COMPOSITIO MATHEMATICA

HERBERT POPP

On moduli of algebraic varieties II

Compositio Mathematica, tome 28, nº 1 (1974), p. 51-81

<http://www.numdam.org/item?id=CM_1974__28_1_51_0>

COMPOSITIO MATHEMATICA, Vol. 28, Fasc. 1, 1974, pag. 51–81
Noordhoff International Publishing
Printed in the Netherlands

ON MODULI OF ALGEBRAIC VARIETIES II

Herbert Popp

Introduction

In part I, [22], in this series of papers we have shown that over the complex numbers coarse moduli spaces exist for polarized K-3 surfaces and for algebraic varieties with a very ample canonical bundle and that these moduli spaces are algebraic spaces of finite type over $\boldsymbol{C}$ in the sense of [2].

In this paper we treat, in Section 2, the moduli problem for surfaces of general type defined over $\boldsymbol{C}$. We will show that for such surfaces coarse moduli spaces exist (theorem 2.7), and that these spaces are algebraic spaces of finite type over $\boldsymbol{C}$. This fact has interesting applications to the structure of the Albanese mapping $\alpha : X \to Alb\,(X)$ of 4-dimensional compact manifolds X without meromorphic functions.

We obtain by using the existence of non constant meromorphic functions on the moduli spaces of surfaces of general type (this holds, because the moduli spaces are algebraic spaces) that the general fibre of α has Kodaira dimension $\leqq 0$. It is this kind of applications to the classification theory of compact complex manifolds (compare the introduction to [22]) which make it important to know that moduli spaces are algebraic spaces and not just analytic spaces.

The key to the considerations is the following theorem on quotients which will be proved in Section 1. (Compare theorem 1.4.)

THEOREM: *Let X be a quasi projective $\boldsymbol{C}$-scheme* [1] *and G an algebraic group which operates on X. Assume that the operation is proper and that all stabilizers are finite. Then the geometric quotient Y of X by G, in the category of algebraic $\boldsymbol{C}$-spaces, exists and Y is of finite type over $\boldsymbol{C}$.*
This is how one constructs Y locally, i.e., an open etale neighborhood of a $\boldsymbol{C}$-valued point $\bar{P} \in Y$.

Let P be a $\boldsymbol{C}$-valued point of X which maps to the point $\bar{P}$ by the (topological) quotient map. Let O_P be the orbit of P with respect to G. We show that there exists a locally closed subscheme U_P of X such that
1. $P \in U_P$, U_P is affine and U_P is transversal to O_P at P.

[1] The assumption, X is quasi projective, is not necessary. X can be a separated algebraic $\boldsymbol{C}$-space locally of finite type over $\boldsymbol{C}$. (compare remark 1.5.)

2. The stabilizer I of P (stabilizer with respect to the action of G on X) operates on U_P.
3. For any C-valued point $Q \in U_P$, U_P is transversal to the orbit O_Q of Q. Furthermore if $g \in G$ and $g(Q) = Q$, then $g \in I$.

Then the geometric quotient U_P^I of U_P by I is an etale neighborhood of $\bar{P} \in Y$.

Unfortunately, our proof of the above stated theorem is not algebraic. The theory of complex spaces is needed. However there should exist along the same lines a purely algebraic proof which applies also in characteristic > 0.

The quotient theorem 1.4 leads, combined with the results of Deligne-Mumford [5], to the existence of an algebraic C-space $\overline{M}_g$ which is a coarse moduli space for stable curves of genus g defined over C. This result is implicitly contained also in [5], 4.21. The space $\overline{M}_g$ is of finite type over C and contains the coarse moduli space M_g for smooth C-curves of genus g as a dense subspace. Furthermore, $\overline{M}_g$ is proper over C and hence compact with respect to the complex topology. [2] To conclude the last fact it is again essential to know that $\overline{M}_g$ is an algebraic space. We explain this in more details at the end of Section 1.

As far as characteristic 0 is concerned the above stated quotient theorem is stronger than the theorems on quotients proved in [22]. We would like to point out that some of the results on quotients in Mumford's book [20] and theorem 7.2 in the paper of Seshadri [24] are related to our theorem 1.4. By Mumford and Seshadri the following holds.

THEOREM: *Let X be a normal* [3] *projective variety over an algebraically closed field k of arbitrary characteristic let G be a reductive group which operates linearly on X with respect to an ample line bundle L on X, and with finite stabilizers. Let $X^s(L)$ be the subscheme of X consisting of stable points. Then the induced operation on $X^s(L)$ by G is proper and the geometric quotient of $X^s(L)$ by G exists and is a quasiprojective k-scheme of finite type.*

However there exist examples, compare [20], p. 83, which show that the reductive group SL (n) can operate properly and with finite stabilizer on a quasi projective scheme X which is simple over C, but there does not exist an ample sheaf L on X such that G operates linearly with respect to L and $X = X^s(L) =$ set of stable points with respect to L.

In such a situation the above stated theorem of Mumford and Seshadri cannot be applied whereas theorem 1.4 leads to an algebraic C-space which is a geometric quotient of X by G. An often very difficult analysis of stability is not needed.

[2] According to Mumford [21], p. 462, $\overline{M}_g$ is even a projective variety.

[3] In characteristic 0 normality is not needed.

Finally we would like to mention that the above stated theorem on quotients and the considerations in [22], chapter II lead, for (smooth) canonical polarized varieties to algebraic C-spaces of finite type which are coarse moduli spaces for these varieties.

I am grateful to the referee for pointing out an error in the proof of proposition 2.3 and for valuable suggestions which helped to improve the paper considerably.

1. Group actions and quotients
The coarse moduli space for stable C-curves of genus g

The notion of an algebraic space is that of [2] or [15]. *All algebraic spaces are C-spaces and assumed to be separated*, unless stated otherwise, where C is the complex number field. Also, all schemes are C-schemes.

To every algebraic space X over C of finite type a complex analytic space X^{an} is associated in a natural way, [2].

DEFINITION (1.1): A complex analytic space Z is called *algebraic* if there exists an algebraic C-space X of finite type such that the associated analytic space X^{an} is isomorphic to Z.

Every irreducible, reduced, analytic space Z which is algebraic has many global meromorphic functions. More precisely, if $m(Z)$ denotes the field of meromorphic functions of such a space, the transcendence degree of the field extensions $m(Z)/C$ is $\geqq \dim Z$ as Chow's Lemma, [15], p. 192, applies, i.e. there exists a quasi projective scheme Z' and a birational proper map $Z' \to Z$.

If the analytic space Z is irreducible, reduced and compact the following theorem holds.

THEOREM (1.2): *Z is algebraic if and only if transcendence degree of $m(Z)/C$ is equal to dimension Z.*

PROOF [2], p. 176 ff.

It is known, [2], that an algebraic space X of finite type over C is proper over C if and only if the associated analytic space X^{an} is compact.

This shows that proper is the correct equivalent to compact. For algebraic spaces a good test for properness is the valuation criterion.

VALUATIVE CRITERION (1.3): Let X be a separated algebraic C-space. X is proper over C if and only if for any discrete valuation ring R over C with field of fractions K and any commutative diagram of morphisms

$$\begin{tikzcd} & & X \arrow[d] \\ \operatorname{Spec}(K) \arrow[r] \arrow[urr, "f"] & \operatorname{Spec}(R) \arrow[r] & \operatorname{Spec}(C), \end{tikzcd}$$

there exists a finite field extension K' of K such that f extends to Spec (R'), where R' is the integral closure of R in K', via the commutative diagram.

$$
\begin{tikzcd}
 & X \arrow[ddr] & \\
\mathrm{Spec}\,(K') \arrow[r] \arrow[d] & \mathrm{Spec}\,(R') \arrow[u, dashed] \arrow[d] & \\
\mathrm{Spec}\,(K) \arrow[r] \arrow[uur, "f"] & \mathrm{Spec}\,(R) \arrow[r] & \mathrm{Spec}\,(\boldsymbol{C}).
\end{tikzcd}
$$

Proof: Notice that Spec $(K') \to$ Spec (K) is an open etale covering of Spec (K) in the sense of algebraic spaces and proceed as in the proof of the valuation criterion for schemes.

We will use this criterion later to show that the coarse moduli space for stable curves of genus $g \geqq 2$, which are defined over $\boldsymbol{C}$, is proper over $\boldsymbol{C}$.

Group actions and quotients will now be considered.

The notation is as in [22], Section 1. G shall be an algebraic group defined over $\boldsymbol{C}$. An algebraic space on which G acts, [22], p. 9, is called a G-space.

Maps between G-spaces are defined as usual.

For a G-space X, the notions 'G-stable etale covering' and 'geometric quotient' of X by G with respect to the category of algebraic $\boldsymbol{C}$-spaces, are as in [22], 1.1, and [22], 1.5, respectively. We formulate the main theorem of this section.

The situation is as follows.

Let X be a quasi projective $\boldsymbol{C}$-scheme and G an algebraic group over $\boldsymbol{C}$. Assume that G operates on X in the sense of schemes, [20], p. 3, and that the operation is proper, i.e., the map

$$G \times X \xrightarrow{\Psi = (\Phi, Id)} X \times X \text{ is proper,}$$

where $\Phi : G \times X \to X$ defines the group operation of G on X and Id is the identity map of X.

$I_P = \{\sigma \in G(\mathrm{Spec}\,(\boldsymbol{C})); \sigma(P) = P\}$ is called the stabilizer of P, P a $\boldsymbol{C}$-valued point of X. We admit only those actions of G on X having finite stabilizer I_P for all $P \in X$. Clearly, if X is considered as an algebraic space, G acts on X.

Theorem (1.4): *In the above situation the geometric quotient Y of X by G exists in the category of algebraic $\boldsymbol{C}$-spaces and is of finite type over $\boldsymbol{C}$.*

REMARK (1.5): The assumption that X is a quasi projective C-scheme is not necessary. Using the methods of Deligne for factoring a finite group which operates on a separated algebraic space ([15], p. 183) the considerations below give a proof of the following more general theorem.

THEOREM: *Let X be a separated algebraic C-space locally of finite type (respectively of finite type) over C. G shall be an algebraic group over C which acts on X such that the action is proper and the stabilizers are finite. Then the geometric quotient Y of X by G exists in the category of algebraic C-spaces and is locally of finite type (respectively of finite type) over C.*

We indicate briefly the main changes in the arguments of the proof of theorem 1.4 which are necessary to obtain the above stated general quotient theorem.

Let $P \in X$ be a point and $I_P = I$ the stabilizer of P. O_P shall be the orbit of P with respect to G. Deligne has shown that there exists an affine scheme X_P of finite type over C, an action of I on X_P and an etale map $X_P \xrightarrow{\chi} X$ which is I-invariant and such that $P \in \chi(X_P)$. Furthermore for an arbitrary point $Q \in X_P$ the stabilizers of Q and $\chi(Q)$ with respect to I are equal.

Consider the smooth I-invariant subspace $\chi^{-1}(O_P)$ of X_P and a point $P^* \in \chi^{-1}(O_P)$ with $\chi(P^*) = P$. Applying lemma 1.8 we obtain a smooth I-invariant subscheme $\hat{U}_{P*} \subset X_P$ which is transversal to $\chi^{-1}(O_P)$ at P. Now one proceeds with $G \times \hat{U}_{P*}$ and the map $G \times \hat{U}_{P*} \xrightarrow{\Phi^*} X$, defined by $(g, u) \to g(\chi(u))$ instead of $G \times \hat{U}_P$ and the map $G \times U_P \xrightarrow{\Phi} X$ as described below.

A difficulty appears in the extension of proposition 1.11 to the general situation. We proceed as follows. Let X_P and $\chi : X_P \to X$ be as above. $\Psi^* : G \times X_P \to X \times X$ shall be the map defined by $\Psi^*((g, Q)) = (g(\chi(Q)), \chi(Q))$ and $\Delta : \hat{U}_{P*} \to X \times X$ the map determined by $\Delta(Q) = (\chi(Q), \chi(Q))$.

We consider the diagram

$$\begin{array}{ccc} U^* = \hat{U}_{P*} \underset{X \times X}{\times} (G \times X_P) & \longrightarrow & G \times X_P \\ \downarrow \Psi_U^* & & \downarrow \Psi^* \\ \hat{U}_{P*} & \xrightarrow{\Delta} & X \times X. \end{array}$$

U^* is a subscheme of $\hat{U}_{P*} \times (G \times X_P)$. Let $U_1^*, \cdots, U_r^*$ be the irreducible components of U^* and $\Psi_U^*(U_i^*) = Z_i$ the image of U_i^*. Let $Z_1, \cdots, Z_s$ be the subsets of $\hat{U}_{P*}$ for which $P \in \bar{Z}_i$ holds, $i = 1, \cdots, s$, ($\bar{Z}_i$ = Zariski closure of Z_i in $\hat{U}_P^*$.) Taking Ψ_U^* instead of Ψ_U the procedure is now almost as on page 59. We have to use again the fact that $P \in \bar{Z}_{i_0}$ and the generalized form of lemma 1.9, which states roughly as follows: Let $Q \in \hat{U}_{P*}$ and $g \in G$ such that $\chi(Q)$ and $g(\chi(Q))$ are near to P (with respect to the complex topology). Then $g(\chi(Q)) \in \chi(\hat{U}_{P*})$ implies $g \in I$.

As we only apply the quotient theorem if X is quasi projective and of finite type over $\boldsymbol{C}$ we obmit the detailed proof of the general theorem.

REMARK (1.6): It will be proved in the following (compare corollary 1.10) that in the situation of theorem 1.4 the operation of G on X is separable in the sense of Holmann [10], definition 15. Hence, by [10], Satz 12, and [12], Satz IV, 9.6, the quotient of X by G exists as an analytic space. More precisely, in the situation of theorem 1.4, G considered as a Lie group, operates on X^{an}, and the geometric quotient as defined in [10] and [12], exists. We will use this fact later for the construction of the quotient of X by G as an algebraic space. Roughly speaking, we show that the analytic quotient of X by G is an algebraic space in the sense of 1.1. The proof of theorem 1.4 requires two lemmas.

LEMMA (1.7): *Let V be an affine $\boldsymbol{C}$-scheme and H a finite group operating on V. There exists an embedding $f: V \to \boldsymbol{C}^m$ of V into an affine space $\boldsymbol{C}^m$ and a finite linear subgroup H^* of $GL(m, \boldsymbol{C})$ which is isomorphic to H, such that $f(V)$ is stable under the action of H^* and the action induced on $f(V)$ by H^* is the same as the action on $f(V)$ induced by H.*

PROOF: Let $g: V \to \boldsymbol{C}^n$ be an arbitrary closed embedding of V into an affine space $\boldsymbol{C}^n$. For every $h \in H$ let V_h be a copy of V. Consider the product variety $W = \prod_{h \in H} V_h$. The group H operates then on W by permutation of the factors. If $h \in H$, the corresponding permutation is $(v_{h_1}, \cdots, v_{h_{|H|}}) \to (v'_{h_1}, \cdots, v'_{h_{|H|}})$ where $v'_{h_i} = v_{h \cdot h_i}$, and $h_1, \cdots, h_{|H|}$ is a fixed ordering of the elements of H. W can be viewed as a subspace of the affine space $\boldsymbol{C}^{|H| \cdot n} = \boldsymbol{C}^m$ via the mapping

$$W \overset{\Pi g}{\to} \boldsymbol{C}^{|H| \cdot n} = \boldsymbol{C}^m,$$

where $|H|$ = order of H.
Then, there exists a finite group H^* of linear transformation of the space $\boldsymbol{C}^m$ (namely certain permutations of the coordinate of $\boldsymbol{C}^m$) which defines the same operation on W as H. Suppose $f: V \to \boldsymbol{C}^{|H| \cdot n} = \boldsymbol{C}^m$ is the morphism defined by

$$f(x) = (g(h_1(x)), g(h_2(x)), \cdots, g(h_{|H|}(x))).$$

The image $f(V)$ is, then, a closed affine subscheme of $\boldsymbol{C}^m$ which is stable under the action of the group H^*. Furthermore, f is an isomorphism from V to $f(V)$ (in the sense of schemes) and transforms the automorphisms of V which are induced by the elements of H to automorphisms of $f(V)$ induced by elements of H^*. Q.E.D.

LEMMA (1.8): *Let V be an affine $\boldsymbol{C}$-scheme on which a finite group H operates, and P be a point of V. Assume that P is kept fixed by all $h \in H$.*

If E is a smooth subvariety of V, $P \in E$ and E is invariant under H, then, there exists an H-invariant subvariety U of V which passes through P and which is transversal [4] *to E.*

PROOF: We may assume, by Lemma 1.7, that $V \subset \boldsymbol{C}^m$ and that the operation of H on V is induced by linear transformations of the $\boldsymbol{C}^m$. Let $T_P(E)$ be the tangent space to the variety E at P, then $T_P(E)$ is invariant with respect to the action of H.

CLAIM: There exists a linear subspace $L \subset \boldsymbol{C}^m$ which passes through P such that

1. H operates on L.
2. $T_P(E)$ and L span the space $\boldsymbol{C}^m$ and $T_P(E) \cap L = \{P\}$.

PROOF OF THE CLAIM: Let Q be a positive definite Hermitian form on $\boldsymbol{C}^m$. Then, $\bar{Q}(x, y) = \sum_{h \in H} Q(h(x), h(y))$ defines a positive definite Hermitian form on $\boldsymbol{C}^m$ which is invariant by H. If we choose for L the linear subspace of $\boldsymbol{C}^m$ perpendicular to the space $T_P(E)$ with respect to $\bar{Q}$, L satisfies the statements of the claim.
$U = L \cap V$ fills the requirements of the lemma. Q.E.D.

We return to the proof of theorem 1.4.

Let $P \in X$ be a point and $I = I_P$ be the stabilizer of P by G. Then, I operates on X and there exists an affine open subscheme V of X which contains P and which is invariant under I. (Use that X is quasi projective and I finite.) If O_P is the orbit of P on X, with respect to G, and $E_P = V \cap O_P$, I operates on E_P.

Applying lemma 1.8 we find that we can choose a closed I-stable subscheme $\hat{U}_P$ of V which contains P and which is transversal to E_P at P.

LEMMA (1.9): *The morphism $G \times \hat{U}_P \xrightarrow{\phi} X$ defined by $(g, Q) \to g(Q)$, considered as an analytic map, is locally at (ε, P) an analytic isomorphism onto an open neighborhood of P in X, ε = unity elements of G. There exists furthermore a complex analytic* neighborhood S_P of P in U_P such that the following holds. S_P is stable with respect to the action of I and if $Q \in S_P$, $g \in G$ and $g(Q) \in S_P$, then $g \in I$.

PROOF: The first statement is proved along the lines of Hilfsatz 1 of Holmann's paper [10]. Compare also [12], Hilfsatz IV, 10.2. One shows that there exists an open neighborhood S_P of P in U_P and an open neighborhood V_ε of ε in G such that the map $V_\varepsilon \times S_P \to X$ is locally at (ε, P) an analytic isomorphism. The proof is obmitted here. For the proof of the

[4] The subschemes W and E of V which pass through $P \in V$ are called transversal at P if the Zariski tangent spaces $t(W)_P$ and $t(E)_P$ are linearly independent as subspaces of $t(V)_P$.

second statement we assume that this statement is false. Then there exists a sequence Q_i of points in U_P with $\lim_{i\to\infty}(Q_i) = P$ and a sequence of points $g_i \in G$, $g_i \notin I$, such that $g_i(Q_i) = Q_i' \in \hat{U}_P$ and $\lim_{i\to\infty}(Q_i') = P$. Consider in $X \times X$ the set $A = \{(Q_i', Q_i), (P, P); i \geqq 1\}$. This set is compact with respect to the complex topology of $X \times X$. By assumption the map $\Psi : G \times X \to X \times X$ is proper as a map of schemes. By [6], p. 323, Ψ is then also proper as an analytic morphism. This implies that $A^* = \Psi^{-1}(\{(Q_i', Q_i), (P, P)\})$ is compact in $G \times X$. Obviously the points (g_i, Q_i) belong to A^*.

Let $(g, Q) \in A^*$ be a limit point of the set $\{(g_i, Q_i), i \geqq 1\}$ and (g_{i_ν}, Q_{i_ν}), $\nu \geqq 1$, a subsequence of (g_i, Q_i) which converges to (g, Q). Then $(g(Q), Q) = \Psi((g, Q)) = \lim_{\nu\to\infty} \Psi((g_{i_\nu}, Q_{i_\nu})) = \lim_{\nu\to\infty}(Q_{i_\nu}', Q_{i_\nu}) = (P, P)$ and we obtain $P = Q$ and $g \in I$.

Take now a complex neighborhood S_P of P in $\hat{U}_P$ such that $V_\varepsilon \times S_P \xrightarrow{\Phi} X$ is an isomorphism onto an open set of X which contains P. (V_ε is a complex neighborhood of the unite element ε of G.) It is easy to see that one can choose S_P in such a way that I operates on S_P. (Use that I operates on $\hat{U}_P$ and that $g(P) = P$ if $g \in I$.) Clearly almost all elements g_{i_ν} are in the neighborhood $V_\varepsilon \cdot g$ of g. Let g_{i_n} be choosen such that $g_{i_n} \in V_\varepsilon \cdot g$, $g_{i_n} \neq g$. Write $g_{i_n} = v_{i_n} \cdot g$ with $v_{i_n} \in V_\varepsilon$, $v_{i_n} \neq \varepsilon$. Then $Q_{i_n}' = g_{i_n}(Q_{i_n})$ implies that $Q_{i_n}' = v_{i_n}(g(Q_{i_n}))$ where $g(Q_{i_n}) \in S_P$. This is a contradiction that to the fact the map $\Phi : V_\varepsilon \times S_P \to X$ is $1-1$. Q.E.D.

Important corollary (1.10): *Under the assumption of theorem* 1.4 *the group G considered as a Lie group acts on* X^{an} *such that the operation is separable in the sense of Holman* [10]. *def.* 13. *Compare also* [12]. *Therefore by* [10], *Satz* 12, *and* [12], *IV*, 9.6, *the geometric quotient of X by G exists as an analytic space.* [5]

Proof: We have only to show that there exists for any two points $Q', Q'' \in X$, which are not equivalent with respect to G, neighborhoods U' and U'' of Q' respectively Q'' such that $g(U')$ and U'' are disjoint for all $g \in G$. But this follows at once from the assumption that $\Psi : G \times X \to X \times X$ is proper and hence the graph of the operation of G on X closed in $X \times X$.

Proposition (1.11): *There exists an open subscheme* U_P' *of* $\hat{U}_P$ *containing P, which is stable with respect to the action of the group I such that* $g \in G(\mathrm{Spec}\,(\boldsymbol{C}))$ *and* $g(Q) = Q$ *for a* $\boldsymbol{C}$*-valued point* $Q \in U_P'$ *implies* $g \in I$.

Proof: Look at the maps $\Psi : G \times X \to X \times X$ and $f : \hat{U}_P \to X \times X$, f

[5] The paper of Holmann deals with a reduced analytic space X. The paper of Kaup extends Holmann's methods to non reduced analytic spaces.

defined by $\hat{U}_P \xrightarrow{i_U} X \xrightarrow{\Delta_X} X \times X$, where i_U is the embedding of $\hat{U}_P$ in X and $\Delta_X : X \to X \times X$ the diagonal map.

Consider the diagram

$$\begin{array}{ccc} U^* = \hat{U}_P \underset{X \times X}{\times} (G \times X) & \longrightarrow & G \times X \\ \downarrow{\scriptstyle \Psi_U} & & \downarrow{\scriptstyle \Psi} \\ \hat{U}_P & \xrightarrow{f} & X \times X. \end{array}$$

Set theoretically $U^*(\mathrm{Spec}\,(\boldsymbol{C})) = \{(g, u);\ g(u) = u,\ u \in \hat{U}_P(\mathrm{Spec}\,(\boldsymbol{C})), g \in G\,(\mathrm{Spec}\,(\boldsymbol{C}))\}$ holds.

Let $U_1^*, \cdots, U_r^*$ be the irreducible components of U^* and $\Psi_U(U_i^*) = Z_i$ be the image of U_i^*. Then Z_i is closed in $\hat{U}_P$, because Ψ_u is proper.

Suppose $Z_1, \cdots, Z_s$ are the subspaces which contain the point P and let $P \notin Z_{s+i}$, $i \geqq 1$. Take $U'_P = \hat{U}_P - \bigcup_{i=1,\cdots,r-s}^{h \in I} h(Z_{s+i} \cap \hat{U}_P)$. Then U'_P is an open I-invariant subscheme of U_P which contains the point P. If $Q \in U'_P$, $g \in G$ and $g(Q) = Q$, there exists by the construction of U'_P a scheme Z_{i_0}, $1 \leqq i_0 \leqq s$, such that $(g, Q) \in Z_{i_0}$. Let $(\hat{g}, \hat{Q})$ be a generic point of Z_{i_0} with respect to $\boldsymbol{C}$ in the sense of A. Weil. If we can show that $\hat{g}$ has coordinates in $\boldsymbol{C}$, we are done, because $(\hat{g}, \hat{Q})$ spezialises then to $(\hat{g}, P)$ with $\hat{g}(P) = P$ and this implies $\hat{g} \in I$.

Why is $\hat{g}$ $\boldsymbol{C}$-valued?

We use a complex open neighborhood S_P of P in U'_P which satisfies lemma 1.9 and get:

If $g \in G$ and $g(Q) = Q$ for a point $Q \in S_P$ then $g \in I$.

Using this fact we conclude that for all spezialisations (g, Q) of $(\hat{g}, \hat{Q})$ such that $Q \in S_P$, the element g belongs to I. The finiteness of I implies that $g = g_0$ for all such g, where (g_0, P) is a point in $U_{i_0}^*$. From this one concludes easily that $\hat{g}$ is $\boldsymbol{C}$-valued. (Roughly speaking, the idea is this. Assume we have chosen an affine embedding of the above situation, i.e. of $G \times X$, locally at (g_0, P). Then any coordinate function of $\hat{g}$ is a polynomial function on $U_{i_0}^*$ which obtains only finitely many values in a complex neighborhood of the point $(g_0, P) \in U_{i_0}^*$. Such a function is constant.)

COROLLARY (1.12): *The subscheme U'_P in proposition* 1.11 *can be choosen to be an affine scheme.*

Look at the scheme $G \times U'_P$, where U'_P is affine and as in corollary 1.12. I acts on $G \times U'_P$ by the rule

$$(g, u) \to (g\alpha^{-1}, \alpha(u)),\ \alpha \in I.$$

The geometric quotient of $G \times U'_P$ by I exists in the sense of schemes, as G is an algebraic group and U'_P is affine. (Use [23], p. 59.) We denote this quotient by $(G\times U'_P)^I$ and denote by

$$\varphi_{G\times U'_P}\colon G\times U'_P \to (G\times U'_P)^I$$

the quotient map.

The group G operates on $(G\times U'_P)^I$ in a natural way.

To obtain this operation, we note first that G operates on $(G\times U'_P)$ by the rule

$$G\times(G\times U'_P) \xrightarrow{\phi_{G\times U_P'}} G\times U'_P$$

$$(g',(g,u)) \to (g'g,u).$$

On $G\times(G\times U'_P)$ the group I operates via the second factor. Furthermore the geometric quotient of $G\times(G\times U'_P)$ by I exists and is isomorphic to $G\times(G\times U'_P)^I$.

Consider the map $G\times(G\times U'_P)\xrightarrow{\varphi_{G\times U'_P}\circ\phi_{G\times U'_P}}(G\times U'_P)^I$, defined by

$$G\times(G\times U'_P)\xrightarrow{\phi_{G\times U'_P}} G\times U'_P \xrightarrow{\varphi_{G\times U'_P}} (G\times U'_P)^I.$$

This map is, obviously, an I-morphism, where I acts trivially on $(G\times U'_P)^I$. Hence, there exists a unique map

$$G\times(G\times U'_P)^I \xrightarrow{\phi_{(G\times U'_P)^I}} (G\times U'_P)^I$$

such that the diagram

$$\begin{array}{ccc} G\times(G\times U'_P) & \xrightarrow{\phi_{G\times U'_P}} & G\times U'_P \\ \Big\downarrow{\scriptstyle (1_G,\varphi_{G\times U'_P})} & & \Big\downarrow{\scriptstyle \varphi_{G\times U'_P}} \\ (G\times(G\times U'_P))^I \cong G\times(G\times U'_P)^I & \xrightarrow{\phi_{(G\times U'_P)^I}} & (G\times U'_P)^I \end{array}$$

is commutative. The map $\Phi_{(G\times U'_P)^I}$ defines an operation of G on $(G\times U_P)^I$, and this operation is proper. The last fact follows at once from the commutative diagram

$$\begin{array}{ccc} G\times(G\times U'_P) & \xrightarrow{(\phi_{G\times U'_P},Id)} & (G\times U'_P)\times(G\times U'_P) \\ \Big\downarrow{\scriptstyle (Id,\varphi_{G\times U'_P})} & & \Big\downarrow{\scriptstyle (\varphi_{G\times U'_P},\varphi_{\times U'_P})} \\ G\times(G\times U'_P)^I & \xrightarrow{\phi_{(G\times U'_P)^I}} & (G\times U'_P)^I\times(G\times U'_P)^I \end{array}$$

where the maps $(\Phi_{G\times U'_P},\mathrm{Id})$, $(\mathrm{Id},\varphi_{G\times U'_P})$, $(\varphi_{G\times U'_P},\varphi_{G\times U'_P})$ are proper, and [7] 5.4.3.

CLAIM I: *The geometric quotient of $(G\times U'_P)^I$ by G exists in the category of schemes and is isomorphic to $\overline{U'_P}$ where $\overline{U'_P} = U'^I_P$ is the geometric quotient of U'_P by I.*

Proof of Claim I: We have the commutative diagram

$$\begin{tikzcd}
G\times U'_P \arrow[r, "\varphi_{G\times U'_P}"] \arrow[d, "pr_2"] & (G\times U'_P)^I \arrow[d, "(\lambda_{G\times U'_P})^I"] \\
U'_P \arrow[r, "\varphi_{U'_P}"] & \overline{U'_P} = U'^I_P .
\end{tikzcd}$$

where $G\times U'_P \xrightarrow{\varphi_{G\times U'_P}} (G\times U'_P)^I$ and $U'_P \xrightarrow{\varphi_{U'_P}} \overline{U'_P}$ are the quotient maps with respect to I.

$\mathrm{pr}_2 : G\times U'_P \to U'_P$ is the quotient map with respect to the action of G on $G\times U'_P$. (Obviously, U'_P is the geometric quotient of $G\times U'_P$ by G.) The map $\lambda_{(G\times U'_P)^I}$ exists and is, uniquely, determined as $\varphi_{U'_P}\circ \mathrm{pr}_2$: $G\times U'_P \to \overline{U'_P}$ is invariant with respect to I and G and $\varphi_{G\times U'_P} : G\times U'_P \to (G\times U'_P)^I$ is a G-map. One checks that U'_P together with $\lambda_{(G\times U'_P)^I} : (G\times U_P)^I \to \overline{U'_P}$ is the geometric quotient of $(G\times U'_P)^I$ by G. Q.E.D.

There is a morphism f from $G\times U'_P$ to X defined by

$$(g, u) \xrightarrow{f} g(u).$$

If $\alpha \in I$, then

$$\alpha((g, u)) = (g\alpha^{-1}, \alpha(u)) \to (g\alpha^{-1}\alpha(u)) = g(u).$$

From this, it follows that f is an I-morphism if one views X together with the trivial action of I.

As a result of the universal mapping property of the quotient $(G\times U'_P)^I$, f must factor through $(G\times U'_P)^I$. We obtain, therefore, a unique map $(G\times U'_P)^I \xrightarrow{h} X$ such that the diagram

$$\begin{tikzcd}
G\times U'_P \arrow[d] \arrow[ddr, "f"] & \\
(G\times U'_P)^I \arrow[dr, "h"'] & \\
 & X
\end{tikzcd}$$

is commutative.

On page 60 an operation of G on $(G\times U'_P)^I$ was defined through the map

$$(g', \overline{(g, u)}) \to \overline{(g'\circ g, u)}, \text{ where } \overline{(g, u)} \in (G\times U'_P)^I.$$

With this operation one finds that the map $h : (G\times U'_P)^I \to X$ is a G-morphism.

Consider now the point $\varphi_{G\times U'_P}(\varepsilon, P) = Q \in (G\times U'_P)^I$ on $(G\times U'_P)^I$.

Claim II: *There is an open G-stable neighborhood W of Q on $(G\times U'_P)^I$ (open with respect to the Zariski topology) such that the map $h : (G\times U'_P)^I \to X$ is etale for every point $Q' \in W$.*

PROOF OF CLAIM II: The map h is of finite type and locally at Q an analytic isomorphism. h is, therefore, etale at Q. By [6], Exposée I, proposition 4.5, we conclude that there exists an open subscheme W of $(G \times U'_P)^I$ which contains Q and on which h is etale. If W is taken maximal, W is G-stable and satisfies claim II. Q.E.D.

As the map $h : W \to X$ is open, [6], $V = h(W)$ is an open subscheme of X. Furthermore, V is G-invariant.

Let $U_P = U'_P \cap V$. Then, U_P is invariant under I and contains the point P. By making U_P smaller, if necessary, we may assume in addition that U_P is open on U'_P, affine and I-invariant.

The group I operates, in this situation, on the affine scheme $G \times U_P$ via

$$(g, u) \to (g\alpha^{-1}), \alpha(u)), \alpha \in I$$

and the geometric quotient $(G \times U_P)^I$ with respect to this action, exists.

The considerations above show that G, again, operates on $(G \times U_P)^I$, as defined on page 60, that $(U_P)^I$ is the geometric quotient of $(G \times U_P)^I$ by this operation of G and, also, that the G-map

$$(G \times U_P)^I \xrightarrow{h} X$$

is etale for every point $Q \in (G \times U_P)^I$.

Furthermore we obtain that for every point $Q \in U_P$, the scheme U_P is transversal to the orbit of Q with respect to G. To make this more precise, we consider the diagram of etale maps which are G-invariant

$$G \times U_P \xrightarrow{\varphi_{G \times U_P}} (G \times U_P)^I \xrightarrow{h} X.$$

As the orbit of the point $(\varepsilon, Q) \in G \times U_P$, with respect to G, is transversal to U_P and the map $h \circ \varphi_{G \times U_P}$ is etale we conclude the above transversality statement.

PROPOSITION (1.13): *Denote by $\overline{X}^{an}$ the analytic quotient of X by G, in the sense of Holmann, existing by* [10] *and* [12], *compare corollary* 1.10. *There is a natural analytic morphism $(U_P^I)^{an} \xrightarrow{h} \overline{X}^{an}$ which is a local isomorphism at every point $\overline{Q} \in (U_P^I)^{an}$ where $(U_P^I)^{an}$ is the analytic space which is associated to the scheme U_P^I.*

PROOF: follows from lemma 1.14.

LEMMA (1.14): *Let $P \in X$, O_P be the orbit of P by G, I be a finite subgroup of G which contains the stabilizer I_P of the point P with respect to G, U_P be a locally closed affine subscheme of X containing P which is transversal to O_P at P and on which I operates, U_P^I be the geometric quotient of U_P by I in the sense of schemes and, finally, $\varphi_U : U_P \to U_P^I$ the quotient map.*

If $\bar{P} = \varphi_U(P)$, *then,* $(U_P^I)^{an}$ *is locally at* $\bar{P}$ *isomorphic to* $\bar{X}^{an}$ *at the point* $\bar{P} = \lambda_X(P)$; *where* $\lambda_X : X^{an} \to \bar{X}^{an}$ *is the analytic quotient map.*

Proof of the lemma: Take the action of G on $(G \times U_P)^I$ as defined on page 60. Then G, as a Lie group, operates on $((G \times U_P)^I)^{an}$. The analytic quotient of this operation exists and is isomorphic to $(U_P^I)^{an}$.

For the proof of this fact we consider the commutative diagram of holomorphic maps

$$\begin{tikzcd} (G \times U_P)^{an} \arrow[r, "\varphi_{G \times U_P}"] \arrow[d, "pr_2"] & ((G \times U_P)^I)^{an} \arrow[d, "\lambda_{(G \times U_P)^I}"] \\ U_P^{an} \arrow[r, "\varphi_{U_P}"] & (U_P^I)^{an}. \end{tikzcd}$$

It is easily seen that $((G \times U_P)^I)^{an}$, respectively, $(U_P^I)^{an}$ are the analytic quotients of $(G \times U_P)^{an}$, respectively, U_P^{an} by I. Also, one checks that U_P^{an} is the quotient of $(G \times U_P)^{an}$ by G. Together this implies that $(U_P^I)^{an}$ is the quotient of $((G \times U_P)^I)^{an}$ by G, in the analytic sense.

Looking back at page 62 we find that there is a G-morphism

$$((G \times U_P)^I)^{an} \xrightarrow{h} X^{an}.$$

By the universal mapping properties of the quotient, we obtain a map $\bar{h} : (U_P^I)^{an} \to \bar{X}^{an}$ such that the diagram

$$\begin{tikzcd} ((G \times U_P)^I)^{an} \arrow[r, "h"] \arrow[d, "\lambda_{(G \times U_P)^I}"] & X^{an} \arrow[d, "\lambda_X"] \\ (U_P^I)^{an} \arrow[r, "\bar{h}"] & \bar{X}^{an} \end{tikzcd}$$

is commutative.

If $(e, P) \in ((G \times U_P)^I)^{an}, h((e, P)) = P$. Lef $\lambda_{(G \times U_P)^I}(e, P) = \overline{P'}, \overline{P'} \in U_P^I$ and $\lambda_X(P) = \bar{P}$. Then, $\bar{h}(\overline{P'}) = \bar{P}$ and $\bar{h}$ is, by [10], p. 421 ff., and [12], locally an isomorphism. Holmann uses in the proof of this statement the important fact that the finite group I contains the stabilizer of P with respect to the action of G. Q.E.D.

Remark (1.15): The main property of the G-stable etale map $h : (G \times U_P)^I \to X$ which is implicitely needed in the above consideration is that for any $\boldsymbol{C}$-valued point $Q \in (G \times U_P)^I$ the stabilizer of Q and $h(Q) \in X$ with respect to G are equal. Notice, this is not true for the G-map $G \times U_P \to X$.

So far our construction was local and related to the point P. We have shown that, for every point $P \in X$, there exists an affine locally closed subscheme U_P of X containing P on which the group I, the stabilizer of P, operates, such that the diagram

$$\begin{tikzcd} ((G\times U_P)^I)^{an} \arrow[r, "h"] \arrow[d, "\lambda_{(G\times U'_P)^I}"] & X^{an} \arrow[d, "\lambda_X"] \\ (U_P^I)^{an} \arrow[r, "\bar{h}"] & \bar{X}^{an} \end{tikzcd}$$

is commutative and the maps h and $\bar{h}$ are etale.

It is not difficult to check that for finitely many appropriate points $P_1, \cdots, P_n$ of X and their corresponding schemes $U_{P_1}, \cdots, U_{P_n}$, X is covered by the open sets $h_i((G\times U_{P_i})^{I_i})$, $i = 1, \cdots, n$, where I_i is the stabilizer of the point P_i and $h_i : (G\times U_{P_i})^{I_i} \to X$ is the map introduced on page 62.

Let $W = \coprod_{i=1}^n (G\times U_{P_i})^{I_i}$ be the direct sum of the schemes $(G\times U_{P_i})^{I_i}$, $W = \prod_{i=1}^n (G\times U_{P_i})^{I_i} \to X$ the surjective etale map induced by the maps h_i, and $\bar{U} = \coprod_{i=1}^n (U_{P_i}^{I_i})$ the direct sum of the scheme $U_{P_i}^{I_i}$. Then the operations of G on $(G\times U_{P_i})^{I_i}$ induce an operation of G on W. This operation is proper and $\bar{U}$ is the geometric quotient of W in the sense of schemes.

The diagram

$$\begin{tikzcd} W = (\coprod (G\times U_i)^{I_i})^{an} \arrow[r, "\amalg h_i = h"] \arrow[d] & X^{an} \arrow[d] \\ \bar{U}^{an} = (\coprod U_{P_i}^{I_i})^{an} \arrow[r, "\amalg \bar{h}_i = \bar{h}"] & \bar{X}^{an} \end{tikzcd}$$

is commutative and $\bar{h} : \bar{U}^{an} \to X^{an}$ is a surjective etale map of analytic spaces. If $R_{\bar{U}}^{an} = \bar{U}^{an} \underset{X^{an}}{\times} \bar{U}^{an}$ is the fibre product with respect to $\bar{h}$, then, by definition, $R_{\bar{U}}^{an}$ is a closed analytic subspace of $\bar{U}^{an}\times\bar{U}^{an}$ which defines an etale equivalence relation on $\bar{U}^{an}$ with $\bar{X}^{an}$ as quotient, i.e., the diagram

$$R_{\bar{U}}^{an} \underset{\bar{\pi}_2}{\overset{\bar{\pi}_1}{\rightrightarrows}} \bar{U}^{an} \xrightarrow{\bar{h}} \bar{X}^{an}$$

is a quotient diagram, and the maps $\bar{\pi}_1$, $\bar{\pi}_2$ are etale as $\bar{h}$ was etale.

CLAIM III: *$R_{\bar{U}}^{an}$ is a scheme. More precisely, the set $R_{\bar{U}}$ which belongs to $R_{\bar{U}}^{an}$ carries the structure of a scheme and the associated analytic space is $R_{\bar{U}}^{an}$. Furthermore $R_{\bar{U}}$ defines an etale equivalence relation on $\bar{U}$.*

PROOF OF CLAIM III: Assume first that the scheme X is reduced. Then also $\bar{U}$ and $\bar{X}^{an}$ are reduced.

Consider the map $\bar{U}^{an}\times\bar{U}^{an} \xrightarrow{\bar{h}\times\bar{h}} \bar{X}^{an}\times\bar{X}^{an}$ and let $\Delta_{\bar{X}}$ be the diagonal of $\bar{X}^{an}\times\bar{X}^{an}$. By definition, $R_{\bar{U}}^{an}$ is the fibre over $\Delta_{\bar{X}}$ of the morphism $\bar{h}\times\bar{h}$. We want to show that the set $R_{\bar{U}}$ is closed in $\bar{U}\times\bar{U}$ with respect to the Zariski topology. For this purpose we look at the diagram

$$(*)\qquad \begin{array}{ccc} W\times W=\coprod(G\times U_{P_i})^{I_i}\times\coprod(G\times U_{P_i})^{I_i} & \xrightarrow{h\times h} & X\times X \\ \big\downarrow & & \big\downarrow{\scriptstyle \lambda_X\times\lambda_X} \\ \bar U\times\bar U & \xrightarrow{\bar h\times\bar h} & \bar X\times\bar X, \end{array}$$

where the objects appearing are viewed as topological spaces in the Zariski topology.

We need to say what the Zariski topology on the set $\bar X\times\bar X$ should be. Suppose $X\times X$ is endowed with the Zariski topology [6]. $G\times G$ acts on $X\times X$ via the factors and $\bar X\times\bar X$ is set theoretically (with respect to the $\boldsymbol{C}$-valued points) the quotient by this action. $\bar X\times\bar X$, in the above diagram, is defined to be the topological quotient of $X\times X$ by $G\times G$ with respect to the Zariski topology of $X\times X$.

The maps which appear in the diagram (*) are continuous. Also, the diagonal $\Delta_{\bar X}$ of $\bar X\times\bar X$ is closed in $\bar X\times\bar X$, as the action of G on X is proper and therefore the graph $\Gamma\subset X\times X$ of this action is closed in $X\times X$, all with respect to the Zariski topology. This implies that $R_{\bar U}=(\bar h\times\bar h)^{-1}(\Delta_{\bar X})$ is a Zariski closed subset of $\bar U\times\bar U$. As $R_{\bar U}$ is closed in $\bar U\times\bar U$, it is, in a canonical way a reduced $\boldsymbol{C}$-scheme of finite type. This $\boldsymbol{C}$-scheme is denoted in the following by $R_{\bar U}$.

It remains to show that the analytic space associated to $R_{\bar U}$ is the analytic space $R_{\bar U}^{an}$.

But this is a consequence of Hilberts Nullstellensatz and the fact that $R_{\bar U}$ and $R_{\bar U}^{an}$ are reduced, have the same underlying sets, and are both subspaces of $(\bar U\times\bar U)^{an}=\bar U^{an}\times\bar U^{an}$.

This settles the reduced case for it is clear that $R_{\bar U}\rightrightarrows\bar U$ defines an equivalence relation on $\bar U$ which is etale.

If X is not reduced we look to the diagram

$$R_{\bar U}^{an}\underset{\bar\pi_2}{\overset{\bar\pi_1}{\rightrightarrows}}\bar U^{an}\xrightarrow{\bar h}\bar X^{an}$$

and pass to the corresponding diagram of reduced spaces

$$(**)\qquad (R_{\bar U}^{an})_{red}\rightrightarrows\bar U_{red}^{an}\longrightarrow\bar X_{red}^{an}.$$

By the above considerations we obtain that $(R_{\bar U}^{an})_{red}$ is a closed subscheme of $\bar U_{red}\times\bar U_{red}$. Denote this scheme by $(R_{\bar U})_{red}$ and consider the diagram

$$(R_{\bar U})_{red}\underset{\bar\pi_2}{\overset{\bar\pi_1}{\rightrightarrows}}\bar U_{red}{}',$$

[6] In the proof of general quotient theorem stated in 1.9 one has to take the Zariski topology of the algebraic space $X\times X$ as defined in [15], p. 132.

where the maps π_i are etale and define an etale equivalence relation on $\bar{U}_{red}$.

Using [7], IV, 18.1, we obtain a uniquely determined C-scheme $\bar{R}$, of finite type over C together with two etale maps $\bar{R} \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bar{U}$ such that the associated diagram of reduced schemes

$$\bar{R}_{red} \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bar{U}_{red}$$

is the diagram $(R_{\bar{U}})_{red} \underset{\bar{\pi}_2}{\overset{\bar{\pi}_1}{\rightrightarrows}} \bar{U}_{red}$.

We show that the analytic space $\bar{R}^{an}$ associated to $\bar{R}$ is isomorphic to the space $\bar{U}^{an} \times_{\bar{X}} \bar{U}^{an}$.

For this purpose we notice that the analytic maps $\bar{h} \cdot \alpha_i : \bar{R}^{an} \to \bar{X}^{an}$, $i = 1, 2$, are equal, as they are etale and the associated reduced maps are equal. We obtain therefore a unique map $\bar{R}^{an} \to \bar{U}^{an} \times_{\bar{X}} \bar{U}^{an}$ which is easily checked to be an isomorphism. (Use that the maps appearing in the diagrams $\bar{R}^{an} \rightrightarrows \bar{U}^{an}$ and $\bar{U}^{an} \times_{\bar{X}} \bar{U}^{an} \to \bar{X}^{an}$ are etale.)

This fact allows to prove that the diagram $\bar{R} \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bar{U}$ defines an etale equivalence relation on $\bar{U}$ as follows.

First, there exists a map $\bar{R} \to \bar{U} \times \bar{U}$. As the associated map $\bar{U}^{an} \times_{\bar{X}} \bar{U}^{an} \cong \bar{R}^{an} \to \bar{U}^{an} \times \bar{U}^{an}$ is a closed embedding, also $\bar{R} \to \bar{U} \times \bar{U}$ is a closed embedding by [6], XII, 3.2, and therefore $\bar{R}$ a closed subscheme of $\bar{U} \times \bar{U}$. That the subscheme $\bar{R}$ of $\bar{U} \times \bar{U}$ defines an equivalence relation on $\bar{U}$ is now easily checked by using again the fact that $\bar{R}^{an} \cong \bar{U}^{an} \times_{\bar{X}} \bar{U}^{an}$ and that $\bar{U}^{an} \times_{\bar{X}} \bar{U}^{an}$ defines an equivalence relation on $\bar{U}^{an}$ in the analytic sense.

This proves the claim.

Putting things together we see that $R_{\bar{U}} = \bar{U} \times_{\bar{X}} \bar{U}$ is a closed subscheme of $\bar{U} \times \bar{U}$ and that the projection maps

$$R_{\bar{U}} \underset{\bar{\pi}_2}{\overset{\bar{\pi}_1}{\rightrightarrows}} \bar{U}$$

are etale and define an etale equivalence relation on $\bar{U}$.

Hence, the diagram $R_{\bar{U}} \rightrightarrows \bar{U}$ of schemes defines a separable algebraic C-space Y of finite type for which the associated analytic space is isomorphic to $\bar{X}^{an}$.

Let W be the scheme from page 64 and $W \xrightarrow{h} X$ the etale map. $R_W = W \times_X W$ shall be the fibre product. Then, $R_W \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} W \to X$ is the G-stable etale covering of X.

We have the diagram

$$\begin{tikzcd}
R_W \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "\lambda'_W"] & W \arrow[r, "h"] \arrow[d, "\lambda_W"] & X \arrow[d, "\lambda"] \\
R_{\bar{U}} \arrow[r, shift left, "\bar{\pi}_1"] \arrow[r, shift right, "\bar{\pi}_2"'] & \bar{U} \arrow[r, "\bar{h}"] & Y,
\end{tikzcd}$$

where $\lambda_W : W \to \bar{U}$ is the quotient morphism and $\lambda_W : R_W \to R_{\bar{U}}$ the morphism induced by λ_W via $R_W \hookrightarrow W \times W \xrightarrow{\lambda_W \times \lambda_W} \bar{U} \times \bar{U}$. ($R_W$ is mapped to $R_{\bar{U}}$, because the analytic map $(\lambda_W \times \lambda_W)^{an}$ maps R_W^{an} to $R_{\bar{U}}^{an}$.) On R_W the group G acts in a natural way by the rule

$$(g, (w_1, w_2)) \to (g(w_1), g(w_2))$$

and with this action the maps π_i are G-maps.

CLAIM IV: *The geometric quotient of R_W by G exists and is isomorphic to $R_{\bar{U}}$.*

PROOF: Consider the diagram

$$\begin{tikzcd}
R_W \arrow[rr, "\pi_i"] \arrow[dr, "\rho_i"] \arrow[dd, "\lambda'_W"'] & & W \arrow[dd] \\
& R_{\bar{U}} \times_{\bar{U}} W \arrow[ur] \arrow[dl] & \\
R_{\bar{U}} \arrow[rr, "\bar{\pi}_i"] & & \bar{U} = \coprod_i U_i^{I_i}
\end{tikzcd}$$

Then the G-map $\rho_i : R_W \to R_{\bar{U}} \times_{\bar{U}} W$ is surjective, etale and radical, hence an isomorphism. Similar as in the proof of claim *I* one concludes that $R_{\bar{U}}$ is the geometric quotient of $R_{\bar{U}} \times_{\bar{U}} W$ by G. Q.E.D.

For the algebraic space Y, defined by $R_{\bar{U}} \overset{\bar{\pi}_1}{\underset{\bar{\pi}_2}{\rightrightarrows}} \bar{U} \to Y$ and the map λ, definition 1.5 in [22] must be verified.

Statement 1 is obviously true for the algebraic space Y. For the proof of statement 2 of definition 1.5 in [22] assume, $\bar{V} \to Y$ is an etale map where $\bar{V}$ is a scheme. We must show (compare [15], p. 104) that for every point $P \in \bar{V}$ there exists an etale neighborhood $\bar{g} : \bar{V}' \to \bar{V}$ of $\bar{P}$, where $\bar{V}'$ is a $\boldsymbol{C}$-scheme, and a G-stable etale morphism $V' \to X$, where V' is also a $\boldsymbol{C}$-scheme on which G operates, such that $\bar{V}'$ is the geometric quotient of V' modulo G in the sense of schemes. In that event, the structure sheaf of $\bar{V}'$ is the fixed sheaf $O_{V'}^G$ of $O_{V'}$.

The etale map $\bar{V} \to Y$ can be described by a commutative diagram

$$
\begin{tikzcd}
R_{\bar V'} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\bar f''"] & \bar V' \arrow[r] \arrow[d, "\bar f'"] & \bar V \arrow[d, "\bar f"] \\
R_{\bar U} \arrow[r, shift left] \arrow[r, shift right] & \bar U \arrow[r] & Y,
\end{tikzcd}
$$

where $\bar V'$ is an affine scheme, $\bar V' \to \bar V$ is a representable etale covering of $\bar V$ and $\bar f'$ is an etale map.

There is a commutative diagram

$$
\begin{tikzcd}
 & & & W\times_{\bar U}\bar V' = V' \arrow[dl, "f'"'] \arrow[dd, "h"] & & \\
R_W \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[dd] & & W \arrow[rr] \arrow[dd] & & X \arrow[dd] & \\
 & R_{\bar V'} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, "\bar f''"] & & \bar V' \arrow[rr] \arrow[dl, "\bar f'"] & & \bar V \arrow[dl, "\bar f"] \\
R_{\bar U} \arrow[rr, shift left] \arrow[rr, shift right] & & \bar U \arrow[rr] & & Y &
\end{tikzcd}
$$

where $W\times_{\bar U}\bar V'$ is the scheme product.
The group G acts on $W\times_{\bar U}\bar V'$ in a natural way (the operation is induced by the one on W) and f' is an etale G-map. Similar to the proof of claim I one concludes that $\bar V'$ is the geometric quotient of $W\times_{\bar U}\bar V'$ by G in the sense of schemes.

For the proof of statement 3 of [22] definition 1.5 we consider a separated $\boldsymbol{C}$-space Z and a G-invariant morphism $f: X \to Z$. Let $R_{Z^*} \rightrightarrows Z^* \to Z$ be a representable etale covering of Z where Z^* is affine. There is a commutative diagram

$$
\begin{tikzcd}
X^*\times_X X^* = R_{X^*} \arrow[r, shift left, "\pi^*_1"] \arrow[r, shift right, "\pi^*_2"'] \arrow[d, "f''"] & X^* = X\times_Z Z^* \arrow[r, "\gamma"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
R_{Z^*} \arrow[r, shift left] \arrow[r, shift right] & Z^* \arrow[r] & Z,
\end{tikzcd}
$$

where f' and f'' are maps of schemes.

The operation of G on X induces an operation of G on the schemes $\hat{X}^*$ and R_{X^*} such that the etale maps π_i^* and γ are G-maps. Furthermore, the diagram $R_{X^*} \rightrightarrows X\times_Z Z^*$ determines X as an algebraic space. For every point $P^* \in X^*$ the stabilizer of P^* with respect to G is equal to the stabilizer of the point $\gamma(P^*) \in X$.

Let U_P, $i = 1, \cdots, n$, be the locally closed subscheme of X constructed on page 64 and I_i be the stabilizer of P_i. $U_i^* = \gamma^{-1}(U_{P_i})$ shall be the inverse image of U_{P_i}. Then I_i operates on U_i^* and also on $G\times U_i^*$ as described on page 59. The arguments used in the proof of claim I-III yield:

1. There exists a surjective etale map $W = \coprod (G\times U_i^*)^{I_i} \xrightarrow{\lambda} X$, where $(G\times U_i^*)^{I_i}$ denotes the quotient of $G\times U_i^*$ by I_i. G operates in a natural way on $(G\times U_i^*)^{I_i}$ such that λ is a G-map.
2. The geometric quotient of $(G\times U_i^*)^{I_i}$ by G exists and is isomorphie to $\coprod U_i^{*I_i}$.

   We obtain the diagram

$$\begin{tikzcd}
\coprod (G\times U_i^*)^{I_i} \arrow[r] \arrow[d] & \coprod (G\times U_{P_i})^{I_i} \arrow[d] & \\
\overline{U^*} = \coprod U_i^{*I_i} \arrow[r] & \coprod U_{P_i}^{I_i} \arrow[r] & Y,
\end{tikzcd}$$

where the composite map $\bar{\gamma} : \overline{U^*} \to Y$ is etale as $\coprod U_i^{*I_i} \to \coprod U_{P_i}^{I_i}$ is etale. Let $R_{\overline{U^*}} = \overline{U^*}\times_Y \overline{U^*}$. Then $R_{\overline{U^*}}$ is a closed subscheme of $\overline{U}^*\times \overline{U}^*$ the diagram

$$R_{\overline{U^*}} \rightrightarrows \overline{U}^*$$

defines Y as an algebraic space.

To prove the existence of a unique map $Y \xrightarrow{\bar{f}} Z$ making the diagram

$$\begin{tikzcd}
X \arrow[dr, "\lambda"] \arrow[ddr, "f"'] & \\
& Y \arrow[d, "\bar{f}"] \\
& Z
\end{tikzcd}$$

commute, we notice that there exists a unique map

$$\overline{U}^* \xrightarrow{\overline{f^*}} Z^*$$

such that

$$\begin{tikzcd}
\coprod (G\times U_i^*)^{I_i} \arrow[dr] \arrow[ddr, "f^*"'] & \\
& \overline{U^*} \arrow[d, "\overline{f^*}"] \\
& Z^*
\end{tikzcd}$$

is commutative.

It follows from the considerations in claim IV that $\bar{f}^*$ induces a unique map $\overline{f^{**}} : \overline{U^*}\times_Y \overline{U^*} = R_{\overline{U^*}} \to R_{Z^*}$.

The maps $\bar{f}^*$ and $\bar{f}^{**}$ define a morphism $\bar{f} : Y \to Z$ via

$$\begin{tikzcd}
R_{\overline{U^*}} = \overline{U^*} \times_Y \overline{U^*} \arrow[d, "\bar{f}^{**}"] \arrow[r, shift left] \arrow[r, shift right] & \overline{U^*} \arrow[r] \arrow[d, "\bar{f}^*"] & Y \arrow[d, "\bar{f}"] \\
R_{Z^*} \arrow[r, shift left] \arrow[r, shift right] & Z^* \arrow[r] & Z
\end{tikzcd}$$

satisfying the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\lambda"] \arrow[dd, "f"'] & \\
& Y \arrow[dl, "\bar{f}"] \\
Z &
\end{tikzcd}$$

Obviously $\bar{f}$ is uniquely determined.

This completes the proof of theorem 1.4.

COROLLARY (1.16): *If in the situation of theorem* 1.4 *the scheme X is normal then also the quotient Y is normal.*

The proof follows immediately from the fact that the Y is normal as *an analytic space*, [10], *Satz* 12. *The algebraic space Y has then also to be normal.*

We explain now how one can obtain an algebraic space $\overline{M}_g$ of finite type and proper over $\boldsymbol{C}$ which is over the complex numbers a coarse moduli space for stable curves of genus g.

The notion of a family of stable curves of genus g over a scheme S is as defined in [5], 1.1.

For a noetherian $\boldsymbol{C}$-scheme S we denote by $\mathscr{M}_g(S)$ the set of classes of families of stable curves of genus g over S up to isomorphism. The collection of sets $\mathscr{M}_g(S)$ form a contravariant functor from the category of noetherian $\boldsymbol{C}$-schemes to the category of sets. $\mathscr{M}_g$ can be extended in a natural way to a functor from the category of noetherian algebraic $\boldsymbol{C}$-spaces to the category of sets which is a sheaf (i.e., sheafify with respect to the etale topology). This functor is called the functor of stable curves of genus g and is denoted also by $\mathscr{M}_g$.

We want to show that $\mathscr{M}_g$ has in the category of algebraic $\boldsymbol{C}$-spaces a coarse moduli space. (Compare [20] or [22] for the definition.) For this purpose we notice that by [5], p. 78, for a family $C \to S$ the canonical sheaf $\omega_{C/S}$ of $C \to S$ is relatively ample and $\omega_{C/S}^{\otimes 3}$ is very ample. Using the sheaf $\omega_{C/S}^{\otimes 3}$ we obtain locally a morphism Φ_3 from $C \to S$ into the projective space $P^{5g-6} \times S$ which is an isomorphism on the fibres of C/S and has the property that the fibres of the image have all the same Hilbert polynomial $h(x) = (6x-1)(g-1)$. Let $\mathrm{Hilb}^{h(x)}_{P^{5g-6}}$ be the Hilbert scheme over $\boldsymbol{C}$ which parametrizes all closed subschemes of P^{5g-6} with $h(x)$ as Hilbert polynomial. One shows then by arguments to be used in the proof of proposition 2.2 that there exists a locally closed subscheme $H_g \subset \mathrm{Hilb}^{h(x)}_{P}$ which parametrizes the 3-canonical stable curves of genus g. (The image of a curve C/S under the map ϕ_3 is called a 3-canonical stable curve.) Let $\Gamma \to \mathrm{Hilb}^{h(x)}_{P^{5g-6}}$ be the universal family over $\mathrm{Hilb}^{h(x)}_{P^{5g-6}}$ and $\Gamma_g \to H_g$ be the pullback of Γ to H_g. On H_g the group PGL $(5g-6)$ operates in a natural way. It follows from [5], lemma 1.12, that this operation is proper and has finite stabilizers as for every $\boldsymbol{C}$-valued point $P \in H_g$ the stabilizer is isomorphic to the automorphism group of the fibre over P in the family $\Gamma_g \to H_g$ and this group is finite.

The geometric quotient $\overline{M}_g$ of H_g by PGL $(5g-6)$ exists therefore by theorem 1.4 and is according to the considerations in [22] the coarse moduli space for stable curves.

Clearly, $\overline{M}_g$ contains the coarse moduli space for smooth curves of genus g as an open subspace.

By the stable reduction theorem for curves ([5], p. 87 ff) and the valuation criterion for proper algebraic spaces (compare page 53) we obtain that $\overline{M}_g$ is proper over $\boldsymbol{C}$.

2. Moduli spaces for surfaces of general type

Let V be a minimal surface of general type defined over $\boldsymbol{C}$, i.e., V is a smooth projective surface of Kodaira [7] dimension 2 which is minimal. K_V denotes the canonical line bundle and $p_a(V)$ the arithmetic genus of V.

K_V^2 is the self-intersection number of the canonical bundle K_V. By [3], K_V^2 is an integer $\geqq 1$.

We will show, in this chapter, that over the complex numbers for minimal surfaces V of general type with $K_V^2 = K^2, p_a(V) = p_a$, where K^2 and p_a are fixed integers, a coarse moduli space, in the sense of definition 2.6 exists.

See theorem 2.7 for a more precise formulation.

We must recall certain facts on algebraic surfaces of general type.

Let V be a minimal surface of general type. The m-genus $p_V(m) = \dim_{\boldsymbol{C}} H^0(V, K_V^{\otimes m})$ of V satisfies the formula

$$p_V(m) = \tfrac{1}{2} \cdot m(m-1) \cdot K_V^2 + p_a(V)$$

for $m \geqq 2$.

[7] Compare [22], p. 2, for this notion.

In [3], it is demonstrated that, for any surface V of general type, the map

$$\Phi_{mK} : V \to P^N, \quad m \geqq 5$$

is a birational morphism.

Also, one finds in [3] that the image variety $\Phi_{mK}(V) = W_m$ is normal and has only isolated singularities which are rational double points, provided $m \geqq 5$. By an unpublished result of Kodaira W_m is even arithmetically normal, provided $m \geqq 8$.

These facts imply that every minimal surface V over $\boldsymbol{C}$ of general type with fixed $K_V^2 = K^2$ and $p_a(V) = p_a$ can be mapped into the same projective space $P^N/\boldsymbol{C}$ by the map Φ_{5K}, $N = 10K^2+p_a-1$, in such a way that the image surface $\Phi_{5K}(V) = W$ has Hilbert polynomial

$$h(x) = \tfrac{5}{2}x\,(5x-1)+p_a.$$

We call such a surface $W = \Phi_{5K}(V)$ a *5-canonical surface* in P^N.

In the following we consider minimal surfaces V of general type defined over $\boldsymbol{C}$ such that $K_V^2 = K^2, p_a(V) = p_a$.

PROPOSITION (2.1): *Let V and V' be minimal surfaces of general type defined over $\boldsymbol{C}$ such that $K_V^2 = K_{V'}^2$, and $p_a(V) = p_a(V')$. Then, V and V' are isomorphic if and only if the surfaces $\Phi_{5K}(V)$ and $\Phi_{5K}(V')$ are projectively equivalent, i.e., there exists a projective transformation $\sigma \in PGL(N)$, $N = 10 \cdot K_V^2+p_a(V)-1$, such that*

$$\sigma : \Phi_{5K}(V) \to \Phi_{5K}(V').$$

PROOF: It is clear that an isomorphism $\sigma : V \to V'$ induces a projective transformation $\sigma \in PGL(N)$ which maps $\Phi_{5K}(V)$ isomorphically onto $\Phi_{5K}(V')$. The converse is also obvious. As a matter of fact if the varieties $\Phi_{5K}(V)$ and $\Phi_{5K}(V')$ are projectively equivalent, the varieties V and V' are birationally equivalent. But V and V' are minimal models and, therefore, isomorphic.

Proposition 2.1 leads us to consider the Hilbert scheme $H_{P^N}^{h(x)}$ which parametrizes the surfaces in P^N with $h(x)$ as Hilbert polynomial. $N = 10 \cdot K^2+p_a-1$, $h(x) = \frac{5}{2}x(5x-1)K^2+p_a$, with K^2 and p_a fixed. Let $\Gamma \to H_{P^N}^{h(x)}$ be the universal family in $P^N \times H_{P^N}^{h(x)}$ of surfaces belonging to $H_{P^N}^{h(x)}$. If P is a $\boldsymbol{C}$-valued point of $H_{P^N}^{h(x)}$, we denote by Γ_P the fibre of $\Gamma \to H_{P^N}^{h(x)}$ over the point P.

PROPOSITION (2.2): *There exists a locally closed subscheme H of $H_{P^N}^{h(x)}$ such that a $\boldsymbol{C}$-valued point $P \in H_{P^N}^{h(x)}$ belongs to H if and only if the sur face Γ_P is a 5-canonical surface in P^N.*

The proof of proposition 2.2 and the construction of H will be obtained in various steps.

We determine first an open subscheme H' of $H_{PN}^{h(x)}$ such that a point $P \in H_{PN}^{h(x)}$ is in H' if and only if the fibre Γ_P is an irreducible, normal surface with only rational double points as singularities. By arguments similar to those used in the proof of [22], proposition 2.12, we determine a maximal open subscheme U_1 of $H_{PN}^{h(x)}$ such that the induced family $\Gamma_1 = \Gamma \times_{H_{PN}(h_x)} U_1 \to U_1$ is smooth with connected fibres. U_1 will be an open subscheme of the scheme H' we are constructing first.

Suppose S is the complement of U_1 with respect to $H_{PN}^{h(x)}$, i.e., $S = H_{PN}^{h(x)} - U_1$.

Let $S_1, \cdots, S_{r*}$ be the irreducible reduced components of S and $\Gamma s_i = \Gamma \times_{H_{PN}\, h(x)} S_i \to S_i$ the pullbacks of the family $\Gamma \to H_{PN}^{h(x)}$ to the schemes S_i. Furthermore let $\Gamma_{S_i} \to S_1, \cdots, \Gamma_{s_r} \to S_r$, $r \leqq r^*$, be those families among the families $\Gamma_{S_i} \to S_i$, $i = 1, \cdots, r^*$, which contain an irreducible surface as fibre with only normal rational double points as singularities.

CLAIM: *The generic fibre of any of the families $\Gamma_{\ i} \to S_i$, $i = 1, \cdots, r$, is a surface with only normal rational double points as singularities. To prove this suppose $P \in S_i$ is a* **C**-*valued point of S_i such that the corresponding geometric fibre Γ_P is a normal surface with rational double points as singularities.*

Let R be a complete discrete valuation ring of rank 1 and β : Spec (R) $\to S_i$ be a morphism which maps the closed point of Spec (R) to $P \in S_i$, and the general point of Spec (R) to the general point of S_i.

Consider

$$\Gamma_R = \Gamma_{S_i} \times_{S_i} \text{Spec}\,(R) \to \text{Spec}\,(R).$$

The general fibre of $\Gamma_R \to$ Spec (R) which is irreducible has at most isolated singularities as the opposite would imply that the closed fibre of $\Gamma_R \to$ Spec (R) is singular along a curve, contrary to the assumption.

We must prove that the general fibre of Γ_R has only rational double points as singularities. We may assume that the singular points of the general fibre of Γ_R are all K-rational, where K is the quotient field of R. Otherwise this situation can be arrived by taking a finite field extension K' of K and the integral closure R' of R in K. If Q is a singular point of the general fibre of Γ_R and $\bar{Q}$ is the specialization of Q over R, then, $\bar{Q}$ is a singular point of the closed fibre of Γ_R and, therefore, a rational double point of the closed fibre. In particular, the multiplicity and the embedding dimension of the local ring of the closed fibre of Γ_R at $\bar{Q}$ are 2, respectively, 3. Also, $\bar{Q}$ is an isolated double point of the closed fibre of Γ_R in the sense of Kirby [14] and [4].

One checks now that the embedding dimension of the general fibre at Q is 3, and, by the arguments of Kirby [14], p. 601 ff, and the general Weierstraß preparation theorem [1], p. 72, that Q is an isolated double point of the general fibre of Γ_R. The multiplicity of Q on the general fibre of Γ_R has to be $\leqq 2$, because its specialization $\bar{Q}$ has multiplicity $= 2$ on the closed fibre. This yields, together with [4], Satz 1, the assertion that Q is a rational double point of the general fibre of Γ_R. The claim is justified.

Let $\tilde{\Gamma}_i$ be the general fibre of the family $\Gamma_{S_i} \to S_i$, $i = 1, \cdots, r$, and $Q_1^{(i)}, \cdots, Q_n^{(i)}$ be the K-rational singular points of $\tilde{\Gamma}_i$. We view $\tilde{\Gamma}_i$ as a subscheme of Γ_{S_i} and the $Q_j^{(i)}$ as points of the scheme Γ_{S_i}. Suppose $Z_1^{(i)} = \langle Q_1^{(i)} \rangle, \cdots, Z_n^{(i)} = \langle Q_n^{(i)} \rangle$ are the closed subschemes of Γ_{S_i} having $Q_1^{(i)}, \cdots, Q_n^{(i)}$, respectively, as generic points.

Blowing up the closed subvarieties $Z_1^{(i)}, \cdots, Z_n^{(i)}$ of Γ_{S_i}, one creates a family

$$\Gamma_{S_i}^{(1)} \to S_i$$

of surfaces. By [17], theorem 4.1, the general fibre of $\Gamma_{S_i}^{(1)} \to S_i$ has, again, only rational double points as singularities. The procedure described above is applied again to the family $\Gamma_{S_i}^{(1)} \to S_i$ and a family $\Gamma_{S_i}^{(2)} \to S_i$ results.

[17], theorem 4.1, implies that this process leads, after finitely many steps, to a family $\Gamma_{S_i}^{(m)} \to S_i$ for which the general fibre is smooth.

Consider S_i', the maximal open subscheme of S_i where the family $\Gamma_{S_i}^{(m)} \to S_i$ is smooth and, moreover, the family $\Gamma_{S_i} \times_{S_i} S_i' \to S_i'$. Then, every fibre of the family $\Gamma_{S_i} \times_{S_i} S_i' \to S_i'$ is a surface of general type with only rational double points as singularities. In this way we obtain locally closed subschemes $S_1', \cdots, S_r'$. The open subscheme of $H_{P^N}^{h(x)}$ determined by the Zariski open subset $U_1 \cup S_1' \cup \cdots \cup S_r'$, this scheme is denoted also by $U_1 \cup S_1' \cup \cdots \cup S_r'$, will be an open subscheme of H'.

We continue the construction of the scheme H' by considering the subscheme $T = H_{P^N}^{h(x)} - (U_1 \cup S_1' \cup \cdots \cup S_r' \cup S_{r+1} \cdots \cup S_{r^*})$.

$T_1, \cdots, T_{s^*}$ shall be the irreducible reduced components of T and $\Gamma_{T_i} \to T_i$ the pullback families. Applying the method used for the families $\Gamma_{S_i} \to S_i$, $i = 1, \cdots, r^*$, to the families $\Gamma_{T_j} \to T_j$, $j = 1, \cdots, s^*$, we find open subschemes T_j' of T_j, $j = 1, \cdots, s$, such that $U_1 \cup S_1' \cup \cdots \cup S_r' \cup T_1' \cup \cdots \cup T_s'$ is an open subscheme of H'. The process comes to an end after finitely many steps and leads to the scheme H'.

For the construction of the scheme H we proceed like in [22], proposition 2.21 or [20], proposition 5.1. By these arguments we find a locally closed subscheme H of H' such that a $\boldsymbol{C}$-valued point $P \in H'$ is in H if and only if the fibre Γ_P is a 5-canonical surface in P^N. Q.E.D.

The group $PGL(N)$ operates on the scheme $H_{PN}^{h(x)}$ in a natural way and H is stable by this operation, as the construction of H implies, hence $PGL(N)$ induces an action on H.

PROPOSITION (2.3): *The induced action of $PGL(N)$ on H is proper. Moreover, for any geometric point $P \in H$ the stabilizer with respect to this action is finite.*

For the proof of proposition 2.3 the following lemma is needed.

LEMMA (2.4): *Let R be a complete discrete valuation ring over $\mathbf{C}$ with quotient field K and residue field k. $\Gamma_i \to Spec\,(R)$, $i = 1, 2$, shall be two families of 5-canonical embedded surfaces of general type over R.*[8] *Assume that the general fibres of Γ_1 and Γ_2 are isomorphic. Then, there exists a finite ramified extension $Spec\,(R') \xrightarrow{\varphi} Spec\,(R)$ such that the families $\Gamma_1 \times Spec\,(R')$ and $\Gamma_2 \times Spec\,(R')$ are isomorphic.*

PROOF: By [19], p. 672, corollary 1, the lemma follows if the families $\Gamma_i \to \text{Spec}\,(R)$ are smooth. In the general case we show that there exists a finite ramified covering $\text{Spec}\,(R') \to \text{Spec}\,(R)$, R' again a complete discrete valuation ring, and a commutative diagram of algebraic spaces

$$(*) \qquad \begin{array}{ccc} \Gamma_i' & \xrightarrow{\Psi} & \Gamma_i \\ \downarrow{\scriptstyle f'_i} & & \downarrow{\scriptstyle f_i} \\ \text{Spec}\,(R') & \xrightarrow{\varphi} & \text{Spec}\,(R) \end{array}$$

with the properties:

1. f_i' is proper and smooth,
2. Ψ is proper and for the points $t \in \text{Spec}\,(R')$ the induced morphisms $(\Gamma_i')_t \to (\Gamma_i)_{\varphi(t)}$ are a minimal resolution of singularities.

We call diagram (*) a resolution of the morphism $f_i \cdot \Gamma_i \to \text{Spec}\,(R)$. If this is proved, the R'-spaces Γ_i', $i = 1, 2$, will be isomorphic over R' by arguments as in [19]. This implies that $\Gamma_i \times \text{Spec}\,(R')$, $i = 1, 2$, are projectively isomorphic as $\Gamma_i \times \text{Spec}\,(R')$ is a 5-canonical embedding of $\Gamma_i' \to \text{Spec}\,(R')$.

Consider $\Gamma_i \to \text{Spec}\,(R)$. There exists a unique map $\text{Spec}\,(R) \xrightarrow{\lambda_i} H$ such that

$$\Gamma_H \times_H \text{Spec}\,(R) = \Gamma_i \text{ holds.}$$

Let C_i be the closure of $\lambda_i(\text{Spec}\,(R))$ in H. Then, C_i is an irreducible curve over $\mathbf{C}$.

In the families $\Omega_i = \Gamma_H \times_H C_i \to C_i$ resolve the singularities of the general fibre of $\Omega_i \to C_i$ by the process described one page 74. Let $\tilde{\Omega}_i \to C_i$

[8] $\Gamma_i \to \text{Spec}\,(R)$, $i = 1, 2$, are flat, proper families over Spec (R) with 5-canonical surfaces as geometric fibres.

be such a resolution which is minimal. $\tilde{\Omega}_i \xrightarrow{h_i} C_i$ has only finitely many singular fibres and these fibres have only rational double points as singularities.

Suppose now that P_i is the image of the closed point of Spec (R) by the map λ_i and consider the fibre $(\tilde{\Omega}_i)_{P_i}$ of $\tilde{\Omega}_i \xrightarrow{h_i} C_i$ over P_i. If these fibres are smooth, we conclude that the pullbacks of $\tilde{\Omega}_i \times \operatorname{Spec}(R) = \Gamma_i \to \operatorname{Spec}(R)$, $i = 1, 2$, are smooth and obtain the lemma.

If the surfaces $(\tilde{\Omega}_i)_{P_i}$ have singularities they will be rational double points. By Brieskorn, see [9], the resolution of the versal analytic deformation of a rational double point exists. Using this result we conclude that there exists an open neighborhood U_i of P on C_i and a ramified covering $U_i' \xrightarrow{\rho'_i} U_i$ together with a smooth family $\tilde{\Omega}_i' \to U_i'$ such that the diagram

$$\begin{tikzcd} \tilde{\Omega}_i' \arrow[r] \arrow[d, "h'_i"'] & \tilde{\Omega}_i \times_{C_i} U_i \arrow[d, "h_i"] \\ U_i' \arrow[r, "\rho_i"'] & U_i \end{tikzcd}$$

resolves the singularities of the morphism $h_i : \tilde{\Omega}_i \times_{C_i} U_i \to U_i$. This fact follows first locally in a neighborhood of a singular point of $(\tilde{\Omega}_i)_{P_i}$ by the universal property of the versal deformation space and its resolution and then also globally, because resolving the morphism $\tilde{\Omega}_i \times_{C_i} U_i \to U_i$ is a local problem. Look at [4], p. 90. One checks that the resolution $\tilde{\Omega}_i' \to U$ of $\tilde{\Omega}_i \times U_i \to U_i$ is an algebraic space. [9]

Let $P_i' \in U_i'$ be a point lying over $P_i \in U_i$ and let $\hat{O}_{P'_i}$ be the completion of the local ring of P_i'. One can then find a complete local ring $R' \supset R$ such that a commutative diagram of the following form exists

$$\begin{tikzcd} \operatorname{Spec}(R') \arrow[r, "\lambda'_i"] \arrow[d] & \hat{U}_i' \arrow[d, "\rho_i"] \\ \operatorname{Spec}(R) \arrow[r, "\lambda_i"] & \hat{U}_i \end{tikzcd}$$

where $\hat{U}_i$ and $\hat{U}_i'$ stand for the germs of analytic spaces determined by U_i (respectively, U_i') in the points P_i (respectively, p_i'). λ_i' maps the closed point of Spec (R') to the point P_i' of U_i'. Taking the pullbacks $\tilde{\Omega}_i' \times_{\hat{U}_i} \operatorname{Spec}(R')$, we obtain two smooth families of surfaces over Spec (R') which satisfy diagram (*). The lemma is proved.

Now to the proof of proposition 2.3.

[9] This fact follows also from the forthcoming paper of M. Artin and M. Schlessinger, Algebraic construction of Brieskorn's resolutions.

It has to be proved that the map $PGL(N)\times H \xrightarrow{\Psi} H\times H$, defined by $(\Psi(g, h)) = (g(h), h)$, is proper. Using the valuation criterion for proper morphism the following has to be shown.

Let R be a complete discrete valuation ring with a residue field containing $\boldsymbol{C}$ and quotient field K. Let $h : \mathrm{Spec}\,(R) \to H\times H$ be a morphism and $g : \mathrm{Spec}\,(R) \to PGL(N)\times H$ be a rational map such that $\Psi \circ g = h$ (as rational maps). Then there exists a morphism $g_R : \mathrm{Spec}\,(R) \to PGL(N)\times H$ such that the diagram

(**)
$$\begin{tikzcd} & PGL(N)\times H \arrow[dd, "\Psi"] \\ \mathrm{Spec}\,(R) \arrow[ur, "g_R"] \arrow[dr, "h"'] & \\ & H\times H \end{tikzcd}$$

is commutative and g_R is equal to g as a rational map.

The map h determines two 5-canonical surfaces over R in P^N/R which shall be denoted by Γ_1 and Γ_2. The morphism $g : \mathrm{Spec}\,(K) \to PGL(N)$ $\times H$ induces a morphism $\mathrm{Spec}\,(K) \to PGL(N)$, i.e., a projective linear transformation of P^N/K which maps the general fibre of Γ_1 isomorphically onto the general fibre of Γ_2.

Also the rational map $g : \mathrm{Spec}\,(R) \to PGL(N)$ induces a unique morphism $g_R : \mathrm{Spec}\,(R) \to \overline{PGL(N)}$, the projective closure of $PGL(N)$. By lemma 2.4 this morphism is a map from $\mathrm{Spec}\,(R)$ into $PGL(N)$. With g_R as map the diagram (**) is commutative.

This proves the first statement of proposition 2.3. For the second statement we notice that the stabilizer group of a geometric point P of H (stabilizer with respect to the action of $PGL(N)$) is isomorphic to the automorphism group of the geometric fibre Γ_P of the family $\Gamma \to H^{h(x)}_{P^N}$ over the point P. But this group is by [18] finite. Q.E.D.

One expects that the geometric quotient of H by $PGL(N)$ is a coarse moduli space for the surfaces V of general type defined over $\boldsymbol{C}$ having $K_V = K^2$, and $P_a(V) = p_a$.

Introduce the following notions.

Definition (2.5): A family $V \to S$ is called a family of normal minimal surfaces of general type with K^2 and p_a as invariants if

1. The family $V \to S$ is flat and proper.
2. The geometric fibres V_P of $V \to S$ are irreducible, normal surfaces without exceptional curves of the first kind and with only rational double points as singularities.
3. For any geometric fibre V_P of $V \to S$, $K(V_P)^2 = K^2$ and $p_a(V_P) = p_a$.

Furthermore, V_P is a surface of general type, i.e., a minimal desingularisation of V_P has Kodaira dimension 2.

Let K^2 and p_a be fixed integers and let $\mathscr{M} = \mathscr{M}^{K^2,p_a}$ be the functor from the category of noetherian $\boldsymbol{C}$-schemes to the category of sets where, for a $\boldsymbol{C}$-Scheme S, $\mathscr{M}(S)$ is the set of isomorphy classes of families of minimal surfaces of general type with K^2 and p_a as invariants. We normal extend $\mathscr{M}$ to a functor from the category of noetherian algebraic $\boldsymbol{C}$-spaces to the category of sets and sheafify with respect to the etale topology. This new functor is also denoted by $\mathscr{M}$ and is considered in the following.

DEFINITION (2.6): An algebraic $\boldsymbol{C}$-space of finite type M and a morphism Φ from $\mathscr{M}$ to the functor $h_M(S) = \text{Hom}(S, M)$, S a noetherian $\boldsymbol{C}$-scheme, is called a coarse moduli space for the functor $\mathscr{M}$ if

1. for every algebraically closed field k which contains $\boldsymbol{C}$, the map $\Phi(\text{Spec}(k)) : \mathscr{M}(\text{Spec}(k)) \to h_M(\text{Spec}(k))$ is an isomorphism.
2. Given an algebraic $\boldsymbol{C}$-space of finite type N and a morphism Ψ from $\mathscr{M}$ to the functor h_N, there exists a unique morphism $\lambda : h_M \to h_N$ such that $\Psi = \lambda \cdot \Phi$.

Return to the scheme H.

Applying theorem 1.4 we conclude that there exists an algebraic $\boldsymbol{C}$-space of finite type which is a geometric quotient of H by $PGL(N)$ in the category of algebraic spaces.

Denote this quotient by M_{K^2,p_a}.

THEOREM (2.7): *The $\boldsymbol{C}$-space M_{K^2,p_a} is a coarse moduli space for the surface of general type with K^2 and p_a as invariants.*

The proof of theorem 2.7 requires certain facts on the canonical invertible sheaf of a flat family $V \to S$ of normal minimal surfaces of general type.

If $V \to S$ is such a family with K^2 and p_a as invariants, the geometric fibres of $V \to S$ are locally complete intersections. f is flat, and, therefore, the morphism f is locally a complete intersection, i.e., locally V is isomorphic to an S-scheme $V(f_1, \cdots, f_{n-2}) \in A^n \times U$, where $U \subset S$ is open and $f_1, \cdots, f_{n-2} \in \Gamma(A^n \times U)$ is a regular sequence. According to [8], chapter III, there exists a canonical invertible sheaf $\omega_{V/S}$ on V with the next properties:

1. For every $T \xrightarrow{\beta} S$, $\omega_{V \times_T T/T}$ is canonically isomorphic to $\beta^*(\omega_{V/S})$. In particular, the pullback of $\omega_{V/S}$ to a geometric fibre of the family $V \xrightarrow{f} S$ is the canonical sheaf of the fibre.
2. $\omega_{V/S}^{\otimes 5}$ defines a birational morphism $\Phi_{V/S}$ from V/S into a projective space P^N/S which induces on the fibres the 5-canonical mappings. This is deduced from the vanishing theorems in [3] and from [20], p. 19.

We conclude:

LEMMA (2.8): *With the above notation the invertible sheaf $\omega_{V/S}^{\otimes 5}$ defines a morphism $\Phi_{V/S}$ from V/S into P^N/S with the following properties: For every point $P \in S$ there exists an open subset U of S containing P such that*

1. *the map $\Phi_{V/S}$ induces a closed imersion of the family $V_U = V \times_S U \to U$ into $P^N \times U$.,*
2. *the image family $\Phi_{V/S}(V_U/U)$ is a family of 5-canonical surfaces in $P^N \times U$, $N = 10K^2 + p_a - 1$.*

Using Lemma 2.8 the proof of theorem 2.7 is on the same lines as the proof of proposition 2.16 in [22].

We look at applications of theorem 2.7.

It is explained in [22] (see the introduction of this paper) that it is important for the classification theory of compact, complex manifolds to study the structure of the Albanese mappings of such manifolds.

Theorem 2.7 and results of Kawai and Iitaka can be employed to determine the structure of the Albanese mapping of a compact, complex Kähler manifold of dimension 4 which has only constant meromorphic functions.

Suppose M is such a manifold and let $\alpha : M \to T = \mathrm{Alb}\,(M)$ be the Albanese mapping, where T is the Albanese torus. The arguments in [13], p. 611, prove that $\dim T =$ irregularity of $M = q(M) \leqq 4$. Iitaka has remarked in [11] that, by the arguments given in [13] the following statements hold.

1. T has no effective divisors.
2. α is proper and surjective.
3. Any general fibre $\alpha^{-1}(t)$, $t \in T$, is connected.
4. The case $q(M) = 1 = \dim T$ is not possible.
5. If $q(M) = 3$, the general fibre $\alpha^{-1}(t)$ is an elliptic or rational curve.

Hence, we have some understanding of the structure of the Albanese mapping of M if $q(M) = 4$ or 3.

But what is the situation if $q(M) = 2$? Here, the general fibre is a smooth connected surface. The question relates to the type of this surface, type in connection with the classification table of Kodaira, [16].

One expects that the general fibres of the map $\alpha : M \to T$ are surfaces of Kodaira dimension $\leqq 0$, i.e., are not surfaces of general type or elliptic surfaces of general type.

Now, Iitaka has shown in [11], theorem V, that a general fibre of $\alpha : M \to T$ cannot be an elliptic surface of general type.

It remains to exclude the possibility that a surface of general type appears in $\alpha : M \to T$ as a general fibre.

But this can be done with the same arguments that Kawai applied in [13], theorem 2. There, Kawai shows that the general fibre of the Albanese map

$$\alpha : M^{(3)} \rightarrow T^{(2)}$$

of a 3-dimensional compact complex manifold with irregularity 2 is not a curve of genus $\geqq 2$.

In his discussion two facts about curves of genus $g \geqq 2$ are important.

1. The curves of genus $g \geqq 2$ have finite automorphism groups.
2. For curves of genus g, $g \geqq 2$, the moduli space is an algebraic space, i.e., a complex space with many meromorphic functions.

As a consequence of theorem 2.7 and its proof, these two statements are also true for surfaces of general type and Kawai's arguments can be applied. (Compare the forthcoming Springer Lecture Notes of K. Ueno on classification theory).

REFERENCES

[1] S. Abhyankar: *Local analytic geometry*. Acad. Press, New York/London, (1964).

[2] M. Artin: *Théorèmes de représentabilité pour les espaces algébri9ues*. Le Presse de Université de Montreal.

[3] E. Bombieri: Canonical models of surfaces of general type. *Public. Math., Vol. 42* (1973) 171–220.

[4] E. Brieskorn: Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. *Math. Ann., 166* (1966) 76–102.

[5] P. Deligne and D. Mumford: The irreducibility of the space of curves of given genus. *Public. Math., Vol. 36* (1969) 75–109.

[6] A. Grothendieck: Séminaire de géometrie algébriques 1960/61. *Lecture notes in Math. 224*. Springer Verlag 1971.

[7] A. Grothendieck et J. Dieudonne: EGA- Eléments de géometrie algébriques. *Public. Math., Vol. 4* (1960).

[8] R. Hartshorne: Residues and duality. *Lecture notes in Math., 20*. Springer Verlag 1966.

[9] F. Huikeshoven: On the versal resolution of deformations of rational double points. *Inventiones Math. 20* (1973) 15–33.

[10] H. Holmann: Quotienten komplexer Räume. *Math. Ann. 142* (1961) 407–440.

[11] S. Iitaka: Deformations of compact complex surfaces III. *J. Math. Soc. Japan 22* (1970) 247–261.

[12] B. Kaup: Äquivalenzrelationen auf allgemeinen komplexen Räumen. *Schriftenreihe Math. Institut der Univ. Münster, Heft 39* (1968).

[13] S. Kawai: On compact complex analytic manifolds of complex dimension 3 : I, II. *J. Math. Soc. Japan 17* (1965) 438–442, *ibid. 21* (1969) 604–626.

[14] D. Kirby: The structure of an isolated multiple point of a surface I, II. *Proc. London Math. Soc. VI* (1956) 597–609.

[15] D. Knutson: Algebraic spaces. *Lecture notes in Math. 203*. Springer Verlag 1971.

[16] K. *Kodaira*: On the structure of complex analytic surfaces IV. *A. J. of Math.*, 90 (1968) 1048–1066.

[17] J. Lipman: Rational singularities with applications to algebraic surfaces and unique factorization. *Public. Math., Vol. 36* (1969) 195–280.

[18] H. Matsumura: On algebraic groups of birational transformations. Lincei-Rend. *Sc. mat. e nat. XXXIV* (1963) 151–154.

[19] T. Matsusaka and D. Mumford: Two fundamental theorems on deformations of polarized varieties. *Am. J. Math. 86* (1964) 668–684.

[20] D. MUMFORD: Geometric invariant theory. *Ergebnisse der Math., 34.* Springer Verlag 1965.
[21] D. MUMFORD: The structure of the moduli spaces of curves and abelian varieties *Actes, Congrès intern. math., Tome 1* (1970) 457–465.
[22] H. POPP: On moduli of algébraic varieties I. *Inventiones Math., 22* (1973) 1–40.
[23] J. P. SERRE: *Groupes algebriques et corps de classes.* Hermann, Paris 1959.
[24] C. S. SESHADRI: Quotient spaces modulo reductive algebraic groups. *Annals of Math., 95* (1972) 511–556.

(Oblatum: 16–IV–1973 & 20–IX–1973)

Institut für Mathematik
und Informatik
Universität Mannheim
68 Mannheim
Schloss