COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 27, nº 2 (1973), p. 217-221
<http://www.numdam.org/item?id=CM_1973__27_2_217_0>

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COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 2, 1973, pag. 217–221
Noordhoff International Publishing
Printed in the Netherlands

ON THE TRIVIALIZATION OF LINE BUNDLES OVER SCHEMES

Knud Lønsted

1. Introduction

Consider a line bundle L over a scheme S, and assume that $L \oplus \cdots \oplus L$ is trivial. We show the existence of a finite flat covering $\rho : T \to S$ such that $\rho^*(L)$ is trivial over T, and give a condition which ensures ρ to be étale. This condition is verified e.g. when S is of finite type over a field of characteristic zero. In particular, if S is connected and simply connected in the algebraic sense, then L is trivial.

The corresponding theorem for complex *topological* line bundles over, say, a 1-connected CW-complex S is essentially trivial, as pointed out to us by J. Dupont and V. Lundsgaard Hansen: Note that the first Chern class yields a bijection from the isomorphism classes of line bundles to $H^2(S, \mathbb{Z})$ (see [4, Chap. 1]). By the universal coefficient theorem one gets $H^2(S, \mathbb{Z})$ torsionfree as $H_1(S, \mathbb{Z}) = 0$. Since in this case we have

$$c_1(L \oplus \cdots \oplus L) = c_1(L^{\otimes n}) = n \cdot c_1(L)$$

then $n \cdot c_1(L) = 0$ implies $c_1(L) = 0$, whence L is trivial.

With a minor change of vocabulary in our proof of the algebraic case below, it becomes a proof of the complex analytic version.

In the remarks at the end of the paper we have inserted a translation into commutative algebra, and there are a few comments on 'universality' of the covering ρ in the cases where S is either the spectrum of a Dedekind ring or a smooth projective curve.

This paper grew out of an unsuccessful attempt to solve Serre's problem on algebraic vector bundles over affine spaces, and we are indebted to Serre for giving a counterexample to the naive generalization of the theorem below to vector bundles of arbitrary rank.

The notations will follow [2]. Note specifically that $\boldsymbol{\mu}$ is used to denote the multiplicative group scheme, instead of the usual $\boldsymbol{G}_m$. All schemes and morphisms of schemes are over some fixed base scheme that never occurs in the notation, and *sheaves* are taken to be sheaves in the *fffp*-topology (faithfully flat finitely presented) relative to the base scheme.

We denote the group scheme of the n'th roots of unity by ${}_n\boldsymbol{\mu}$ and ${}_n\boldsymbol{\mu}_S$

equals ${}_n\mu \times S$ for any scheme S. Finally, if S is noetherian and connected, $\pi_1(S)$ denotes the algebraic fundamental group of S [3], and for any prime p, $\pi_1(S)^{(p)}$ denotes the non-p-part of $\pi_1(S)$. We set $\pi_1(S)^{(1)} = \pi_1(S)$.

2. Results

THEOREM: *Let S be any scheme and $\mathscr{L}$ a quasi-coherent locally free $\mathcal{O}_S$-Module of rank 1 such that $\mathscr{L} \oplus \cdots \oplus \mathscr{L}$ (n summands) is free, for some integer n. Then there exists a finite flat morphism $\rho : T \to S$ of degree at most n such that $\rho^*(\mathscr{L})$ is a free $\mathcal{O}_T$-Module.*

Furthermore, if ${}_n\mu_S$ is étale over S, then ρ may be chosen étale.

COROLLARY: *If, in addition to all the hypothesis of the theorem, S is connected and of finite type over a field of exponent p, so that $(p, n) = 1$ and $\pi_1(S)^{(p)} = 0$, then $\mathscr{L}$ is a free $\mathcal{O}_S$-Module.*

This conclusion holds in particular for $p = 1$ and S connected and simply connected.

PROOFS: We may assume that $n > 1$. Let $\mathfrak{G}$ be a *fffp*-sheaf of groups over the base scheme, and $\mathfrak{H} \subset \mathfrak{G}$ an invariant subgroup. To the exact sequence

$$1 \to \mathfrak{H} \to \mathfrak{G} \to \mathfrak{G}/\mathfrak{H} \to 1$$

belongs an exact sequence of pointed sets (see [3, III.4] for this and the sequel) of which we shall need the following piece

$$\tilde{H}^0(S, \mathfrak{G}) \to \tilde{H}^0(S, \mathfrak{G}/\mathfrak{H}) \xrightarrow{\partial} \tilde{H}^1(S, \mathfrak{H}) \to \tilde{H}^1(S, \mathfrak{G}) \tag{1}$$

Recall that one defines $\tilde{H}^0(S, \mathfrak{G}) = \mathrm{Hom}\,(S, \mathfrak{G})$ and that $\tilde{H}^1(S, \mathfrak{G})$ is the set of isomorphism classes of *fffp*-torseurs over S under $\mathfrak{G}$. It is an immediate consequence of the definitions of the maps in (1) that the sequence is functorial in S. Hence for any morphism $\rho : T \to S$ we have a commutative diagram:

$$\begin{array}{ccccccc} \tilde{H}^0(S, \mathfrak{G}) & \to & \tilde{H}^0(S, \mathfrak{G}/\mathfrak{H}) & \xrightarrow{\partial} & \tilde{H}^1(S, \mathfrak{H}) & \to & \tilde{H}^1(S, \mathfrak{G}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{H}^0(T, \mathfrak{G}) & \to & \tilde{H}^0(T, \mathfrak{G}/\mathfrak{H}) & \xrightarrow{\partial} & \tilde{H}^1(T, \mathfrak{H}) & \to & \tilde{H}^1(T, \mathfrak{G}) \end{array} \tag{2}$$

We apply these generalities to $\mathfrak{G} = \mathfrak{GL}_n$, $\mathfrak{H} = \mu$, and $\mathfrak{G}/\mathfrak{H} = \mathfrak{PGL}_{n-1}$. Then $\tilde{H}^1(S, \mu) = \mathrm{Pic}\,(S)$ and the map $\beta : \mathrm{Pic}\,(S) \to \tilde{H}^1(S, \mathfrak{GL}_n)$ sends the isomorphism class of an invertible sheaf $\mathscr{M}$ of $\mathcal{O}_S$-Modules onto the isomorphism class of $\mathscr{M} \oplus \cdots \oplus \mathscr{M}$ (n summands). Denote the isomorphism class of $\mathscr{L}$ in Pic (S) by l. By assumption $\beta(l) = e$, where e is the marked point in $\tilde{H}^1(S, \mathfrak{GL}_n)$, whence $l = \partial(x)$ for some $x \in \tilde{H}^0(S, \mathfrak{PGL}_{n-1})$. We set $l' = \tilde{H}^1(\rho, \mu)(l)$, and have $l' = \partial(x')$ with $x' = \tilde{H}^0(\rho,$

$\mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1})(x)$. In order to get $l' = 0$ we must have $x' = \alpha(y)$, where $y \in \tilde{H}^0(T, \mathfrak{G}\mathfrak{L}_n)$ and $\alpha : \tilde{H}^0(T, \mathfrak{G}\mathfrak{L}_n) \to \tilde{H}^0(T, \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1})$ is the canonical map, coming from the projection $\pi : \mathfrak{G}\mathfrak{L}_n \to \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1}$. This means that we want to lift the morphism $x' : T \to \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1}$ through π to some morphism $y : T \to \mathfrak{G}\mathfrak{L}_n$.

Let $i : \mathfrak{S}\mathfrak{L}_n \to \mathfrak{G}\mathfrak{L}_n$ denote the inclusion of the unimodular group. Then we have an exact sequence

$$1 \to {}_n\mu \to \mathfrak{S}\mathfrak{L}_n \xrightarrow{p} \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1} \to 1$$

where $p = \pi \cdot i$. Now *define* the morphism $\rho : T \to S$ as the pull-back of p by $x : S \to \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1}$,

$$\begin{array}{ccc} T & \longrightarrow & \mathfrak{S}\mathfrak{L}_n \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow[x]{} & \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1} \end{array} \tag{3}$$

Since p is finite and flat, so is ρ, and by construction the morphism $x' = x \circ \rho$ factors through $\mathfrak{S}\mathfrak{L}_n$, hence through $\mathfrak{G}\mathfrak{L}_n$. Furthermore, if we pull the cartesian diagram (3) back over S, we get

$$\begin{array}{ccc} T & \longrightarrow & \mathfrak{S}\mathfrak{L}_{n,S} \\ \downarrow & & \downarrow{\scriptstyle p_S} \\ S & \xrightarrow[(x,1_S)]{} & \mathfrak{P}\mathfrak{G}\mathfrak{L}_{n-1,S} \end{array}$$

where ${}_n\mu_S = \text{Ker}\,(p_S)$, and $p_S = (p, 1_S)$, thus p_S is étale if and only if ${}_n\mu_S$ is an étale S-group scheme. This proves the theorem.

The corollary follows from the observation that under the mentioned hypothesis the morphism $\rho : T \to S$ constructed above has a section, which provides a lifting $S \to \mathfrak{G}\mathfrak{L}_n$ of x, i.e. $\partial(x) = 0$.

3. Remarks

In terms of commutative algebra the theorem states: Given a commutative ring A and an invertible A-module P such that $P \oplus \cdots \oplus P$ (n summands) is free. Then there exists a finite flat (possibly unramified) extension $A \to B$ such that $P \otimes_A B$ is a free B-module.

Suppose that A is a Dedekind ring. It follows from the isomorphism $K_0(A) = \mathbb{Z} \oplus \text{Pic}\,(A)$ (see e.g. [1, Chap. XIII]) and the cancellation law, that for an invertible A-module P the conditions $P \oplus \cdots \oplus P$ (n summands) free, and $P^{\otimes n}$ free, are equivalent. If A is of arithmetic type, e.g. the ring of integers in a number field, then $\text{Pic}\,(A)$ is finite, and we conclude that *every* invertible P becomes trivial after some finite flat extension of A (thus this is true for every projective A-module, in view of their structure), and the knowledge of the order of P in Pic (A) enables one to predict where this extension might ramify. Furthermore, one may even take a fixed 'universal' extension of A that works for all projective A-modules: choose an extension for each element in Pic (A) and take their tensor product over A.

Assume finally that S is a smooth projective connected curve defined over an algebraically closed field k of exponent p, and let n be an integer for which $(p, n) = 1$. In this case there exists a universal finite étale covering $\rho_n : S_n \to S$ of degree n^{2g}, where g is the genus of S, and S_n is a smooth projective connected curve, such that $\rho_n^*(\mathscr{L})$ is trivial for *every* invertible sheaf $\mathscr{L}$ on S for which $\mathscr{L} \oplus \cdots \oplus \mathscr{L}$ (n summands) is trivial. This is seen as follows: Let $\mathscr{L}$ be such a sheaf. Then the étale covering $\rho : T \to S$ constructed in the proof of the theorem is an abelian Galois covering of S with Galois group ${}_n\mu(k)$. We replace T by one of it's connected components, which again is an abelian Galois covering of S, the Galois group of which is a subgroup of ${}_n\mu(k)$. This new covering is also denoted by $\rho : T \to S$. By a theorem due to Serre [5, Chap. VI p. 128] every connected abelian covering of S is the pull-back of some separable isogeny of it's Jacobian $\boldsymbol{J}$, via one of the canonical immersions $S \to \boldsymbol{J}$. Let $\boldsymbol{A} \to \boldsymbol{J}$ be an isogeny that gives the covering ρ. Then the kernel is obviously killed by n, and one readily sees that there exists an isogeny $\boldsymbol{J} \to \boldsymbol{A}$ such that the composition $\boldsymbol{J} \to \boldsymbol{A} \to \boldsymbol{J}$ is multiplication by n. Now *define* $\rho_n : S_n \to S$ to be the pull-back of $\boldsymbol{J} \xrightarrow{\times n} \boldsymbol{J}$. Since Ker $(\boldsymbol{J} \xrightarrow{\times n} \boldsymbol{J}) = {}_n\boldsymbol{J}$ is a finite constant subgroup of $\boldsymbol{J}$ of rank n^{2g}, the morphism ρ_n has the properties claimed above, and we clearly have a morphism $\sigma : S_n \to T$ for which $\rho \circ \sigma = \rho_n$. Thus $\rho_n^*(\mathscr{L})$ is trivial.

The hypothesis k algebraically closed may be weakened to the assumptions that S has a k-rational point, and that ${}_n\mu$ and ${}_n\boldsymbol{J}$ are constant group schemes over k.

Added in proof

R. Fossum has called our attention to a paper by G. Garfinkel: Generic Splitting Algebras for Pic, Pacific J. Math. 35 (1970), 369–380, which deals with torsion elements of the Picard group in the affine case, also treated in [2, p. 376]. It contains our theorem in the case of a Dedekind domain.

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(Oblatum 30–IV–1973)

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