, $\mathfrak{B}$ *of* T,

$$\mathfrak{A} \prec_r \mathfrak{B} \Rightarrow \mathfrak{A} \prec_{r+1} \mathfrak{B}.$$

PROOF: Notice that the equivalence (i) $\Leftrightarrow$ (ii) and the implications (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) are obvious. Thus it is sufficient to prove the implications (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). Suppose (iii) and consider any formula $\phi(\boldsymbol{v})$. Let

$$\Psi(\boldsymbol{v}) = \{\psi(\boldsymbol{v}) \in \forall_{r+1} : T \vdash (\forall \boldsymbol{v})(\phi(\boldsymbol{v}) \rightarrow \psi(\boldsymbol{v})]\}$$

It is sufficient to show that

$$T \cup \Psi(\boldsymbol{v}) \vdash \phi(\boldsymbol{v}).$$

Consider any structure $\mathfrak{A}$ such that

$$\mathfrak{A} \models T \cup \Psi(\boldsymbol{a})$$

for some point $\boldsymbol{a}$ of $\mathfrak{A}$. We must show that

$$\mathfrak{A} \models \phi(\boldsymbol{a}).$$

Let $\vec{a}$ be an enumeration of $\mathfrak{A}$ and consider the set

$$T \cup [Th\,(\mathfrak{A}, \boldsymbol{a}, \vec{a}) \cap \forall_r] \cup \{\phi(\boldsymbol{a})\}.$$

This set is consistent, for if not we have

$$T \vdash \phi(\boldsymbol{a}) \rightarrow \neg\, \alpha(\boldsymbol{a}', \boldsymbol{a})$$

for some formula $\alpha(\boldsymbol{u}, \boldsymbol{v}) \in \forall_r$ and point $\boldsymbol{a}'$ of $\mathfrak{A}$ (which does not overlap $\boldsymbol{a}$) such that

$$\mathfrak{A} \models \alpha(\boldsymbol{a}', \boldsymbol{a}).$$

Thus we get

$$T \vdash (\forall \boldsymbol{v})[\phi(\boldsymbol{v}) \rightarrow (\forall \boldsymbol{u})\neg\alpha(\boldsymbol{u}, \boldsymbol{v})]$$

and so $(\forall \boldsymbol{u})\neg\alpha(\boldsymbol{u}, \boldsymbol{v}) \in \Psi(\boldsymbol{v})$. Hence

$$\mathfrak{A} \models (\forall \boldsymbol{u})\neg\alpha(\boldsymbol{u}, \boldsymbol{a})$$

which give a contradiction.

From the consistency of the above set we get

$$\mathfrak{A} \prec_r \mathfrak{B} \models \phi(\boldsymbol{a})$$

for some model $\mathfrak{B}$ of T. But, by (iii), this gives $\mathfrak{A} \prec \mathfrak{B}$, and so

$$\mathfrak{A} \models \phi(\boldsymbol{a})$$

as required.

(iv) ⇒ (iii). Suppose (iv) and that $\mathfrak{A} \prec_r \mathfrak{B}$ for models $\mathfrak{A}, \mathfrak{B}$ of T. To show that $\mathfrak{A} \prec \mathfrak{B}$ it is sufficient to show that $\mathfrak{A} \prec_k \mathfrak{B}$ for all positive integers k. This we prove by induction on k.

The initial cases $k \leqq r+1$ are trivial, so it is sufficient to prove the induction step $k \mapsto k+1$ where $r+1 \leqq k$. Notice that for these k we have $r \leqq k-1$.

We have $\mathfrak{A} \prec_k \mathfrak{B}$, and so

$$\mathfrak{A} \prec \mathfrak{C}, \mathfrak{B} \prec_{k-1} \mathfrak{C}$$

for some $\mathfrak{C}$. Thus, applying the induction hypothesis to $\mathfrak{B} \prec_{k-1} \mathfrak{C}$ (which implies $\mathfrak{B} \prec_r \mathfrak{C}$) we have $\mathfrak{B} \prec_k \mathfrak{C}$. Hence we get $\mathfrak{A} \prec_{k+1} \mathfrak{B}$, as required.

This theorem leads to the following definition.

Definition: A theory T is r-complete if T satisfies (i-iv) of theorem 4.1.

Notice that any r-complete theory is $\forall_{r+2}$-axiomatizable. The 0-complete theories are exactly the model complete theories.

For r-complete theories there is just one kind of injective-like model (for a fixed κ)

Theorem (4.2): *Suppose T is r-complete. Then*

$$\kappa - WI_r = \kappa - WJ_r = \kappa - WK_r = \kappa - WI_\infty .$$

Proof: This follows since all $\prec_r$-like injections between models of T are elementary.

There is a converse to this theorem.

Theorem (4.3): *Suppose T is such that*

$$\kappa - WI_\infty \subseteq \kappa - WJ_r .$$

Then T is r-complete.

Proof: We use part (iv) of theorem 4.1.

Consider any two models $\mathfrak{A}', \mathfrak{B}'$ of T such that $\mathfrak{A}' \prec_r \mathfrak{B}'$. Consider any two $\mathfrak{A}, \mathfrak{B}$ such that

$$\mathfrak{A} \prec \mathfrak{A}', \ \mathfrak{A} \prec_r \mathfrak{B} \prec \mathfrak{B}', \ \mathrm{card}\,(\mathfrak{B}) < \kappa.$$

It is sufficient to show that $\mathfrak{A} \prec_{r+1} \mathfrak{B}$. (For by considering all such $\mathfrak{A}, \mathfrak{B}$ we then get $\mathfrak{A}' \prec_{r+1} \mathfrak{B}'$.)

Using corollary 2.1 we have $\mathfrak{A} \prec \mathfrak{M}$ for some $\mathfrak{M}$ which is $\kappa - WI_\infty$ over T. Hence $\mathfrak{M}$ is $\kappa - WJ_r$ over T. Thus we have a commuting diagram

$$\begin{array}{ccc} \mathfrak{A} & \prec & \mathfrak{M} \\ \curlyvee_r & \nearrow h & \\ \mathfrak{B} & & \end{array}$$

for some $\prec_r$-like injection h. This gives $\mathfrak{A} \prec_{r+1} \mathfrak{B}$, as required.

Just as we have generalized the notion of a model complete theory so we can generalize the notion of a model companion of a theory.

DEFINITION: For any theory T, a theory $T^{(r)}$ is an r-companion if
(i) $T \subseteq T^{(r)}$,
(ii) for each model $\mathfrak{A}$ of T there is a model $\mathfrak{B}$ of $T^{(r)}$ such that $\mathfrak{A} \prec_r \mathfrak{B}$,
(iii) $T^{(r)}$ is r-complete.
Notice that 0-companions are model companions

THEOREM (4.4): *Any theory T has at most one r-companion.*

PROOF: Suppose $T_1^{(r)}$, $T_2^{(r)}$ are both r-companions of T. Thus both $T_1^{(r)}$ $T_2^{(r)}$ are $\forall_{r+2}$-axiomatizable.

Consider any model $\mathfrak{A}$ of $T_1^{(r)}$. Then $\mathfrak{A} \models T$ and so, by property (ii), we have

$$\mathfrak{A} \prec_r \mathfrak{B} \models T_2^{(r)}$$

for some $\mathfrak{B}$. Similarly we have

$$\mathfrak{B} \prec_r \mathfrak{C} \models T_1^{(r)}$$

for some $\mathfrak{C}$. Now we have $\mathfrak{A} \prec_r \mathfrak{C}$ and both are models of $T_1^{(r)}$ so that $\mathfrak{A} \prec \mathfrak{C}$. This gives

$$\mathfrak{A} \prec_{r+1} \mathfrak{B} \models T_2^{(r)}$$

and hence $\mathfrak{A} \models T_2^{(r)}$, since $T_2^{(r)}$ is $\forall_{r+2}$-axiomatizable.

Thus we have

$$T_2^{(r)} \subseteq T_1^{(r)}.$$

A similar argument shows the reverse inclusion, and so we get the required equality.

It is known that the existence of a model companion of a theory T is closely connected with the behaviour of the structures which are existentially closed over T, (see [5]). There are corresponding results for r-companions.

DEFINITION: A structure $\mathfrak{A}$ is $\exists_{r+1}$-closed over a theory T if $\mathfrak{A} \models T$ and, for each model $\mathfrak{B}$ of T

$$\mathfrak{A} \prec_r \mathfrak{B} \Rightarrow \mathfrak{A} \prec_{r+1} \mathfrak{B}.$$

Starting from this definition the whole of [5] can be generalized in a straight forward manner. As an example of this generaliastion we prove the following theorem.

THEOREM (4.5): *Suppose T has an r-companion $T^{(r)}$. For any structure $\mathfrak{A}$ the following are equivalent.*
(i) *$\mathfrak{A}$ is a model of $T^{(r)}$.*
(ii) *$\mathfrak{A}$ is $\exists_{r+1}$-closed over T.*

PROOF: (i) $\Rightarrow$ (ii). Suppose $\mathfrak{A} \models T^{(r)}$ and that $\mathfrak{A} \prec_r \mathfrak{B} \models T$. (We must show that $\mathfrak{A} \prec_{r+1} \mathfrak{B}$.)

Now we have

$$\mathfrak{B} \prec_r \mathfrak{C} \models T^{(r)}$$

for some $\mathfrak{C}$, and since $T^{(r)}$ is r-complete this gives $\mathfrak{A} \prec \mathfrak{C}$. Hence $\mathfrak{A} \prec_{r+1} \mathfrak{B}$ as required.

(ii) $\Rightarrow$ (i). Suppose (ii). Now we have

$$\mathfrak{A} \prec_r \mathfrak{B} \models T^{(r)}$$

for some B, and so

$$\mathfrak{A} \prec_{r+1} \mathfrak{B} \models T^{(r)}.$$

But $T^{(r)}$ is $\forall_{r+2}$-axiomatizable, and so $\mathfrak{A} \models T^{(r)}$, as required.

The next theorem shows the relevance of $\exists_{r+1}$-closed structures to injective-like structures.

THEOREM (4.6): *For any model $\mathfrak{M}$ of T the following are equivalent*

(i) *$\mathfrak{M}$ is $\kappa - WJ_r$ over T.*

(ii) *$\mathfrak{M}$ is $\kappa - WK_r$ and $\exists_{r+1}$-closed over T.*

PROOF: The implication (i) $\Rightarrow$ (ii) follows immediately from the definitions involved. To prove the implication (ii) $\Rightarrow$ (i) we argue as follows

Suppose

$$\mathfrak{M} \prec_r \mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

where $\mathfrak{M}$ is $\exists_{r+1}$-closed over T and $\kappa - WK_r$ over T, $\mathfrak{N}$ is a model of T, and $\Sigma(\vec{m}, \vec{v})$ is a $\kappa - \exists_{r+1}$-type over T. Since $\mathfrak{M}$ is $\exists_{r+1}$-closed over T we have $\mathfrak{M} \prec_{r+1} \mathfrak{N}$ and so

$$\mathfrak{M} \prec \mathfrak{D}, \mathfrak{N} \prec_r \mathfrak{D}$$

for some $\mathfrak{D}$. Thus

$$\mathfrak{M} \prec \mathfrak{D} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

and so, since $\mathfrak{M}$ is $\kappa - WK_r$ we have

$$\mathfrak{M} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

which gives the required result.

A large portion of this paper was inspired by the following theorem. This theorem derives from a remark of Paul Bacsich.

THEOREM (4.7): *Suppose T has an r-companion $T^{(r)}$. Then $\kappa - WJ_r \subseteq \kappa - WI_\infty$.*

PROOF: Suppose $\mathfrak{M}$ is $\kappa - WJ_r$ over T and that

$$\mathfrak{M} \prec \mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

where $\Sigma(\vec{m}, \vec{v})$ is a κ-type over $\mathfrak{M}$. We have $\mathfrak{M} \models T^{(r)}$ (by theorems 4.6 and 4.5) and so $\mathfrak{N} \models T^{(r)}$. Thus (using theorem 4.1) we may assume that $\Sigma(\vec{m}, \vec{v})$ is a $\kappa - \exists_{r+1}$-type over $\mathfrak{M}$. The assumption on $\mathfrak{M}$ now gives

$$\mathfrak{M} \models (\exists \vec{v}) \wedge \Sigma(\vec{m}, \vec{v})$$

as required.

5. The back and forth argument

The back and forth argument and injective-like structures are tailor made for each other. In this section we give several examples of this duo in action. These examples are intended to be illustrative and we do not attempt to be exhaustive nor prove theorems in their full generality.

Throughout this section T is some fixed theory. The word 'small' is used to mean 'of cardinality $< \kappa$'.

There are two uses of the back and forth argument. One is to construct isomorphisms and the other is to construct elementary equivalences (in some first order or high order language). First we consider examples of the former use.

THEOREM (5.1): *For any situation*

$$\begin{array}{ccc} \mathfrak{M} & & \mathfrak{N} \\ \curlyvee & & \curlyvee \\ \mathfrak{A}_0 & \xrightarrow{f_0} & \mathfrak{B}_0 \end{array} \qquad (S_3)$$

where $\mathfrak{M}$, $\mathfrak{N}$ are $\kappa - WJ_r$ over T of cardinality κ, $\mathfrak{A}_0$, $\mathfrak{B}_0$ are small models of T, and f_0 is $\prec_r$-like, there is an isomorphism f between $\mathfrak{M}$, $\mathfrak{N}$ which extends f_0.

PROOF: Consider any small sets X, Y such that

$$A_0 \subseteq X \subseteq M, \; B_0 \subseteq Y \subseteq N$$

and consider any small $\mathfrak{B}' \prec \mathfrak{N}$ such that $Y \subseteq B'$. Thus we have

$$\begin{array}{ccc} \mathfrak{M} & & \mathfrak{B}' \\ \curlyvee & & \curlyvee \\ \mathfrak{A}_0 & \xrightarrow{f_0} & \mathfrak{B}_0 \end{array}$$

But $\mathfrak{M}$ is $\kappa - WJ_r$ and so we have some $\prec_r$-like injection $\mathfrak{B}' \xrightarrow{g} \mathfrak{M}$ such that

$$\begin{array}{ccc} \mathfrak{M} & & \\ \curlyvee^r & & \\ \mathfrak{A}' & \xleftarrow{g} & \mathfrak{B} \\ \curlyvee^r & & \curlyvee \\ \mathfrak{A}_0 & \xrightarrow{f_0} & \mathfrak{B}_0 \end{array}$$

commutes, where $A' = \text{Image}\ (g)$.

Next consider any small $\mathfrak{A}_1 \prec \mathfrak{M}$ such that $X \subseteq A_1$ and $\mathfrak{A}' \prec_r \mathfrak{A}_1$. Since $\mathfrak{N}$ is $\kappa - WJ_r$ we can repeat the above argument to get a commuting diagram

$$\begin{array}{ccc} \mathfrak{M} & & \mathfrak{N} \\ \curlyvee & & \curlyvee \\ \mathfrak{A}_1 & \overset{f_1}{\rightarrowtail} & \mathfrak{B}_1 \\ \curlyvee^r & & \curlyvee^r \\ \mathfrak{A}' & \overset{g}{\leftarrowtail} & \mathfrak{B} \\ \curlyvee^r & & \curlyvee \\ \mathfrak{A}_0 & \overset{f_0}{\rightarrowtail} & \mathfrak{B}_0 \end{array} \qquad (S_4)$$

where f is $\prec_r$-like and $\mathfrak{A}_1, \mathfrak{B}_1$ are small models of T. In particular f_1 extends f_0.

If we now iterate this construction we will obtain the required isomorphism.

THEOREM (5.2): *Let $\mathfrak{M}$, $\mathfrak{N}$ be $\kappa - WJ_r$ over T of cardinality κ. Then*

$$\mathfrak{M} \equiv\!\!\rangle (\exists_{r+1})\mathfrak{N} \Rightarrow \mathfrak{M} \cong \mathfrak{N}.$$

PROOF: Suppose that $\mathfrak{M} \equiv\!\!\rangle (\exists_{r+1})\mathfrak{N}$. Consider any small $\mathfrak{A} \prec \mathfrak{M}$. By theorem 5.1 it is sufficient to construct *a* $\prec_r$-like injection $\mathfrak{A} \rightarrowtail \mathfrak{N}$. To do this we use the compact-like property of $\mathfrak{N}$. (See theorem 3.2)

Let $\vec{a}$ be an enumeration of $\mathfrak{A}$ and consider the (parameter free) $\kappa - \exists_{r+1}$-type $\Sigma(\vec{v})$ over $\mathfrak{M}$ given by

$$\Sigma(\vec{a}) = Th\,(\mathfrak{A}, \vec{a}) \cap \forall_r.$$

Now $\Sigma(\vec{v})$ contains no parameters from $\mathfrak{M}$, so $\Sigma(\vec{v})$ is also a $\kappa - \exists_{r+1}$-type over $\mathfrak{N}$. To obtain the required injection it is sufficient to show that

$$\mathfrak{N} \models (\exists \vec{v}) \wedge \Sigma(\vec{v}).$$

Consider any finite part $\Delta(\vec{v})$ of $\Sigma(\vec{v})$. We have

$$\mathfrak{M} \models (\exists \vec{v}) \wedge \Delta(\vec{v})$$

hence, since $\mathfrak{M} \equiv\!\!\rangle (\exists_{r+1})\mathfrak{N}$, we have

$$\mathfrak{N} \models (\exists \vec{v}) \wedge \Delta(\vec{v}).$$

Thus $\Sigma(\vec{v})$ is finitely satisfiable in $\mathfrak{N}$, and so, since $\mathfrak{N}$ is $\kappa - \exists_{r+1}$-complete, we get that $\Sigma(\vec{v})$ is satisfiable in $\mathfrak{N}$, as required.

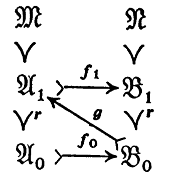
The next theorem is the analogue of theorem 5.1 for $\kappa - WI_r$ structures.

THEOREM (5.3): *For any situation*

$$\begin{array}{ccc} \mathfrak{M} & & \mathfrak{N} \\ \curlyvee^r & & \curlyvee^r \\ \mathfrak{A}_0 & \overset{f_0}{\rightarrowtail} & \mathfrak{B}_0 \end{array}$$

where $\mathfrak{M}$, $\mathfrak{N}$ *are* $\kappa - WI_r$ *over* T *of cardinality* κ, $\mathfrak{A}_0$, $\mathfrak{B}_0$ *are small models of* T, *and* f_0 *is* $\prec_r$*-like, there is an isomorphism* f *between* $\mathfrak{M}$, $\mathfrak{N}$ *which extends* f_0.

PROOF: One journey back and forth produces a commuting diagram



where f_1 is $\prec_r$-like and $\mathfrak{A}_1, \mathfrak{B}_1$ are small models of T. We can now apply theorem 5.1.

The above three results are concerned with isomorphisms. We now consider some results concerned with elementary equivalences. The next theorem and corollaries are stated in terms of first order elementary equivalence, however they can be strengthened to theorems stated in terms of high order elementary equivalence.

THEOREM (5.4): *For any situation* (S_3) *where* $\mathfrak{M}$, $\mathfrak{N}$ *are* $\kappa - WJ_r$ *over* T, $\mathfrak{A}_0$, $\mathfrak{B}_0$ *are small models of* T, *and* f_0 *is* $\prec_r$*-like, then in fact* f_0 *is elementary.*

PROOF: As in the proof of theorem 5.1 we obtain a commuting diagram (S_4) where f_0, g, f_1 are $\prec_r$-like. We now prove, by induction on n, that f_0, f, are $\prec_n$-like for all $n \geqq 0$ (and hence are elementary).

The initial cases are trivial, so it is sufficient to consider the induction step $n \mapsto n+1$. We assume that f_0, f_1 are $\prec_n$-like and first show that f_0 is $\prec_{n+1}$-like.

Consider any formula $\phi(\boldsymbol{u}, \boldsymbol{v}) \in \exists_n$ such that

$$\mathfrak{A}_0 \models (\forall \boldsymbol{u})\phi(\boldsymbol{u}, \boldsymbol{a})$$

for some point $\boldsymbol{a}$ of $\mathfrak{A}_0$. Then, since $\mathfrak{A}_0 \prec \mathfrak{A}_1$ we have

$$\mathfrak{A}_1 \models (\forall \boldsymbol{u})\phi(\boldsymbol{u}, \boldsymbol{a})$$

so that

$$\mathfrak{A}_1 \models \phi(g(\boldsymbol{b}), \boldsymbol{a})$$

for any point $\boldsymbol{b}$ of $\mathfrak{B}_0$. But f_1 is $\prec_n$-like and so

$$\mathfrak{B}_1 \models \phi(f_1 \circ g(\boldsymbol{b}), f_1(\boldsymbol{a}))$$

i.e. using the commuting properties of (S_4)

$$\mathfrak{B}_1 \models \phi(\boldsymbol{b}, f_0(\boldsymbol{a})).$$

Hence, since $\mathfrak{B}_0 \prec \mathfrak{B}_1$, we have

$$\mathfrak{B}_0 \models \phi(\boldsymbol{b}, f_0(\boldsymbol{a})).$$

Thus we get

$$\mathfrak{B}_0 \models (\forall \boldsymbol{u})\phi(\boldsymbol{u}, f_0(\boldsymbol{a}))$$

and so f_0 is $\prec_{n+1}$-like.

Now f_0 is arbitrary and so by the same argument f_1 is also $\prec_{n+1}$-like, as required.

COROLLARY (5.5): *Let $\mathfrak{M}$, $\mathfrak{N}$ be $\kappa - WJ_r$ over T. Then*

$$\boldsymbol{M} \equiv> (\exists_{r+1})\mathfrak{N} \Rightarrow \mathfrak{M} \equiv \mathfrak{N}$$

PROOF: In the manner of theorem 5.2.

COROLLARY (5.6): *Let $\mathfrak{M}$, $\mathfrak{N}$ be $\kappa - WJ_r$ over T. Then*

$$\mathfrak{M} \prec_r \mathfrak{N} \Rightarrow \mathfrak{M} \prec \mathfrak{N}.$$

PROOF: Almost immediately from the theorem.

The injective-like property of the $\kappa - WI_r$ can be slightly strengthened, as is shown in the next theorem. I do not know whether or not a similar theorem holds for the $\kappa - WJ_r$.

THEOREM (5.7): *For any situation*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f} & \mathfrak{M} \\ & \searrow^{g} & \\ & & \mathfrak{B} \end{array}$$

where $\mathfrak{M}$ is $\kappa - WI_r$ over T, $\mathfrak{A}$, $\mathfrak{B}$ are models of T with card $(\mathfrak{A}) < \kappa$, card $(\mathfrak{B}) \leqq \kappa$, and f, g are both $\prec_r$-like, there is a $\prec_r$-like injection $\mathfrak{B} \xrightarrow{h} \mathfrak{M}$ such that $f = h \circ g$.

PROOF: Let $\mathfrak{B}_0$ be the range of g and let

$$\mathfrak{A} \xrightarrow{g} \mathfrak{B} = \mathfrak{A} \xrightarrow{g_0} \mathfrak{B}_0 \prec_r \mathfrak{B}.$$

Now let

$$\mathfrak{B} = \cup \{\mathfrak{B}_i : i < \kappa\}$$

where, for each $i < \kappa$,

$$\mathfrak{B}_i \prec_r \mathfrak{B} \text{ and card } (\mathfrak{B}_i) < \kappa.$$

By induction we can construct for each $0 < i < \kappa$ a commuting diagram

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{f_i} & \mathfrak{A}_i & \prec_r & \mathfrak{M} \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g_i} & & \\ \mathfrak{B}_0 & \prec_r & \mathfrak{B}_i & \prec_r & \mathfrak{B} \end{array}$$

where

$$\mathfrak{A} \rightarrowtail^{f} \mathfrak{M} = \mathfrak{A} \rightarrowtail^{f_i} \mathfrak{A}_i \prec_r \mathfrak{M},$$

g_i is $\prec_r$-like, and for each $j < ig_i$ extends g_j. The required injection is now obtained from $\cup \{g_i : i < \kappa\}$.

In [4], Abraham Robinson has introduced into model theory generic structures using forcing with infinite conditions. However it turns out that these generic structures can be constructed, and their properties developed, without using any forcing techniques. The crucial facts are theorem 2.8 and corollary 5.6 (both for the case $r = 0$). These generic structures are characterized in the following theorem (which we state without proof).

THEOREM (5.8): *Suppose T is $\forall_2$-axiomatizable. For any model $\mathfrak{A}$ of T the following are equivalent*

(i) $\mathfrak{A}$ *is a generic model of T.*

(ii) $\mathfrak{A} \prec \mathfrak{M}$ *for some $\mathfrak{M}$ which is $\kappa - WJ_0$ over T.*

6. Homogeneous-universal and saturated models

We have seen that the $\kappa - WJ_\infty$ models of T are exactly the κ-saturated models (see theorems 3.3 and 3.5). In this section we will connect $\kappa - WJ_0$ models with the κ-homogeneous-universal models.

First we consider the notion of universality.

DEFINITION: The model $\mathfrak{M}$ of T is $\kappa - U_r$ over T if for each model $\mathfrak{A}$ of T with card $(\mathfrak{A}) < \kappa$ there is a $\prec_r$-like injection $\mathfrak{A} \rightarrowtail \mathfrak{M}$.

Notice that the $\kappa^+ - U_0$ models of cardinality κ are what are usually called the universal models of cardinality κ.

THEOREM (6.1): *Suppose T has JEP_r. Then every $\kappa - WJ_r$ model is $\kappa - U_r$, and every $\kappa - WI_r$ model is $\kappa^+ - U_r$.*

PROOF: Suppose $\mathfrak{M}$ is $\kappa - WJ_r$ over T and $\mathfrak{C}$ is any model of T with card $(\mathfrak{C}) < \kappa$. Let $\mathfrak{A} \prec \mathfrak{M}$ where card $(\mathfrak{A}) < \kappa$. Using JEP_r we obtain a situation

$$\begin{array}{ccc} & \mathfrak{A} \prec \mathfrak{M} & \\ & \searrow{\scriptstyle g} & \\ \mathfrak{C} & \xrightarrow[k]{} & \mathfrak{B}. \end{array}$$

for some model $\mathfrak{B}$ with card $(\mathfrak{B}) < \kappa$ and $\prec_r$-like injections g, k. But $\mathfrak{M}$ is $\kappa - WJ_r$ so we have a $\prec_r$-like injection $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$. Thus $h \circ k$ gives the required injection.

If $\mathfrak{M}$ is $\kappa - WI_r$ a similar argument using theorem 5.7 shows that $\mathfrak{M}$ is $\kappa^+ - U_r$.

The following theorem shows that to construct universal models joint embedding is necessary.

THEOREM (6.2): *If T has some $\lambda^+ - U_r$ model then T has JEP_r.*

PROOF: Let $\mathfrak{M}$ be $\lambda^+ - U_r$ over T and consider any two models $\mathfrak{A}$, $\mathfrak{B}$ of T. We wish to find a model $\mathfrak{C}$ of T together with $\prec_r$-like injections

$$\mathfrak{A} \rightarrowtail \mathfrak{C}, \mathfrak{B} \rightarrowtail \mathfrak{C}.$$

A simple compactness argument shows that we may assume card $(\mathfrak{A}) \leqq \lambda$, card $(\mathfrak{B}) \leqq \lambda$. Thus we can take $\mathfrak{C} = \mathfrak{M}$.

We now consider the connection between $\kappa - WI_r$ and $\kappa - WJ_r$. Of course we always have $\kappa - WI_r \subseteq \kappa - WJ_r$, the next two theorems show that either we have equality or $\kappa - WI_r$ is empty.

THEOREM (6.3): *Suppose T has AP_r. then $\kappa - WI_r = \kappa - WJ_r$.*

PROOF: Since $\kappa - WI_r \subseteq \kappa - WJ_r$ it is sufficient to show that $\kappa - WJ_r \subseteq \kappa - WI_r$ whenever T has AP_r.

Suppose T has AP_r and consider any $\mathfrak{M}$ which is $\kappa - WJ_r$ over T. Consider any situation (S_1) where both f, g are $\prec_r$-like. We must produce a $\prec_r$-like fill in $\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M}$.

Consider any $\mathfrak{A}' \prec \mathfrak{M}$ such that card $(\mathfrak{A}) < \kappa$ and $\mathrm{Im}\,(f) \subseteq \mathfrak{A}'$ (i.e. $\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{A}'$). Using AP_r we obtain a commuting diagram

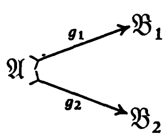
$$\begin{array}{ccc} \mathfrak{A} & \overset{f}{\rightarrowtail} & \mathfrak{A}' \prec \mathfrak{M} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathfrak{B} & \underset{f'}{\rightarrowtail} & \mathfrak{B}' \end{array}$$

where card $(\mathfrak{B}) < \kappa$ and f', g' are both $\prec_r$-like. But $\mathfrak{M}$ is $\kappa - WJ_r$ and so we have a $\prec_r$-like fill in $\mathfrak{B}' \overset{h'}{\rightarrowtail} \mathfrak{M}$. Thus we may put $h = h' \circ f'$.

The next theorem shows the inevitability of AP_r.

THEOREM (6.4): *If each model of T of cardinality λ is $\prec_r$-embeddable in some $\lambda^+ - WI_r$ model of T then T has AP_r.*

PROOF: Suppose we wish to amalgamate



where $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ are models of T and g_1, g_2 are $\prec_r$-like. By compactness we can assume that card $(\mathfrak{A}) \leqq \lambda$, card $(\mathfrak{B}_1) \leqq \lambda$, and card $(\mathfrak{B}_2) \leqq \lambda$.

Consider $\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{M}$ where $\mathfrak{M}$ is $\lambda^+ - WI_r$ over T and f is $\prec_r$-like. This model $\mathfrak{M}$ provides the required amalgam.

Finally we consider the notion of homogeneity.

Definition: The model $\mathfrak{M}$ of T is $\kappa - H_r$ over T if for any two $\prec_r$-like injections

$$\mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{M}, \mathfrak{A} \overset{f'}{\rightarrowtail} \mathfrak{M}$$

where $\mathfrak{A}$ is a model of T with card $(A) < \kappa$, there is an isomorphism $\mathfrak{M} \overset{k}{\rightarrowtail} \mathfrak{M}$ such that $f = k \circ f'$.

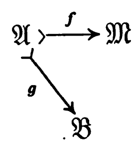
Theorem 5.3 immediately gives us the following.

Theorem (6.5): *If $\mathfrak{M}$ is $\kappa - WI_r$ over T and has cardinality κ then $\mathfrak{M}$ is $\kappa - H_r$ over T.*

On the other hand we have the following.

Theorem (6.6): *Suppose $\mathfrak{M}$ is both $\kappa - H_r$ and $\kappa - U_r$ over T. Then $\mathfrak{M}$ is $\kappa - WI_r$ over T.*

Proof: Consider any situation



where $\mathfrak{A}, \mathfrak{B}$ are models of T with card $(\mathfrak{B}) < \kappa$, and f, g are both $\prec_r$-like. Since M is $\kappa - U_r$ we have a $\prec_r$-like injection
$\mathfrak{B} \rightarrowtail \mathfrak{M}$. Let

$$\mathfrak{A} \overset{f'}{\rightarrowtail} \mathfrak{M} = \mathfrak{A} \overset{g}{\rightarrowtail} \mathfrak{B} \rightarrowtail \mathfrak{M}$$

so that f' is $\prec_r$-like. Now $\mathfrak{M}$ is also $\kappa - H_r$ hence we have $f = k \circ f'$ for some isomorphism k of M. Thus putting

$$\mathfrak{B} \overset{h}{\rightarrowtail} \mathfrak{M} = \mathfrak{B} \rightarrowtail \mathfrak{M} \overset{k}{\rightarrowtail} \mathfrak{M}.$$

we obtain the required injection.

All this gives us the following theorem.

THEOREM (6.7): *Suppose T is* $\forall_{r+2}$*-axiomatizable.*

(1) *If T has some* $\lambda^+ - H_r - U_r$ *model then T has* $JEP_r + AP_r$.

(2) *If T has* $JEP_r + AP_r$ *and* κ *is a regular cardinal such that* $\kappa^{\kappa^*} = \kappa > \lambda$ *then T has some* $\kappa - H_r$, $\kappa^+ - U_r$ *model of cardinality* κ.

PROOF: (1)By theorems 6.2, 6.6, 6.4.

(2) By theorem 2.8 we have a model $\mathfrak{M}$ of cardinality κ which is $\kappa - WJ_r$ over T. Theorems 6.3, 6.1 show that $\mathfrak{M}$ is $\kappa^+ - U_r$, and theorem 6.5 shows that $\mathfrak{M}$ is $\kappa - H_r$.

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