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THE DERIVED JOIN THEOREM AND COALESCENCE IN LIE ALGEBRAS

R. K. Amayo

1. Introduction

1.1. In this paper we investigate the coalescence of certain classes of Lie algebras and answer a question of Ian Stewart [8], p. 98.

1.2. Notation. We employ the notation and terminology of [8]. *Unless otherwise stated all Lie algebras referred to in this paper are of finite or infinite dimension over an arbitrary field* $\mathfrak{k}$.

Let L be a Lie algebra and H a subspace of L (*qua* vector space over $\mathfrak{k}$). By $H \leqq L$, $H \lhd L$, H si L, $H \lhd^m L$ we shall mean respectively that H is a subalgebra, ideal, subideal (in the sense of Hartley [2] p. 257), m-step subideal (m a non-negative integer) of L. $L^{(n)}$ and L^n will denote respectively the nth terms of the derived series and lower central series of L. We define inductively $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$ and $L^1 = L$, $L^{n+1} = [L^n, L]$. Square brackets [,] will denote Lie multiplication, and triangular brackets $\langle , \rangle$ will denote the subalgebra generated by their contents. If A, B are subsets of L, $[A, B]$ is the subspace spanned by all $[a, b]$ with $a \in A$, $b \in B$; $\langle A^B \rangle$ is the smallest subalgebra containing A and left invariant under Lie multiplication by the elements of B. If $H \leqq L$ the *ideal closure series* of H in L is defined inductively by $H_0 = L$, $H_{n+1} = \langle H^{H_n} \rangle$. Thus H_{n+1} is the smallest ideal of H_n which contains H.

By a class $\mathfrak{X}$ of Lie algebras over $\mathfrak{k}$ we shall mean a class in the usual sense, whose elements are Lie algebras over $\mathfrak{k}$, with the further properties that $0 \in \mathfrak{X}$, and if $H \cong K \in \mathfrak{X}$ then $H \in \mathfrak{X}$. An algebra (ideal, subideal, $\cdots$) lying in $\mathfrak{X}$ may be called an $\mathfrak{X}$-algebra ($\mathfrak{X}$-ideal, $\mathfrak{X}$-subideal, $\cdots$). A *closure operation* A assigns to each class X another class $\mathrm{A}\mathfrak{X}$ in such a manner that for all classes $\mathfrak{X}$, $\mathfrak{Y}$ we have $\mathfrak{A}0 = 0$, $\mathfrak{X} \leqq \mathrm{A}\mathfrak{X}$, $\mathrm{A}(\mathrm{A}\mathfrak{X}) = \mathrm{A}\mathfrak{X}$, and if $\mathfrak{X} \leqq \mathfrak{Y}$ then $\mathrm{A}\mathfrak{X} \leqq \mathrm{A}\mathfrak{Y}$. (Here 0 denotes the class of all 0-dimensional algebras, $\leqq$ is class inclusion).

$\mathfrak{X}$ is A-*closed* if $\mathfrak{X} = \mathrm{A}\mathfrak{X}$. Given two closure operations A and B we can define another closure operation $\{\mathrm{A}, \mathrm{B}\}$ by letting $\{\mathrm{A}, \mathrm{B}\}\mathfrak{X}$ be the smallest class $\mathfrak{Y} \geqq \mathfrak{X}$ such that $\mathfrak{Y} = \mathrm{A}\mathfrak{Y} = \mathrm{B}\mathfrak{Y}$. Given two classes $\mathfrak{X}$, $\mathfrak{Y}$ we define a (non-associative, non-commutative) product $\mathfrak{X}\mathfrak{Y}$ comprising all Lie algebras L having an ideal $I \in X$ such that $L/I \in Y$. Inductively we define $\mathfrak{X}_1 \cdots \mathfrak{X}_n$ to be $(\mathfrak{X}_1 \cdots \mathfrak{X}_{n-1})\mathfrak{X}_n$ and put $\mathfrak{X}^n = \mathfrak{X} \cdots \mathfrak{X}$ (to n factors). We also define an associative and commutative sum $\mathfrak{X}+\mathfrak{Y}$ comprising all Lie algebras L with a finite series $0 = L_m \lhd L_{m-1} \lhd \cdots \lhd L_1 = L$ such that each factor is in $\mathfrak{X}$ or $\mathfrak{Y}$.

We shall need the closure operations S, L, Q, E, L, N_0, defined as follows: $\mathrm{s}\mathfrak{X}$, $\mathrm{I}\mathfrak{X}$, and $\mathrm{Q}\mathfrak{X}$ consist respectively of all subalgebras, subideals, and quotient algebras of $\mathfrak{X}$-algebras. $\mathrm{E}\mathfrak{X} = \cup_{n=1}^{\infty} \mathfrak{X}^n$ (thus $\mathfrak{X}+\mathfrak{Y} = \mathrm{E}(\mathfrak{X} \cup \mathfrak{Y})$). $L \in \mathrm{L}\mathfrak{X}$ if and only if every finite subset of L lies inside an $\mathfrak{X}$-subalgebra of L. $\mathfrak{X}$ is N_0-closed if whenever H, K are $\mathfrak{X}$-ideals of L then $H+K \in \mathfrak{X}$; in general $N_0\mathfrak{X}$ is the smallest N_0-closed class containing $\mathfrak{X}$. We say a class $\mathfrak{X}$ is *subjunctive* if $\mathfrak{X} = \{\mathrm{I}, \mathrm{N}_0\}\mathfrak{X}$.

As regards classes, $\mathfrak{A}$ denotes abelian Lie algebras, $\mathfrak{N}$ nilpotent, $\mathfrak{N}_c$ nilpotent of class $\leqq c$, $\mathfrak{F}$ finite dimensional, $\mathfrak{F}_m$ finite dimensional of dimension $\leqq m$, $\mathfrak{G}$ finitely generated. Thus $\mathrm{E}\mathfrak{A}$ is the class of soluble Lie algebras and $\mathfrak{A}^n$ denotes soluble Lie algebras of derived length $\leqq n$.

Let L be a Lie algebra. If H and K are subspaces of L we write $H+K$ to mean the vector subspace of L generated by H and K (it consists of elements $h+k$, $h \in H$ and $k \in K$). By $J \leqq H+K$ we mean J is a subalgebra of L contained in the subspace $H+K$. We say H and K are *permutable* (or *H permutes* with K) if $[H, K]$ is contained in $H+K$. Thus if $H, K \leqq L$ and H permutes with K, then $\langle H, K\rangle = H+K$. Let $H, K \leqq L$ then we define; $I_K(H) = \{x \in K | [H, x] \subseteq H\}$. $I_K(H)$ is the idealiser of H in K and is a subalgebra of K. The *permutizer* of H in K, $P_K(H) = \langle X \leqq K | [X, H] \subseteq X+H\rangle$. Clearly $P_K(H)$ is the largest subalgebra of K which permutes with H.

Let $A, B, A_1, A_2 \cdots$ be subspaces and $a, b, a_1, a_2, \cdots$ be elements of a Lie algebra L. Then we define inductively $[A, {}_0B] = [A, {}_nB] = [[A, {}_{n-1}B], B]$ and $[A_1, \cdots, A_n, A_{n+1}] = [[A_1, \cdots, A_n], A_{n+1}]$. Similarly for $[a, {}_nb]$ and $[a_1, \cdots, a_n, a_{n+1}]$.

1.3. Basic Results.

LEMMA (1): *Let $H \lhd^m L$, $K \lhd^n L$, and $J = \langle H, K\rangle$. If H permutes with K, then $J \lhd^r L$, where $r \leqq (m+n)!$.*

PROOF: See Amayo [1].

It is well known and easy to prove that if $H, K \leqq L$ and K idealises H (i.e $[H, K] \subseteq H$) then K idealises every term of the ideal closure series of H in L. Furthermore H si L if and only if H equals some term of its ideal closure series, and if $H \leqq K$, then every term of the ideal closure series of H in L is contained in the corresponding one for K. Finally if $H \lhd^m L$ and $K \lhd^n L$, then $H \cap K \lhd^r L$, where $r \leqq \max(m, n)$.

LEMMA (2): *Let* $H \lhd^m L$, $K \lhd^n L$ *and* $P = P_H(K)$. *Then*

$$P \lhd^{r_{mn}} L,$$

where $r_{0n} = 0$, $r_{1n} = 1$, *and* $r_{mn} = (r_{m-1,n+1}+n+2)!$.

PROOF: The result is trivial for $m = 0, 1$. Let $m > 1$ and assume inductively that the result is true for $m-1$, for all n. Put $H_1 = \langle H^L \rangle$, $K_1 = \langle P^K \rangle \cap K$, and $P_1 = P_H(K_1)$. Then $K_1 \leqq H_1$, $K_1 \lhd K \lhd^n L$, and $H \lhd^{m-1} H_1$. Thus $K_1 \lhd^{n+1} H_1$ and so by induction

$$P_1 \lhd^r H_1 \lhd L,$$

where $r = r_{m-1,n+1}$. If $Q = \langle P_1, K_1 \rangle$, then $Q = P_1 + K_1$ and by lemma 1, $Q \lhd^s L$, where $s = (r+n+2)!$. Since $\langle P, K \rangle = P+K$ and $P \leqq U = \langle P^K \rangle$, then

$$U = P + U \cap K = P + K_1,$$

which implies that P permutes with K_1 and so $P \leqq P_1$. Thus $U = P + U \cap K = P + K_1 \leqq P_1 + K_1 = Q$. Hence the sth term, U_s, of the ideal closure series of U in L is contained in $Q_s = Q$. But $K_1 = U \cap K \leqq U \leqq U_s \leqq Q = P_1 + K_1$, and so $U_s = P_1 \cap U_s + K_1$. Furthermore K idealises U and so K idealises U_s. Therefore $\langle U_s, K \rangle = U_s + K = (U_s \cap P_1 + K_1) + K = U_s \cap P_1 + K$. This implies that $U_s \cap P_1$ permutes with K and so

$$U_s \cap P_1 \leqq P \leqq U \cap P_1 \leqq U_s \cap P_1.$$

Thus $P = U_s \cap P_1$. Finally $U_s \lhd^s L$ and $P_1 \lhd^{r+1} L$ and $s = (r+n+2)! > r+1$, and the result follows.

We recall that a class $\mathfrak{X}$ is subjunctive if $\mathfrak{X} = \{\mathrm{I}, \mathrm{N}_0\}\mathfrak{X}$.

THEOREM (3): *Suppose that* $\mathfrak{X}$ *is a subjunctive class and* H, K *are permutable* $\mathfrak{X}$*-subideals of* L. *Then* $J = \langle H, K \rangle$ *is an* $\mathfrak{X}$*-subideal of* L.

PROOF: By lemma 1, J si L and $J = H + K$. Let $H \lhd^m L$ and $K \lhd^n L$. We induct on $m+n$ to show that $J \in \mathfrak{X}$. W.L.O.G we may assume that $m, n > 0$. If $m+n \leqq 2$, then J is a sum of two $\mathfrak{X}$-ideals and so $J \in \mathfrak{X}$. Assume that $m+n > 2$ and the result true for $m+n-1$. Let $H_1 = \langle H^J \rangle$. Then $H_1 = H_1 \cap J = H + H_1 \cap K$, and so H permutes with $H_1 \cap K$. Now $H \lhd^m J$ and so $H \lhd^{m-1} H_1$, and $H_1 \cap K \lhd^n H_1$, and $H_1 \cap K \lhd K \in \mathfrak{X} = \mathrm{I}\mathfrak{X}$, and so by induction $H_1 \in \mathfrak{X}$. Similarly $K_1 = \langle K^J \rangle \in \mathfrak{X}$. Finally $J = H_1 + K_1$, a sum of two $\mathfrak{X}$-ideals and so $J \in \mathfrak{X}$. This completes the proof.

REMARK: These results are the Lie theoretic analogues of the ones in Roseblade [6]. In a somewhat similar vein we can prove:

LEMMA (4): *Let H, $K \in \mathfrak{X}$, a subjunctive class, and suppose that H, K si L. If $A \leqq H$, $B \leqq K$ and A permutes with B, then there exists $X \in \mathfrak{X}$, X si L with $\langle A, B\rangle \leqq X \leqq H+K$.*

From Amayo [1] we have;

LEMMA (5): *Let $H, A, B \leqq L$ and $J = \langle A, B\rangle$. If A idealises H and permutes with B, then $\langle H^J\rangle = \langle H^B\rangle$.*

Finally we need the simple result,

LEMMA (6): *Let L be a Lie algebra, H si L and $P \lhd L$ such that $L = H+P^2$. Then $L = H+P^{(n)}$ for all positive integers n.*

PROOF: Since $P^{(n)}$ *ch* $P \lhd L$, then $P^{(n)} \lhd L$, and so by lemma 1, $K = (H+P^{(n)})$ si L. Let $K_1 = \langle K^L\rangle$. Then L/K_1 is soluble. But we also have

$$L/K_1 = (H+P^2)/K_1 \leqq (K_1+L^2)/K_1 = (L/K_1)^2.$$

Therefore $L = K_1$ and, since K si L, we must have $L = K$.

PROOF: (of lemma 4)

Let $J = \langle H, K\rangle$, $C = \langle A, B\rangle$ and suppose that $H \lhd^m L$ and $K \lhd^n L$. We induct on m. For $m = 0$ or 1 we have $H \lhd L$ so $H \lhd J$, whence J is an $\mathfrak{X}$-subideal of L by theorem 3. So we may take J for X. (since also $J = H+K$). Let $m > 1$ and suppose that the result holds for $m-1$ in place of m. Put $H_1 = \langle H^L\rangle$ and $A_1 = \langle A^B\rangle$. Evidently $C = A+B$ so $A_1 = C \cap A_1 = A+B \cap A_1$ and $B \cap A_1 \leqq K \cap H_1$. Now $H \lhd^{m-1} H_1$, $K \cap H_1 \lhd^n H_1$ and $K \cap H_1 \lhd K$ so that $K \cap H_1 \in \mathrm{I}\mathfrak{X} = \mathfrak{X}$. Furthermore $A_1 \leqq L$ and $A_1 = A+B \cap A_1$ implies that A permutes with $B \cap A_1$. So by the inductive hypothesis we can find $X_1 \in \mathfrak{X}$ with $X_1 \lhd^p L$ for some p and

$$A_1 \leqq X_1 \subseteq H+K \cap H_1.$$

Let Q_{p+1} be the $(p+1)th$ term of the ideal closure series of A_1 in L. Since $X_1 \lhd^p H_1 \lhd L$ then $Q_{p+1} \leqq X_1$. We also have $Q_{p+1} \lhd^{p+1} L$. Now B idealises A_1 and so by the remarks in section 1.2 we have that B idealises Q_{p+1} and so

$$B \leqq P = P_K(Q_{p+1}).$$

By lemma 2 we have $P \lhd^r L$ for some r. Since P permutes with Q_{p+1} then $\langle Q_{p+1}, P\rangle = Q_{p+1}+P$ si L by lemma 1. Put

$$X = Q_{p+1}+P.$$

We have $Q_{p+1} \leqq X_1$ so Q_{p+1} si X_1 whence $Q_{p+1} \in \mathrm{I}\mathfrak{X} = \mathfrak{X}$. Furthermore $P \leqq K$ so P si K and so $P \in \mathfrak{X}$. Thus by theorem 3 $X \in \mathfrak{X}$. Now $C = \langle A, B\rangle = A_1+B \leqq Q_{p+1}+P = X$ and $X \subseteq X_1+P \subseteq (H+K \cap H_1)$

$+\ K \subseteq H+K$. Hence the result holds for m and our proof is complete.

2. The derived join theorem

Here and in the sequel the symbols $\lambda(m, n, \cdots)$ and $\lambda_i(m, n, \cdots)$, for $i = 1, 2, \cdots$, will denote non-negative integers depending only on the arguments shown.

In [1] we proved:

THEOREM (A): (*The Derived Join Theorem*). *Suppose that* $J = \langle H_1, \cdots, H_n \rangle$ *and* $H_i \lhd^{h_i} L$ *for* $i = 1, 2, \cdots, n$. *Then there exists* $\lambda = \lambda(h, r)$ *such that*

$$J^{(\lambda)} \lhd^{\lambda} L$$

and

$$J^{(\lambda)} \leqq H_1^{(r_1)} + \cdots + H_n^{(r_n)}$$

whenever $h_1 + \cdots + h_n \leqq h$ *and* $r_1 + \cdots + r_n \leqq r$.

By lemma 2 if $H_1 \lhd^{h_1} L$ and $H_2 \lhd^{h_2} L$, then there exists $\lambda_1 = \lambda_1(h)$ such that

$$P_{H_1}(H_2) \lhd^{\lambda_1} L \tag{1}$$

whenever $h_1 + h_2 \leqq h \cdot \lambda_1(0) = 0$, $\lambda_1(h) = r_{h_1 h_2}$ as defined in lemma 2.

Let $\lambda_2(h) = \lambda(h, 0)$ in theorem A and let $n = 2$. Then

$$J^{(\lambda_2)} \leqq H_1 + H_2 .$$

Put $M = J^{(\lambda_2)}$ and $K = \langle M, H_2 \rangle$. Then $K = M + H_2 \leqq H_1 + H_2$ and so $K = K \cap H_1 + H_2$. This implies that $K \cap H_1$ permutes with H_2. Thus

$$H_1^{(\lambda_2)} \leqq M \cap H_1 \leqq K \cap H_1 \leqq P_{H_1}(H_2).$$

This proves

COROLLARY (A1): *Let* $H_1 \lhd^{h_1} L$, $H_2 \lhd^{h_2} L$. *Then there exists* $\lambda_2 = \lambda_2(h)$ *such that*

$$H_1^{(\lambda_2)} \leqq P_{H_1}(H_2), \tag{2}$$

whenever $h_1 + h_2 \leqq h$.

Theorem B. *Suppose that* $J = \langle H_1, \cdots, H_n \rangle$, *where* $H_i \lhd^{h_i} L$ *for* $i = 1, \cdots, n$. *Then there exists* $\lambda_3 = \lambda_3(h)$ *and subideals* $P_1, \cdots, P_n$ *of* $H_1, \cdots, H_n$ *respectively such that*

$$\langle P_1, \cdots, P_{i+1} \rangle = P_1 + \cdots + P_{i+1},\ 1 \leqq i \leqq n-1.$$

$$J^{(\lambda_3)} \lhd^{\lambda_3} L$$

and

$$J_{(\lambda_3)} \leqq P_1 + \cdots + P_n$$

whenever $h_1 + \cdots + h_n \leqq h$.

PROOF: If some $h_i = 0$, then $J = H_i = L$. We may assume that $i = 1$ and put $P_1 = H_1$, $P_j = H_j (j > 1)$ and $\lambda_3 = 0$. W.L.O.G assume that $h_i > 0$ for all i. We define inductively subalgebras P_i, Q_i of J and integers p_i, q_i as follows:

(3) $$P_1 = Q_1 = H_1$$

(4) $$P_{i+1} = P_{H_{i+1}}(Q_i)$$

(5) $$Q_{i+1} = \langle P_{i+1}, Q_i \rangle = Q_i + P_{i+1}$$

By induction on i it follows that

(6) $$Q_i = P_1 + \cdots + P_i \quad 1 \leqq i \leqq n$$

Let

(7) $$p_1 = q_1 = h$$

(8) $$p_{i+1} = \lambda_1(h + q_i) \quad 1 \leqq i \leqq n-1$$

(9) $$q_{i+1} = (p_{i+1} + q_i)! \quad 1 \leqq i \leqq n-1$$

Suppose inductively that

(10) $$P_i \lhd^{p_i} L$$

and

(11) $$Q_i \lhd^{q_i} L$$

Since $h_{i+1} < h$ it follows from (1) and (8) that $P_{i+1} \lhd^{p_{i+1}} L$. Now P_{i+1} and Q_i are permutable and so by lemma 1, (5) and (9), we have $Q_{i+1} \lhd^{q_{i+1}} L$. But $P_1 = Q_1 = H_1 \lhd^{h_1} L$, and so (10) and (11) hold for all i, $1 \leqq i \leqq n$.

Let $r_1 = 0$, $r_{i+1} = \lambda_2(h + q_i)$ for $1 \leqq i \leqq n-1$. We apply corollary A1 to the pair H_{i+1}, Q_i to obtain

(12) $$H_{i+1}(r_{i+1}) \leqq P_{H_{i+1}}(Q_i) = P_{i+1}$$

Finally we define $\lambda_3(h) = \lambda(h, r_1 + r_2 + \cdots + r_n)$. Then by (3), (12) and theorem A it follows that

$$J^{(\lambda_3)} \lhd^{\lambda_3} L$$
$$J^{(\lambda_3)} \leqq H_1^{(r_1)} + \cdots + H_n^{(r_n)} \subseteq P_1 + \cdots + P_n.$$

This proves theorem B.

Suppose that $\mathfrak{X}$ is a subjunctive class of Lie algebras (i.e. $\mathfrak{X} = \mathrm{I}\mathfrak{X} = \mathrm{N}_0\mathfrak{X}$) and each $H_i \in \mathfrak{X}$. Then by theorem 3 and a simple induction on i (for each $P_i \in \mathrm{I}\mathfrak{X} = \mathfrak{X}$) it follows that each $Q_i \in \mathfrak{X}$. Since $J^{(\lambda_3)} \lhd J$, and so $J^{(\lambda_3)} \lhd Q_n$, we have,

COROLLARY (B1): *Suppose that* $J = \langle H_1, \cdots, H_n \rangle$ *and* $H_i \lhd^{h_i} L\,(1 \leqq i \leqq n)$ *and each* H_i *lies in a subjunctive class* $\mathfrak{X}$. *Then*

$$J^{(\lambda_3)} \in \mathfrak{X}$$

and so

$$J \in \mathfrak{X}\mathfrak{A}^{\lambda_3}.$$

We recall that a class $\mathfrak{X}$ is said to be Q-closed if quotients of $\mathfrak{X}$-algebras are also $\mathfrak{X}$-algebras.

LEMMA 7. *Let* $H \lhd^m L$, $K \leqq L$, *and* $H, K \in \mathfrak{X} = \{\mathrm{I}, \mathrm{Q}\}\mathfrak{X}$. *If* $J = \langle H,K \rangle$ *and* K *permutes with* H, *then* $J \in \mathfrak{X}^{1+m}$.

PROOF: By induction on m. If $m = 0$, then $J = H \in \mathfrak{X}$. Let $m > 0$, and assume the result for $m-1$. Now $J = H+K$, since K permutes with H. Let $H_1 = \langle H^J \rangle$. Then $H_1 = H + H_1 \cap K$, and so $H_1 \cap K$ permutes with H. Furthermore $H_1 \cap K \lhd K \in \mathfrak{X}$ implies $H_1 \cap K \in X$. We also have $H \lhd^{m-1} H_1$ and so by induction, $H_1 \in \mathfrak{X}^{1+m-1}$. But $H_1 \lhd J$, and $J/H_1 \cong K/H_1 \cap K \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$, and so $J \in \mathfrak{X}^{1+m}$.

We note that if $J = \langle H, K \rangle$, $H \lhd^m J$, $K \lhd^n J$, and H permutes with K, then $J \in \mathfrak{X}^{1+r}$, where $r = \min(m, n)$.

Trivially, if $\mathfrak{X}$ is $\{\mathrm{I}, \mathrm{Q}\}$-closed, then so is $\mathfrak{X}^m$ for every $m > 0$.

COROLLARY (B2): *Let* $J = \langle H_1, \cdots, H_n \rangle$, $H_i \lhd^{h_i} J$ *and* $H_i \in X = \{\mathrm{I}, \mathrm{Q}\}X$ *for* $i = 1, 2, \cdots, n$. *Then there exists* $\lambda_4 = \lambda_4(h)$ *such that*

$$J^{(\lambda_3)} \in \mathfrak{X}^{\lambda_4}$$

and so

$$J \in \mathfrak{X}^{\lambda_4}\mathfrak{A}^{\lambda_3},$$

whenever $h_1 + \cdots + h_n \leqq h$.

PROOF: Using the notation of theorem B we have $P_1 = Q_1 = H_1 \in \mathfrak{X}$. Define $m_1 = 1$ and $m_{i+1} = (1+q_i)m_i$, $i = 1, 2, \cdots, n$. Assume inductively that $Q_i \in \mathfrak{X}^{m_i}$ and apply lemma 7 to the pair P_{i+1}, Q_i (for P_{i+1} si H_{i+1} implies that $P_{i+1} \in \mathfrak{X} \leqq \mathfrak{X}^{m_i}$). Then

$$Q_{i+1} \in (\mathfrak{X}^{m_i})^{1+q_i} \leqq \mathfrak{X}^{m_{i+1}}.$$

Hence $Q_i \in \mathfrak{X}^{m_i}$ for all i, in particular $Q_n \in \mathfrak{X}^{m_n}$. Put $\lambda_4(h) = m_n$. By theorem B, $J^{(\lambda_3)} \lhd Q_n$ and the result follows.

3. Local coalescence

Let $\mathfrak{X}$ be any class of Lie algebras. We say that $\mathfrak{X}$ is *coalescent* if and only if in any Lie algebra the join of a pair of $\mathfrak{X}$-subideals is always an $\mathfrak{X}$-subideal. We say that $\mathfrak{X}$ is *locally coalescent* if and only if whenever H and K are $\mathfrak{X}$-subideals of a Lie algebra L then every finitely generated subalgebra C of $J = \langle H, K\rangle$ is contained in some $\mathfrak{X}$-subideal X of L with $C \leqq X \leqq J$.

In [2] Hartley has proved that over fields of characteristic zero, the class $\mathfrak{N}$ of nilpotent Lie algebras is locally coalescent (This is false for characteristic $p > 0$). We shall prove the following:

THEOREM (C): *Over fields of characteristic zero, the universal class* $\mathfrak{D}$ *of all* Lie *algebras is locally coalescent.*

By the derived join theorem the join of a pair of soluble subideals is always soluble. Thus we have

COROLLARY (C1): *Over fields of characteristic zero, the classes* $\mathfrak{G}$ *and* $\mathfrak{G} \cap E\mathfrak{A}$ *are coalescent.*

COROLLARY (C2): (*Over fields of characteristic zero*) *Let* $J = \langle H, K\rangle$, H *si* L *and* K *si* L. *If* $J/J^2 \in \mathfrak{G}$, *then* J *si* L.

PROOF. Since $J/J^2 \in \mathfrak{G}$ then there exists $C \in \mathfrak{G}$ with $J = C+J^2$. By theorem C we can find X si L with $C \leqq X \leqq J$. Thus $J = X+J^2$. By the derived join theorem there exists r such that $J^{(r)}$ si L. Finally since X si J, then by lemma 6, $J = X+J^{(r)}$ and so by lemma 1, J si L.

REMARK: The result above holds if J is the join of finitely many subideals. We say a class $\mathfrak{X}$ is *persistent* if in any Lie algebra the join of a pair of $\mathfrak{X}$-subideals is always an $\mathfrak{X}$-algebra. For example by the derived join theorem the class E$\mathfrak{A}$ of soluble Lie algebras is persistent. From theorem C we have

COROLLARY (C3): *Over fields of characteristic zero, every* I*-closed persistent class is locally coalescent.*

Finally we mention a result which generalises corollary C2. First we note that if $\mathfrak{X}$ is locally coalescent then $\mathfrak{G} \cap \mathfrak{X}$ is coalescent. If $\mathfrak{X}$ is a subjunctive and locally coalescent and $H, K \in \mathfrak{X}$ in corollary C2, then the subideal X may be taken to be in $\mathfrak{X}$. Finally by corollary B1 we may choose r such that $J^{(r)} \in \mathfrak{X}$ and so by theorem 3, $J = X+J^{(r)} \in \mathfrak{X}$. Thus we have (if $\mathfrak{C}$ is the class of Lie algebras L with $L/L^2 \in \mathfrak{F}$)

COROLLARY (C4): (*Over fields of characteristic zero*) *Suppose that* $\mathfrak{X}$ *is a subjunctive and locally coalescent class. Let* $H, K \in \mathfrak{X}$, H *si* L, K *si* L, *and* $J = \langle H, K\rangle$. *If* $J \in \mathfrak{C}$, *then* J *si* L, $J \in \mathfrak{C} \cap \mathfrak{X}$. *In particular the class* $\mathfrak{C} \cap \mathfrak{X}$ *is coalescent.*

PROOF: (of theorem C).
First we need the simple result,

LEMMA (8): *(Any field) Suppose that we have a chain of Lie algebras* $A_n \leqq A_{n-1} \leqq \cdots \leqq A_1 \leqq A_0 = A$ *such that* $B_i \leqq A_i$ *and* B_i *si* A_{i-1} *for* $i = 1, \ldots, n$. *Then*

$$B = \bigcap_{i=1}^{n} B_i \text{ si } A.$$

PROOF: Trivial by induction on n.

In order to prove theorem C we seek the truth of the following statement (over fields of characteristic zero):

If H, K *si* L, $J = \langle H, K\rangle$ *and* $C \leqq J$, $C \in \mathfrak{G}$, *then there exists* X *si* L *with* $C \leqq X \leqq J$. *

We establish the truth of (*) by first considering some special cases. We assume throughout that $H \lhd^m L$ and $K \lhd^n L$.

Case 1. $L = A+J$, $A \lhd L$, $A^2 = 0$, $A \cap J = 0$. We have $[A, H^m] \leqq [A, {}_mH] \leqq A \cap H = 0$ and so A centralises H^m. By lemma 5,

$$\langle (H^m)^L\rangle = \langle (H^m)^J\rangle \leqq J.$$

Similarly $\langle (K^n)^L\rangle \leqq J$. Put $F = \langle (H^m)^L\rangle + \langle (K^n)^L\rangle$. Then $F \lhd L$, $F \leqq J$. Clearly J/F is the join of two $\mathfrak{N}$-subideals of L/F and, as $\mathfrak{N}$ is locally colescent (see Hartley [2] p. 259, theorems 2–4), there exists X/F si L/F with $(C+F)/F \leqq X/F \leqq J/F$. Thus X si L and $C \leqq X \leqq J$. This proves (*) for case 1.

Case 2. $L = B+J$, $B^2 = 0$, $B \lhd L$.
Since B is abelian then $U = B \cap J \lhd B$. But $U \lhd J$ and so $B \lhd L$. Now L/U satisfies the hypothesis of case 1, and so there exists X/U si L/U with $(C+U)/U \leqq X/U \leqq J/U$. Hence X si L and $C \leqq X \leqq J$ and (*) is proved for case 2.

Case 3. $L^{(d)} = 0$ for some d.

Case 4. $H \lhd^m L$, $K \lhd^n L$, and $J \in \mathrm{E}\mathfrak{A}$.
Let $H \lhd H_{m-1} \lhd \cdots \lhd L$ and $K = K_n \lhd K_{n-1} \cdots \lhd L$. We use induction on $m+n$. If $m+n \leqq 2$, then J si L (by lemma 1) and so we may take J for X. If one of m, n is 0 then $J = L$ and there is nothing to prove. Assume that $m, n > 0$, $m+n > 2$ and (*) is true for $m+n-1$. Let $J_1 =$

* Is trivially true if $d = 0$ or 1. Let $d > 0$ and assume inductively that (*) is true for d—1 in place of d. Let $B = L^{(d-1)}$. Then by induction there exists Y/B si L/B with $(C+B)/B \leqq Y/B \leqq (J+B)/B$. Thus Y si L and $C \leqq Y \leqq J+B$ and $C \leqq J$, $C \in G$. Apply case 2 to $J+B$ (for $B^2 = L^{(d)} = 0$, and $B \lhd L$). Then we can find X_1 si $J+B$ with $C \leqq X_1 \leqq J$. Put $X = X_1 \cap Y$. Then by lemma 8, X si L. We also have $C \leqq X_1 \cap Y \leqq J$. This completes our induction on d and proves (*) for case 3.

$\langle H_{m-1}, K\rangle$, and $J_2 = \langle H, K_{n-1}\rangle$. As H and K are soluble then by the derived join theorem ($H_{m-1} \lhd^{m-1} L$ and $K_{n-1} \lhd^{n-1} L$), for a sufficiently large r, we have

$$D = J_1^{(r)} \leqq H_{m-1}, D \lhd^r L$$

and

$$E = J_2^{(r)} \leqq K_{n-1}, E \lhd^r L.$$

Let

$$D = D_r \lhd D_{r-1} \lhd \cdots \lhd D_1 \lhd D_0 = L$$

be the ideal closure series of D in L. Since $D \lhd J_1$, then by an earlier remark (see remark before lemma 2) J_1 idealises every D_i. Let us fix i, $1 \leqq i \leqq r$. Then $(J_1+D_i)/D_i \in \mathrm{E}\mathfrak{A}$, $(J_1+D_i)/D_i = \langle (H_{m-1}+D_i)/D_i, (K+D_i)/D_i\rangle \leqq (J_1+D_{i-1})/D_i$, $(H_{m-1}+D_i)/D_i \lhd^{m-1} (J_1+D_{i-1})/D_i$, and $(K+D_i)/D_i \lhd^n (J_1+D_{i-1})/D_i$. Therefore by the induction on $m+n-1$ there exists Y_i/D_i si $(J_1+D_{i-1})/D_i$ with $(C+D_i)/D_i \leqq Y_i/D_i \leqq (J_1+D_i)/D_i$. Thus for each i, $1 \leqq i \leqq r$, we have Y_i si J_1+D_{i-1} and $C \leqq Y_i \leqq J_1+D_i$. Let $Y = \cap_{i=1}^r Y_i$. Then by lemma 8 Y si $J+D_0 = L$, and $C \leqq Y \leqq J_1+D_r = J_1$.

Similarly by considering E, we get some Z si L with $C \leqq Z \leqq J_2$. Let $X_1 = Y \cap Z$ and $F = J_1 \cap J_2$. Clearly $J \leqq F$, $C \leqq X_1 \leqq F$ and X_1 si L. Now H_{m-1} idealises H and K_{n-1} idealises K, and so $H_{m-1} \cap K_{n-1}$ idealises both H and K and so idealises J. Furthermore from above, $F^{(r)} \leqq J_1^{(r)} \cap J_2^{(r)} \leqq H_{m-1} \cap K_{n-1}$. Thus $F^{(r)}$ idealises J. We apply case 3 to $F/F^{(r)}$ to obtain an $X_2/F^{(r)}$ si $F/F^{(r)}$ with $(C+F^{(r)})/F^{(r)} \leqq X_2/F^{(r)} \leqq (J+F^{(r)})/F^{(r)}$. Thus X_2 si F and $C \leqq X_2 \leqq J+F^{(r)}$. But $J \lhd (J+F^{(r)})$ and so $J \cap X_2 \lhd X_2$, which implies that $J \cap X_2$ si F. Finally let $X = X_1 \cap J \cap X_2$. Then $C \leqq X \leqq J$ and since X_1 si L, $X_1 \leqq F$, then by lemma 8, X si L. This completes our induction on $m+n$ and proves (*) for case 4.

Case 5. The general case.
We have H si L, K si L, $J = \langle H, K\rangle$ and $C \leqq J$, $C \in \mathfrak{G}$. By the derived join theorem there exists an integer r such that $Y = J^{(r)} \lhd^r L$. Let $Y = Y_r \lhd Y_{r-1} \lhd \cdots \lhd Y_1 \lhd Y_0 = L$ be the ideal closure series of Y in L. Since $Y \lhd J$ then J idealises Y_i and $\langle J, Y_i\rangle = J+Y_i$ for all i, $0 \leqq i \leqq r$.

We fix i, $1 \leqq i \leqq r$. Then $(J+Y_i)/Y_i \in \mathfrak{A}^r$ and is the join of a pair of subideals of $(J+Y_{i-1})/Y_i$. Hence applying case 4 to $(J+Y_i)/Y_i$ we get an X_i/Y_i si $(J+Y_{i-1})/Y_i$ with $(C+Y_i)/Y_i \leqq X_i/Y_i \leqq (J+Y_i)/Y_i$. Thus for each i, $1 \leqq i \leqq r$, there exists X_i si $J+Y_{i-1}$ with $C \leqq X_i \leqq J+Y_i$. Let $X = \cap_{i=1}^r X_i$. By lemma 8 X si $J+Y_0 = L$. We also have $C \leqq X \leqq J+Y_r = J$.

This proves (*) in the general case and so proves theorem C.

REMARK: In [6] Roseblade and Stonehewer prove that every subjunctive class of groups is locally coalescent. It is still an open question for Lie algebras, though most of the more familiar classes of Lie algebras are locally coalescent (see Amayo [10]).

Let $\mathfrak{X}$ be any $\{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}$-closed class of Lie algebras and suppose that $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is locally coalescent. Clearly $\mathfrak{X}$ is also a subjunctive class. In the proof of case 5, let H, K be $\mathfrak{X}$-subideals of L, $J = \langle H, K \rangle$. By corollary B1 we can choose r such that $J^{(r)} \lhd^r L$ and $J^{(r)} \in \mathfrak{X}$. By local coalescence of $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ we can choose $X_i/Y_i \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ (for $\mathfrak{X}$ is Q-closed implies that $(H+Y_i)/Y_i \in \mathfrak{X}$ and $(K+Y_i)/Y_i \in \mathfrak{X}$ for all i; and $(J+Y_i)/Y_i \in \mathrm{E}\mathfrak{A}$). In particular $X_r/Y_r \in \mathfrak{X}$ and since $Y_r = Y = J^{(r)} \in \mathfrak{X}$, then by E-closure $X_r \in \mathfrak{X}$ and so $X \in \mathfrak{X}$ as X si L implies that X si X_r ($\mathfrak{X} = \mathrm{I}\mathfrak{X}$).

Thus we have proved;

COROLLARY (C5): *Let $\mathfrak{X} = \{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}\mathfrak{X}$ be a class of Lie algebras. Then $\mathfrak{X}$ is locally coalescent if and only if $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is locally coalescent.*

We note that by corollary B1 or B2 we may replace 'locally coalescent' in corollary C5 by 'persistent'. In the next section we will show that we may even replace 'locally coalescent' by 'coalescent'.

We recall that if $\mathfrak{X}$, $\mathfrak{Y}$ are classes of Lie algebras, then $\mathfrak{X}+\mathfrak{Y}$ denotes the class of Lie algebras L with a finite series $0 = L_0 \lhd L_1 \lhd \cdots \lhd L_n = L$ such that the factors are in X or Y. It is easily verified that $I(\mathfrak{X}+\mathfrak{Y}) \leqq \mathrm{I}\mathfrak{X}+\mathrm{I}\mathfrak{Y}$, $\mathrm{Q}(\mathfrak{X}+\mathfrak{Y}) \leqq \mathrm{Q}\mathfrak{X}+\mathrm{Q}\mathfrak{Y}$, and $(\mathfrak{X}+\mathfrak{Y}) \cap \mathrm{E}\mathfrak{A} = \mathfrak{X} \cap \mathrm{E}\mathfrak{A}+\mathfrak{Y} \cap \mathrm{E}\mathfrak{A}$. Obviously $\mathfrak{X}+\mathfrak{Y}$ is always E-closed. Thus from these properties and corollary C5 we have,

COROLLARY (C6): *Let $\mathfrak{X}$, $\mathfrak{Y}$ be any $\{\mathrm{I}, \mathrm{Q}\}$-closed classes. Then $\mathfrak{X}+\mathfrak{Y}$ is locally coalescent (resp. persistent) if and only if $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}+\mathfrak{Y} \cap \mathrm{E}\mathfrak{A}$ is locally coalescent (resp. persistent).*

4. Coalescent classes with finiteness conditions

Using the notation of [8] and [9] we let

(a) *Min*, *Min-*$\lhd$, *Min-si*, *Min-asc*

denote the classes of Lie algebras satisfying the minimal condition for (respectively) subalgebras, ideals, subideals, and ascendant subalgebras (H asc L if there is a chain $H = H_0 \lhd H_1 \lhd \cdots H_\sigma = L$, for some ordinal σ so that for all $\alpha < \sigma$, $H_\alpha \lhd H_{\alpha+1}$ and for all limit ordinals $\beta \leqq \sigma$, $H_\beta = \cup_{\alpha<\beta} H_\alpha$). We let

(b) *Max*, *Max-*$\lhd$, *Max-si*, *Max-asc*

denote the corresponding maximal condition classes. We also denote by

(c) $\mathfrak{G}^I$

the class of Lie algebras in which every subideal is finitely generated.

For fields of characteristic zero, it follows from Hartley ([2] p. 259, theorem 5) that the class $\mathfrak{F}$ of finite dimensional Lie algebras is coalescent; and from Stewart [8], the class *Min-si* is also coalescent and the question as to whether *Max-si* is coalescent is raised.

We will show that all the classes in (a), (b) and (c) apart from possibly *Max*-$\lhd$, are coalescent (for fields of characteristic zero, false for characteristic $p > 0$).

It is easy to show that if $\mathfrak{X}$ denotes any of the classes in (a), (b) and (c) then (for $X \neq$ Min-$\lhd$)

(d) $$\mathfrak{X} = \{\mathrm{Q}, \mathrm{E}\}\mathfrak{X},\ \mathfrak{X} \cap \mathrm{E}\mathfrak{A} = \mathfrak{G} \cap \mathrm{E}\mathfrak{A}$$

If $X \neq$ *Min*-$\lhd$ or *Max*-$\lhd$, then

(e) $$\mathfrak{X} = \{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}\mathfrak{X},\ \mathfrak{X} \cap \mathrm{E}\mathfrak{A} = \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$$

(Min-$\lhd$ is $\{\mathrm{Q}, \mathrm{E}\}$-closed.)

By Corollary C1 the class $\mathfrak{G} \cap \mathrm{E}\mathfrak{A}$ is coalescent. Now a Lie algebra with a finite series each factor of which is finitely generated is necessarily also finitely generated. Thus if $\mathfrak{X}$, $\mathfrak{Y}$ denote any of the classes in (a), (b) or (c), and $\mathfrak{X}$, $\mathfrak{Y} \neq$ Min-$\lhd$ then

$$(\mathfrak{X}+\mathfrak{Y}) \cap \mathrm{E}\mathfrak{A} = \mathfrak{X} \cap \mathrm{E}\mathfrak{A} + \mathfrak{Y} \cap \mathrm{E}\mathfrak{A} = \mathfrak{G} \cap \mathrm{E}\mathfrak{A},$$

a coalescent class (for characteristic zero; false for characteristic $p > 0$). We note that $\mathfrak{X}+\mathfrak{Y}$ is the smallest E-closed class containing $\mathfrak{X}$ and $\mathfrak{Y}$. Thus if $\mathfrak{X}$ is E-closed then $\mathfrak{X}+\mathfrak{X} = \mathfrak{X}$.

THEOREM (D): *Let $\mathfrak{X}$ be a* Q*-closed class of Lie algebras such that $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is coalescent. Suppose that H si L, K si L, $J = \langle H, K\rangle$, and there exists $A \lhd J$, A si L, with $(H+A)/A \in \mathfrak{X}$ and $(K+A)/A \in \mathfrak{X}$. Then J si L and there exists $B \lhd J$ with $J/B \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$.*

PROOF. By the derived join theorem, there is an integer r such that $J^{(r)}$ si L. Since $J^{(r)} \lhd J$, then $B = (A+J^{(r)}) \lhd J$. Clearly J/B is a join of two $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$-subideals and so is in $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$($\mathfrak{X}$ is Q-closed). By lemma 1, $B \lhd^n L$ for some n.

Let $B = B_n \lhd B_{n-1} \lhd \ldots \lhd B_1 \lhd B_0 = L$ be the ideal closure series of B in L. Then J idealises each B_i for $i = 0, 1, \cdots, n$. Let us fix i, $0 \leqq i \leqq n-1$. Since $\mathfrak{X} = \mathrm{Q}\mathfrak{X}$ and $J^{(r)} \leqq B \leqq B_{i+1}$, then $(H+B_{i+1})/B_{i+1} \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ and $(K+B_{i+1})/B_{i+1} \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$. Furthermore $(J+B_{i+1})/B_{i+1} = \langle (H+B_{i+1})/B_{i+1},\ (K+B_{i+1})/B_{i+1}\rangle$, the join of two $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$-subideals of $(J+B_i)/B_{i+1}$. Therefore as $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is coalescent, $(J+B_{i+1})/B_{i+1}$

si $(J+B_i)/B_{i+1}$, which implies that

$$J+B_{i+1} \text{ si } J+B_i \quad 0 \leqq i \leqq n-1.$$

So we have a series $J = J+B_n$ si $J+B_{n-1}$ si $\cdots$ si $J+B_0 = L$. Thus J si L, and theorem D is proved.

REMARK: To obtain J si L in theorem D, it is enough to insist that $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ be a class satisfying; in any Lie algebra, the join of a pair of $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$-subideals is always a subideal.

The derived join theorem is also true for groups (see Roseblade [5]) and so is lemma 1 (see Robinson [3]). So we have the group theoretic analogue of theorem D,

COROLLARY (D1): *Let $\mathfrak{X}$ be a class of groups closed under the taking of quotients and such that the class of soluble $\mathfrak{X}$-groups is coalescent. Let H sn G, K sn G, $J = \langle H, K\rangle$, and $A \lhd J$, A sn G such that HA/A, KA/A $\in \mathfrak{X}$. Then there exists $B \lhd J$ with $J/B \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ and J sn G. (where $\mathrm{E}\mathfrak{A}$ denotes the class of soluble groups).*

THEOREM (E): *Suppose that $\mathfrak{X} = \{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}\mathfrak{X}$ is a class of Lie algebras. Then $\mathfrak{X}$ is coalescent if and only if $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is coalescent.*

PROOF: Clearly $\mathfrak{X} = \mathrm{N}_0\mathfrak{X}$. By the derived join theorem $\mathrm{E}\mathfrak{A}$ is persistent. Thus if $\mathfrak{X}$ is coalescent then so is $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$. On the other hand let $J = \langle H, K\rangle$, H si L, K si L and $H, K \in \mathfrak{X}$. By the derived join theorem, there is an r such that $J^{(r)} \lhd^r L$, and $J^{(r)} \in \mathfrak{X}$(for $\mathfrak{X}$ is subjunctive, and by corollary B1 or B2). Let $J^{(r)}$ be the A of theorem D. Then the hypothesis of theorem D is satisfied (taking $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ as coalescent) and so J si L. But $J/J^{(r)} \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$, being the join of two $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$-subideals, and since $\mathfrak{X} = \mathrm{E}\mathfrak{X}$ and $J^{(r)} \in \mathfrak{X}$, then $J \in \mathfrak{X}$. So $\mathfrak{X}$ is coalescent. This proves theorem E.

REMARK: A similar statement holds for groups and provides an alternative method for proving the coalescence for such classes whose soluble groups are well behaved.

Clearly if $\mathfrak{X}$, $\mathfrak{Y}$ are both $\{\mathrm{I}, \mathrm{Q}\}$-closed then $\mathfrak{X}+\mathfrak{Y}$ is $\{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}$-closed. So we have

THEOREM (F): *If $\mathfrak{X}$ and $\mathfrak{Y}$ are $\{\mathrm{I}, \mathrm{Q}\}$-closed classes then $\mathfrak{X}+\mathfrak{Y}$ is coalescent if and only if $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}+\mathfrak{Y} \cap \mathrm{E}\mathfrak{A}$ is coalescent.*

From this and the remarks before theorem D we have

COROLLARY (F1): *Let $\mathfrak{X}$, $\mathfrak{Y}$ denote any of the classes Min, Min-si, Min-asc, Max, Max-si, Max-asc* and $\mathfrak{G}^I$. *Then (over fields of characteristic zero) $\mathfrak{X}+\mathfrak{Y}$ is coalescent. In particular $\mathfrak{X}$ is coalescent.*

We note that the class $Min\text{-}si + Max\text{-}si$ is the Lie theoretic analogue of the class of *minimax* groups which is also coalescent (see Robinson [4]).

From (d) and theorem D we note that if $H, K \in Max\text{-}\lhd$ and H, K si L, then (over fields of characteristic zero) if $J = \langle H, K \rangle$, J si L. We will show that $Min\text{-}\lhd$ is coalescent (we still cannot decide whether $Max\text{-}\lhd$ is persistent).

Let $H, K \in Min\text{-}\lhd$, H si L, K si L, $J = \langle H, K \rangle$. Let

$$A = \bigcap_{n=1}^{\infty} H^n \text{ and } B = \bigcap_{n=1}^{\infty} K .$$

By a result of Schenkman [7], $A \lhd L$ and $B \lhd L$. Let $C = A+B$. By $Min\text{-}\lhd$, there exist integers m, n with $A = H^m$, $B = K^n$. Thus $H/A \in Min\text{-}\lhd \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$, which is coalescent (by Hartley [2]). Hence $J/C \in \mathfrak{F} \cap \mathfrak{N}$ and J/C si L/C. Now A has Min-H (the minimal condition on ideals of H contained in A) and so Min-J. Similarly B has Min-J and since $B \cap A \lhd J$, $B/B \cap A \cong C/A$ has Min-J (as a J-module). Thus C has Min-J and so $J \in Min\text{-}\lhd$ and J si L. This completes the proof of $Min\text{-}\lhd$ is coalescent (over fields of characteristic zero). We note that if $Min\text{-}\lhd^n$ denotes the class of Lie algebras satisfying the minimal condition on n-step subideals, then $Min\text{-}\lhd^n$ is coalescent, by a similar proof. $(n > 0)$.

REMARK: Over fields of characteristic $p > 0$, the classes in corollary F1 are not persistent. This follows from corollary C6 and a forthcoming paper of mine which shows that the class $\mathfrak{F}$ is not persistent in characteristic $p > 0$.

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