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REDUCTION OF THE PROOF OF THE NON-RATIONALITY OF A NON-SINGULAR CUBIC THREEFOLD TO A RESULT OF MUMFORD

J. P. Murre

Let X be a non-singular cubic threefold in 4-dimensional projective space $\boldsymbol{P}_4$, defined over an algebraically closed field k.

If k is the field $\boldsymbol{C}$ of complex numbers Clemens and Griffiths [4] have proved that X is not a rational variety. After this another proof, again for $k = \boldsymbol{C}$, has been given by Mumford; this proof is outlined in Appendix C of [4]. The principal tool in both proofs is the intermediate Jacobian of the threefold; this is, in this case, a principally polarized abelian variety. One shows that the rationality assumption for X has as a consequence that the intermediate Jacobian of the threefold is isomorphic, *as polarized abelian variety*, to a product of Jacobians of curves ([4], 3.26). The impossibility of this consequence is obtained via an investigation of the singularities of the 'θ-divisors'. Mumford proves that the intermediate Jacobian of X is isomorphic, as polarized abelian variety, to a so-called Prym variety. This Prym variety is associated with X via the geometry of lines on X (see section 2.1 for a precise description). From his very detailed study of the singularities of the 'θ-divisor' on Prym varieties (see [4], Appendix C page 354 and 355) Mumford concludes that the Prym variety associated with X is not the product of Jacobians of curves. This last part of Mumford's proof is essentially algebraic.

In the case of a field of arbitrary characteristic we don't have the intermediate Jacobian at our disposal. However in [12] we have shown that the Prym variety associated with X can also be studied via the Chow group of 1-dimensional cycle-classes on X. Moreover, by Mumford's general theory of Prym varieties, a Prym variety has a canonical principal polarization (see [11]). In the case $k = \boldsymbol{C}$ the polarization on the intermediate Jacobian is studied via the classical cohomology on X; it is therefore natural to use, in the case of an arbitrary field, the étale cohomology on X in order to get information concerning the polarization of the Prym variety. In doing so we get the following theorem, which is the main result of this paper:

THEOREM: *Let char. $(k) \neq 2$. The assumption that X is a rational variety implies that the canonically polarized Prym variety associated with X,* *is isomorphic, as polarized abelian variety, to a product of Jacobian varieties of curves.*

Combining this with the last part of Mumford's proof, one has the following: [1 2])

COROLLARY (*of the theorem and Mumford's proof*): *Let char.* $(k) \neq 2$. *Let X be a non-singular cubic threefold in 4-dimensional projective space defined over k. Then X is not a rational variety.*

In Section 1 we have collected some auxiliary results; in Section 2 we state the results of [12] which are needed for our present paper. In Section 3 we adopt the rationality assumption and prove the above theorem. Finally, in an appendix, we answer a question raised by Mumford concerning a universal property of the Prym associated with X.

I should like to thank Mumford, Deligne and Jouanolou for stimulating correspondence or discussion on the topic of this paper.

1. Notations and auxiliary results

1.1. *Notations*

Let k be an algebraically closed field of characteristic $p \neq 2$. Let l be a prime number, $l \neq p$. Choose, once for all, a (non canonical) identification

$$\boldsymbol{Z}_l(1) = \boldsymbol{\mu} = \varprojlim_n \boldsymbol{\mu}_{l^n} \xrightarrow{\sim} \boldsymbol{Z}_l.$$

In the following canonical isomorphism means: *canonical after choice of this identification.*

For an abelian variety A the Tate group is denoted by $T_l(A)$:

$$T_l(A) = \varprojlim_n A_{l^n}$$

and put

$$E_l(A) = T_l(A) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l.$$

[1] The part of Mumford's proof which is needed is the part dealing with the question when polarized Pryms are Jacobians. For this see [11] § 7, in particular the last paragraph preceding the appendix.

[2] Manin has informed me that Tjurin also has proved that the Prym variety associated with a cubic is not a Jacobian of a curve and that an outline of this proof is in Tjurin's paper in Uspekhi, 1972, No. 5, on p. 30–31. Since, at the time of writing this footnote, the translation is not yet available, I don't know in how far Turin's methods overlap or supplement the one in this paper. (Forthcoming translation in Russian Math. Surveys).

For a variety (or scheme) X write

$$H^i(X) = \varprojlim_n H^i(X, \mathbf{Z}/l^n\mathbf{Z}) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$$

where the cohomology is with respect to the étale topology.

Finally, $A(X)$ denotes the Chow ring of X in the sense of Chow [3]:

$$A(X) = \oplus A^i(X)$$

where $A^i(X)$ is the group of cycle classes, with respect to *rational equivalence*, of *co*dimension i.[3] Moreover by $A^i_{\text{alg}}(X)$ we denote those classes which are *algebraically equivalent* to zero (and which are of codimension i).

1.2. *Correspondences between curves*

Let C and C' be irreducible, non-singular curves, proper over k and let $\Sigma \subset C \times C'$ be a correspondence between C and C' with dim. $\Sigma = 1$. In general a divisorial correspondence defines a homomorphism of abelian varieties $\text{Alb}(C) \to \text{Pic}\,(C')$; in our case of curves this may also be considered as a homomorphism $\sigma : \text{Pic}\,(C) \to \text{Pic}\,(C')$. Therefore Σ defines:

$$\sigma_{\text{alg}} : E_l(\text{Pic}\,(C)) \to E_l(\text{Pic}\,(C')).$$

On the other hand, using Poincaré duality, Σ defines also (cf. [7], 1.2 and 1.3):

$$\sigma_{\text{top}} : H^1(C) \to H^1(C').$$

Note that *formally* both maps are defined by the same formula:

$$\text{class}\,(\mathfrak{A}) \to q_*\{p^*(\text{class}\,(\mathfrak{A}))\,.\,\text{class}\,(\Sigma)\} \tag{1}$$

where p (resp. q) denotes the projection from $C \times C'$ to C (resp. to C'). Furthermore, for the group of points of order l^n one has canonically ([2], cor. 4.7):

$$\text{Pic}\,(C)_{l^n} \xrightarrow{\sim} H^1(C, \mu_{l^n})$$

and this gives 'canonically' $E_l(\text{Pic}\,(C)) \cong H^1(C)$ and similarly for C'.

LEMMA 1: *With the above canonical identifications $\sigma_{\text{alg}} = \sigma_{\text{top}}$ (and we write σ in the following).*

[3] In [12], page 197 we have used subscripts for the $A(-)$ to indicate the *dimension* of the cycles. Since in this paper we have to use mappings of the Chow groups into cohomology we prefer, now, to use superscripts to indicate *codimension.*

PROOF: *Case 1.* Suppose $\Sigma = \Gamma_\phi$ with $\phi : C' \to C$ a morphism. In that case we have that σ_{alg} is induced from $\phi^*_{\text{alg}} : \text{Pic}\,(C) \to \text{Pic}\,(C')$ and σ_{top} from $\phi^*_{\text{top}} H^1(C, \boldsymbol{G}_m) \to H^1(C', \boldsymbol{G}_m)$. Looking to the description of these maps in terms of invertible sheaves on the one hand and cocycles on the other hand we have $\phi^*_{\text{alg}} = \phi^*_{\text{top}}$ after the usual identifications Pic $(C) = H^1(C, \boldsymbol{G}_m)$ and Pic $(C') = H^1(C', \boldsymbol{G}_m)$.

Case 2. Suppose $\Sigma = {}^t\Gamma_\phi$ with $\phi : C \to C'$ a morphism. Then σ_{alg} is, by definition, induced (via the points of order l^n) by the homomorphism of Albanese varieties $\phi_* : \text{Alb}\,(C) \to \text{Alb}\,(C')$. The dual homomorphism is $\phi^* : \text{Pic}\,(C') \to \text{Pic}\,(C)$, i.e. the one coming from ${}^t\Sigma$ and therefore σ_{alg} is the dual of $({}^t\sigma)_{\text{alg}}$ where ${}^t\sigma$ belongs to ${}^t\Sigma$ (see formula I, p. 186, [10]). On the other hand let $\phi_* : H^1(C) \to H^1(C')$ be the usual map for cohomology (see [7] 1.2), then $\sigma_{\text{top}} = \phi_*$ by [7], 1.3.7 (iii); hence it is the dual of $({}^t\sigma)_{\text{top}} = \phi^*$ (again by [7], 1.3.7(iii)). The assertion follows now by duality from Case 1.

Case 3. Suppose Σ is an irreducible, *non-singular*, curve on $C \times C'$. Put $i : \Sigma \to C \times C'$, $p_1 = p \cdot i$ and $q_1 = q \cdot i$. The mappings are defined by formula (1) above; using the so-called projection formula (see, for instance, [7], p. 362 and 363) the right hand side of (1) can be written as $q_*[i_*\{i^*p^*(\text{class}\ (\mathfrak{A})) \cdot 1\}] = (q_1)_*\ [(p_1)^*(\text{class}\ (\mathfrak{A}))]$, both in the sense of algebraic cycle classes and in the sense of cohomology. The assertion follows then from case 1 and 2, applied respectively to $p_1 : \Sigma \to C$ and to $q_1 : \Sigma \to C'$.

Case 4. Σ arbitrary. By formula (1) in both cases the homomorphisms are linear in the class of Σ and they depend only on the linear equivalence class of Σ on $C \times C'$ (in the case of cohomology this follows from [7] 1.2.1). By [9], lemma 2 the linear system $|\Sigma + H_n|$, where H_n denotes a hypersurface section of degree n, contains a non-singular irreducible curve Σ' provided n is large. The assertion follows now from case 3 applied to Σ' and to H_n.

1.3. *Resumé of some results on monoidal transformations*

Here we collect some results which are essentially contained in [13], [5] and [6]. In this section X denotes a projective, non-singular, irreducible 3-dimensional variety and $s : Y \to X$ a non-singular, irreducible curve in X (lemma 2 and 3 hold more generally for dim $X = n$, dim $Y = n-2$). Let $X' = B_Y(X)$ be obtained by blowing up X along Y. Let Y' be the total transform of Y in X'.

$$(2)\qquad \begin{array}{ccc} Y' & \overset{r}{\hookrightarrow} & X' = B_Y(X) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \underset{s}{\hookrightarrow} & X \end{array}$$

(a) *Behaviour of the Chow groups*

LEMMA 2: *For the additive structure there are isomorphisms α and β, inverse to each other, as follows*

$$A^{\cdot}(X) \oplus A^{\cdot-1}(Y) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A^{\cdot}(X')$$

with $\alpha = (f_*, -g_*r^*)$ and $\beta = f^* + r_*g^*$. Moreover the same is true if $A(-)$ is replaced by $A_{\mathrm{alg}}(-)$. [4]

PROOF: [13], proposition 13 and lemma 1 on page 481 (this reads in our present terminology $g_*r^*r_*g^* = -id_Y$). [5]

(*b*) *Behaviour of the cohomology groups*

LEMMA 3: *For the additive structure there are isomorphisms α and β, inverse to each other, given by the same formulas as in lemma 2, as follows*

$$H^{\cdot}(X) \oplus H^{\cdot-2}(Y) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} H^{\cdot}(X')$$

PROOF: This is [6], 4.2.2. There the additional assumption is made that Y is the intersection of two hyperplanes; however, that assumption is only used to prove the following (formula 4.2.10 in [6]):

$$(3)\qquad g_*r^*r_*g^* = -id_Y$$

Therefore it suffices here to prove this formula. We borrowed the arguments from [13], p. 481–482.

First, consider *in the Chow group* $A^1(X')$ the class of Y', i.e. $r_*(1_{Y'})$. We claim that in the Chow group $A^1(Y')$

$$(4)\qquad r^*r_*(1_{Y'}) = -\zeta + \zeta_1$$

where $g_*(\zeta_1) = 0$ and $g_*(\zeta) = 1_Y$. In order to prove (4) we take a

[4] The above formula should be interpreted explicitly as follows:

$$A^q(X) \oplus A^{q-1}(Y) \rightleftarrows A^q(X)$$

for $q = 0, 1, 2$ or 3 with $A^{-1}(-) = 0$. Similar interpretations for similar formulas below.

[5] Jouanolou has informed me that lemma 2, and similar statements for Y of higher codimension, can also be obtained from his results in [5] section 9. His method works also for cohomology.

sufficiently general linair space L in the ambient projective space $\boldsymbol{P}_N$ of X, of dimension (N-dim Y-2). Consider the cone $C = C_{(Y, L)}$ with vertex L and base Y. Consider a generic point Q of Y, the tangent space $T_{X, Q}$ to X in Q meets L only in one point; from this it follows easily that X and the cone are transversal in Q. Therefore we have

$$X \cdot C = D = D_1 + D_2$$

where the divisor D is the sum of a variety D_1, going through Y and such that Q is simple on D_1, and a divisor D_2 such that $Q \notin \text{Supp}\ (D_2)$. Hence $D_2 \cdot Y$ is defined. Finally, we take $D^* \sim D$ such that $D^* . Y$ is defined. From the fact that $D^* . Y$ and $D_2 . Y$ are defined follows that $g_*(Y' . f^{-1}(D^*)) = 0$ and $g_*(Y' . f^{-1}(D_2)) = 0$. Furthermore

$$f^{-1}(D_1) = 1 . Y' + 1 . f^{-1}[D_1]$$

where $f^{-1}[D_1]$ is the so-called *proper* transform ([15], p. 4). The assertion about the coefficient of Y' is justified because of the fact that in a generic point Q^* of Y' we have for the tangentspaces

$$T_{\Gamma_f, Q^*} \not\subset T_{D_1, Q} \times T_{X', Q'}$$

where Q, resp. Q', is the projection of Q^* on X, resp. on X'. Namely, if we take in X a 'generic arc' through Q we get by lifting in Γ_f an arc hitting Y' in Q^* and the tangent to this arc is not vertical and its projection on X is the tangent to the original arc, hence outside $T_{D_1, Q}$. Moreover we have

$$f^{-1}[D_1] . Y' = Z + Z_*$$

where Z is a variety with $g_*(Z) = Y$ and $g_*(Z_*) = 0$. In order to see this we note (see [15], 18) that the points of Y' above a point $P \in Y$ correspond 1–1 with the linear subspaces contained and of codimension 1 in the tangentspace $T_{X, P}$ to X in P and containing the tangentspace $T_{Y, P}$. Now a generic point $\bar{Q}$ of Z corresponds with the tangentspace $T_{D_1, Q}$ where Q is the projection of $\bar{Q}$ on X; $T_{D_1, Q}$ is rational over $k(Q)$ and from this follows $k(\bar{Q}) = k(Q)$. Finally, the component Z_* has as projection on X singular points of D_1 and from this follows easily $g_*(Z_*) = 0$ Now we have

$$Y' \sim f^{-1}(D^*) - f^{-1}(D_2) - f^{-1}[D_1]. \tag{5}$$

Therefore if we put

$$\zeta = \text{class } (Z) \text{ and } \zeta_1 = \text{class } \{Y' . f^{-1}(D^*) - Y' . f^{-1}(D_2) - Z_*\} \tag{6}$$

then the relation (4) is fulfilled because $r^* r_*(1_{Y'})$ is the class of the intersection of the right hand side of (5) with Y' when the class of that intersection is *considered as class on* Y'.

Returning to cohomology we now remark that the relation (4) *also holds if considered in* $H^2(Y')$, with the same relations $g_*(\zeta) = 1_Y$ and $g_*(\zeta_1) = 0$. This follows by applying the 'cycle maps' $\gamma : A^{\cdot}(-) \to H^{2\cdot}(-)$([7], p. 363).

In order to prove (3) we use the crucial formula, proved by Jouanolou ([5], th. 4.1),

$$r^* r_*(y') = y' \cdot c_1(N_{Y'/X'}) \tag{7}$$

where $y' \in H^{\cdot}(Y')$ and where c_1 is the first Chern class of the normal bundle of Y' in X'. Keeping in mind that $c_1(N_{Y'/X'}) = r^*r_*(1_{Y'})$, we see that in order to prove (3) we must prove

$$g_*\{g^*(y) \,.\, r^*r_*(1_{Y'})\} = -y \tag{3'}$$

for $y \in H^{\cdot}(Y)$ and where we have applied (7) to $y' = g^*(y)$. Using (4), in the cohomological sense, and the projection formula for $g : Y' \to Y$ we get $g_*\{g^*(y) \,.\, r^*r_*(1_{Y'})\} = -g_*\{g^*(y) \,.\, \zeta\} + g_*\{g^*(y) \,.\, \zeta_1\} = -y \,.\, g_*(\zeta) + y \,.\, g_*(\zeta_1) = -y \,.\, 1_Y + 0 = -y$. This completes the proof of lemma 3.

(c) *Behaviour of the cohomology ring*

In fact here we need only some special results. We use the following notation: use . for the product sign in $H^{\cdot}(-)$; however if we are in complementary dimension, then *after application of the orientation map* we use the symbol $\cup$; i.e. $a \cup b$ is always an element of $\boldsymbol{Q}_l$. Furthermore, for convenience, rewrite (7) as

$$r^*r_*(y') = y' \,.\, r^*r_*(1_{Y'}) \tag{7'}$$

LEMMA 4: *With the above notations, let* $y_1, y_2 \in H^1(Y)$. *Then*:

(i) $r_*g^*(y_1) \,.\, r_*g^*(y_2) = r_*\{r^*r_*(1_{Y'}) \,.\, g^*(y_1 \,.\, y_2)\}$;

(ii) $r_*g^*(y_1) \cup r_*g^*(y_2) = -y_1 \cup y_2$.

PROOF: First note that now our assumptions are dim $X = 3$ and dim $Y = 1$.

(i) Using the projection formula and (7') we have $r_*g^*(y_1) \,.\, r_*g^*(y_2) = r_*\{r^*(r_*g^*(y_1)) \,.\, g^*(y_2)\} = r_*\{r^*r_*(1_{Y'}) \,.\, g^*(y_1) \,.\, g^*(y_2)\} = r_*\{r^*r_*(1_{Y'}) \,.\, g^*(y_1 \,.\, y_2)\}$.

(ii) The left hand side of (ii) is obtained from the left hand side of (i) after application of the orientation map for X'. Since the orientation map of X' and Y' commute with r_* the result is the same if we apply the orientaton map of Y' on $r^*r_*(1_{Y'}) \,.\, g^*(y_1 \,.\, y_2)$. Applying the same remark to $g : Y' \to Y$ we see that we get $\delta_Y[g_*\{r^*r_*(1_{Y'}) \,.\, g^*(y_1 \,.\, y_2)\}]$

where δ_Y is the orientation map for Y. Using the relations (4) and the projection formula we get

$$\delta_Y[g_*\{(-\zeta+\zeta)\,.\,g^*(y_1\,.\,y_2)\}] = \delta_Y[g_*(-\zeta+\zeta_1)\,.\,(y_1\cdot y_2)] = \\ = -\delta_Y[y_1\,.\,y_2] = -y_1 \cup y_2.$$

(d) *Remark*

If we take for *Y a point instead of a curve* then we have decompositions similar as in lemmas 2 and 3, both for the Chow ring and for cohomology (cf. also footnote 4). We don't give these lemmas in detail here, partly because we are eventually only interested in $A^2_{\mathrm{alg}}(\ldots)$ and $H^3(\ldots)$ and a point Y does not give contributions to these terms.

1.4. *Algebraic families of cycles*

DEFINITION: *Let U be a non-singular, quasi-projective variety. A map $\rho : U \to A^q(X)$ is called algebraic if for every $u_0 \in U$ there exists an open Zariski neighbourhood U_0 and $\mathfrak{z} \in A^q(U_0 \times X)$ such that for $u \in U_0$ we have $\rho(u) = \mathfrak{z}(u)$ where $\mathfrak{z}(u) =$ class $[pr_X\{(u\times X)\,.\,\mathfrak{z}\}]$.*

LEMMA 5: *The assumptions are as in* 1.3. *Let $\rho : U \to A^q(X')$ be an algebraic map. Then $pr_X\,.\,\alpha\,.\,\rho : U \to A^q(X)$ and $pr_Y\,.\,\alpha\,.\,\rho : U \to A^{q-1}(Y)$ are also algebraic. There are similar statements with algebraic families on X, resp. on Y, and where we make the composite with β.*

PROOF: Consider for instance $U \to A^{q-1}(Y)$. This is defined (in U_0) by the correspondence

$${}^t\Gamma_g\,.\,\Gamma_r\,.\,\mathfrak{z} \in A^{q-1}(U_0 \times Y)$$

where Γ denotes the graph.

2. Resumé of some results of [12]

2.1. *From now on X denotes a non-singular, cubic threefold* in $\boldsymbol{P}_4$ defined over an algebraically closed field of characteristic not two.

Fix a sufficiently general line l on X (see [12], prop. 1.25 for precise conditions). The 2-dimensional linear spaces (shortly 2-planes) L through l are parametrized by a projective space $\boldsymbol{P}_2$. Let $\Delta \subset \boldsymbol{P}_2$ be the set of 2 planes as follows:

$$\Delta = \{L;\, X\,.\,L = l+l'+l'',\, l' \text{ and } l'' \text{ lines on } X\}.$$

Then Δ is a non-singular, absolutely irreducible curve in $\boldsymbol{P}_2$ of degree 5 and genus 6 ([12], 1.25ii).[6] Furthermore let

[6] Δ and $\hat{\Delta}$ are denoted in [12] by H and $\mathscr{H}$ respectively.

$$\hat{\Delta} = \{l'; l' \text{ line on } X \text{ such that } l \cap l' \neq \emptyset\},$$

then $\hat{\Delta}$ is an absolutely irreducible curve on the Fano surface of lines on X([12] 1.25iv). There is a natural morphism $q : \hat{\Delta} \to \Delta$ given by $q(l') = L$, where L is the 2-plane spanned by l and l'; clearly $q^{-1}(L) = \{l', l''\}$. In fact, due to the assumption that l is sufficiently general it follows that $q : \hat{\Delta} \to \Delta$ is an *étale*, 2–1 *covering* ([12] 1.25iv). By Mumford's general theory of Prym varieties [11] we have therefore a *Prym variety*

$$i : P(\hat{\Delta}/\Delta) \hookrightarrow J(\hat{\Delta}) \tag{8}$$

where $J(\hat{\Delta})$ is the Jacobian variety of $\hat{\Delta}$ (cf. also [12], p. 198). We call this the *Prym variety associated with* X, obtained via the geometry of lines on X.

2.2. Consider the restriction to l of the tangent bundle over X and let V be the bundle of associated projective spaces of 1-dimensional linear subspaces.

For $S \in l$ consider the fibre V_S; in V_S there are $5+1$ special points corresponding with the 6 lines on X through S (and l is one of them) ([12] 1.25vi) Varying S over l the 5 points give a curve in V, this curve is *non-singular* ([12], prop. 2.5) and can be identified with $\hat{\Delta}$ ([12] 2.4). The 6$^{\text{th}}$-point in V_S, corresponding with l itself, gives rise to a rational, non-singular curve I and $I \cap \hat{\Delta} = \emptyset$ in V([12], 2.5). Let X be obtained by applying to V a monoidal transformation with centre $\hat{\Delta} \cup I$; then X' is non-singular and by [12], equation (51) we have

$$A^2_{\text{alg}}(X') \cong J(\hat{\Delta})$$

where $J(\hat{\Delta})$ is the Jacobian variety of $\hat{\Delta}$. Furthermore there is a morphism ([12], 4.2) $\phi : X' \to X$, which is generically 2–1 [12], 4.6). Consider the corresponding homorphisms for the Chow groups

$$A^2_{\text{alg}}(X) \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} A^2_{\text{alg}}(X') = J(\hat{\Delta}). \tag{9}$$

Now the main results, 10.8 and 10.10, of [12] can be summarized as:

LEMMA 6:
(i) $\phi_* . \phi^* = 2$
(ii) ϕ^* *is factorized* (*cf.* (8)):

$$\begin{tikzcd} A^2_{\text{alg}}(X) \arrow[rr, "\phi^*"] \arrow[dr, "\phi^*"'] & & J(\hat{\Delta}) \\ & P(\hat{\Delta}/\Delta) \arrow[ur, hook, "i"'] & \end{tikzcd}$$

and ϕ^* *is onto the* $P(\hat{\Delta}/\Delta)$.

(iii) $A^2_{\mathrm{alg}}(X) = P \oplus T$ *with*

  (a) $T = \ker(\phi^*)$ *consists of 2-torsion elements,*

  (b) $P = \mathrm{Im}\,(\phi_*)(P(\hat{\Delta}/\Delta))$ *and* $(\phi^*|P): P \xrightarrow{\sim} P(\hat{\Delta}/\Delta)$

(iv) $(\phi^* \,.\, \phi_*)|P(\hat{\Delta}/\Delta) = 2$

For the statements which are not explicit in [12] 10.8 or 10.10 we refer to the proof of [12] 10.10 on page 201.

3. Consequences of the assumption that X is rational

From now on we make the assumption that X is birational with $\boldsymbol{P}_3$ and we study the consequences for the Prym $P(\hat{\Delta}/\Delta)$ associated with X and for its canonical polarization.

3.1. According to Abhyankar [1] there exists a commutative diagram of the following type

$$\begin{tikzcd}[column sep=small] & X^* = B_{\{\Omega_i\}}(\boldsymbol{P}_3) \arrow[dl, "\lambda"'] \arrow[dr, "f"] & \\ X & & \boldsymbol{P}_3 \arrow[ll, dashed] \end{tikzcd}$$

where the dotted arrow is the given birational transformation, where f is a *sequence of monoidal transformations* with as centres *non-singular curves* $\Omega_i(i = 1, \cdots, 2)$ or *points* (which we have suppressed in the notation) and where λ is a *birational* morphism.

A. Consequences of the rationality assumption for $A^2_{\mathrm{alg}}(X)$.

3.2. Let X' and X^* be as in 2.2 and 3.1 respectively. Put as abbreviation $J' = J(\hat{\Delta})$ and $J^* = \prod_i J(\Omega_i)$. Using lemma 3 we have $A^2_{\mathrm{alg}}(X') = J(\hat{\Delta})$ and $A^2_{\mathrm{alg}}(X^*) = \prod_i J(\Omega_i)$.[7] The situation can be summarised in the following diagram (cf. also lemma 6):

(10)
$$\begin{tikzcd} & P(\hat{\Delta}/\Delta) \arrow[r, hook, "i"] \arrow[dd, dashed, "\rho"] & J' = A^2_{\mathrm{alg}}(X'). \arrow[dl, "\phi_*"] \\ P \oplus T = A^2_{\mathrm{alg}}(X) \arrow[ur, "\phi^*"] \arrow[dr, hook, "\lambda^*"'] & & \\ & J^* = A^2_{\mathrm{alg}}(X^*) & \end{tikzcd}$$

[7] An equality of this kind has to be interpreted as follows: Take a sufficiently large algebraically closed overfield K of k (a so-called 'universal domain'). Take on the one hand the group of the cycle classes which have representatives rational over K and on the other hand the group of the K-rational points of the abelian variety.

Note that $\lambda_* . \lambda^* = 1$ since λ is a birational morphism. Furthermore by lemma 6 (iii) $(\phi^*|P)$ is an isomorphism.

LEMMA 7: *Assuming X to be rational we have:*

(i) $\lambda^* . (\phi^*|P)^{-1} : P(\hat{\Delta}/\Delta) \to J^*$ *defines a homomorphism of abelian varieties* $\rho : P(\hat{\Delta}/\Delta) \to J^*$.

(ii) $T = 0$.

PROOF:

(i) Let ξ be a generic point of $P(\hat{\Delta}/\Delta)$ over k; put $\zeta = (\phi^*|P)^{-1}(\xi)$, then $\zeta \in P$. We want to prove first

$$k(\lambda^*(\zeta)) \subset k(\xi). \tag{11}$$

Let $\mathscr{D} \subset J(\hat{\Delta}) \times \hat{\Delta}$ be the Poincaré divisor of $\hat{\Delta}$. Then $\lambda^* . \phi_* . (\mathscr{D}|P(\hat{\Delta}/\Delta))$ defines, by lemma 5, an *algebraic* map: $P(\hat{\Delta}/\Delta) \to A^2_{\text{alg}}(X^*)$. This map is defined by the following formula (where $\sigma \in P(\hat{\Delta}/\Delta)$ and (0) is the neutral element on $P(\hat{\Delta}/\Delta)$):

$$\sigma \mapsto \lambda^*\{\phi_*(\mathscr{D}(\sigma) - \mathscr{D}(0))\}.$$

REMARK: For the sake of simplicity we have suppressed some other morphisms which also enter in the definition of this map, namely we have

$$P(\hat{\Delta}/\Delta) \hookrightarrow J(\hat{\Delta}) \xrightarrow{\sim} A^1_{\text{alg}}(\hat{\Delta}) \hookrightarrow A^2_{\text{alg}}(X') \underset{\phi_*}{\to} A^2_{\text{alg}}(X) \underset{\lambda^*}{\to} A^2_{\text{alg}}(X^*) \to \prod_i A^1_{\text{alg}}(\Omega_i)$$

where $A^1_{\text{alg}}(\hat{\Delta}) \to A^2_{\text{alg}}(X')$ is defined via the map β in lemma 3 and where $A^2_{\text{alg}}(X^*) \to \prod_i A^1_{\text{alg}}(\Omega_i)$ is defined via the map α in lemma 3; it is precisely for these maps that lemma 5 is used.

Returning to the proof of (i) the above *algebraic* map defines a *homomorphism* of abelian varieties $\psi : P(\hat{\Delta}/\Delta) \to J^*$.[8] Take $\bar{\xi} \in P(\hat{\Delta}/\Delta)$ such that $2\bar{\xi} = \xi$; by lemma 6(iii) and (iv) we have $\phi_*(\bar{\xi}) = \zeta$, hence $\psi(\bar{\xi}) = \lambda^*(\zeta)$ and hence

$$k(\lambda^*(\zeta)) \subset k(\bar{\xi}).$$

Varying $\bar{\xi}$ such that $2\bar{\xi} = \xi$ we have that $\lambda^*(\zeta)$ is invariant, therefore it is invariant under the action of the Galois group of $k(\bar{\xi})/k(\xi)$. Hence it has its coordinates in $k(\xi)$ itself; i.e. $k(\lambda^*(\zeta)) \subset k(\xi)$.

This gives a morphism $\rho : P(\hat{\Delta}/\Delta) \to J^*$ such that

$$\rho(\xi) = \lambda^*\{(\phi^*|P)^{-1}(\xi)\} \tag{12}$$

for a *generic* point ξ on $P(\hat{\Delta}/\Delta)$. However, then by a specialization argument we get that (12) holds for any point ξ' on $P(\hat{\Delta}/\Delta)$. Namely extend the specialization $\xi \to \xi'$ to $(\xi, \bar{\xi}, \zeta) \to (\xi', \bar{\xi}', \zeta')$, then $2\bar{\xi}' = \xi'$ and

[1] Remember that $\phi_* \neq (\phi^*)^{-1}$; in fact $\phi^* . \phi_* = 2$ on $P(\hat{\Delta}/\Delta)$!

$\zeta' = \phi_*(\bar{\xi}')$, hence $\phi^*(\zeta') = \phi^*\phi_*(\bar{\xi}') = 2\bar{\xi}' = \xi'$, i.e. $\zeta' = (\phi^*|P)^{-1}(\xi')$. Furthermore $\rho(\xi') = \psi(\bar{\xi}')$ because both are unique. Finally $\rho(\xi') = \psi(\bar{\xi}') = \lambda^*\phi_*(\bar{\xi}') = \lambda^*(\zeta')$ and this proves the assertion. Applying (12) to the neutral element 0 we see that ρ actually is a homomorphism.

(ii) Let $t_1 \in T$ and $\tau : U \to A^2_{\mathrm{alg}}(X)$ be an algebraic map, with U a non-singular connected variety and u_0, $u_1 \in U$ such that $\tau(u_0) = 0$, $\tau(u_1) = t_1$. Consider the mapping $\nu : U \to J^*$ defined by $\nu = \lambda^* . \tau - \rho . \phi^* . \tau$. For any point $u \in U$, we have by lemma 6 (iii) $\tau(u) = (p, t)$, $p \in P$, $t \in T$ and $\nu(u) = \lambda^*(p)+\lambda^*(t)-\rho\phi^*(p) = \lambda^*(t)$ by the definition of ρ (lemma 7). Hence $Im\,(\nu) \subset J_2^*$. Furthermore $\nu(u_0) = 0$. Since ν is a morphism and U is connected we have that $\nu(u) = 0$ for all $u \in U$. In particular $\lambda^*(t_1) = \nu(u_1) = 0$. Since λ^* is injective we have $t_1 = 0$. Hence $T = 0$.

3.3. Identify P and $P(\hat{\Delta}/\Delta)$ by means of $(\phi^*|P)^{-1}$; then $\rho = \lambda^*$. Using the result $T = 0$, the diagram (10) simplifies with these identifications to (10′):

(10′)
$$\begin{tikzcd}
& A^2_{\mathrm{alg}}(X') = J' \arrow[dl, shift left, "\phi_*"] \\
P(\hat{\Delta}/\Delta) = A^2_{\mathrm{alg}}(X) \arrow[ur, hook, shift left, "i = \phi^*"] \arrow[dr, hook, "\rho = \lambda^*"'] & \\
& A^2_{\mathrm{alg}}(X^*) = J^*
\end{tikzcd}$$

B. Consequences of rationality assumption for $P(\hat{\Delta}/\Delta)$ and its canonical polarization.

3.4. *General result of Mumford* [11]. Let $i : P(\hat{\Delta}/\Delta) \to J(\hat{\Delta})$ be a Prym variety and θ the canonical theta divisor of $J(\hat{\Delta})$. Then

$$i^{-1}(\theta) = 2\Xi \tag{13}$$

where Ξ is a *principal polarization* on $P(\hat{\Delta}/\Delta)$.

3.5. *Main problem of this paper*: Apply this result to the present situation $i : P(\hat{\Delta}/\Delta) \to J(\hat{\Delta})$ as described in 2.1, i.e., to the Prym variety associated with the cubic threefold X. In order to get a coherent notation we write θ' for the canonical theta divisor on J'; hence (13) reads:

$$i^{-1}(\theta') = 2\Xi. \tag{13′}$$

On the other hand we have on $J^* = \prod_i J(\Omega_i)$(see 3.1) the *principal* polarization θ^* with $\theta^* = \sum \theta_i^*$ and

$$\theta_i^* = J(\Omega_1)\times \cdots \times J(\Omega_{i-1})\times \theta_i \times J(\Omega_{i+1})\times \cdots \times J(\Omega_q),$$

where θ_i is the canonical polarization on the Jacobian $J(\Omega_i)$. Furthermore we have the homomorphism $\lambda^* : P(\hat{\Delta}/\Delta) \to J^*$ (see diagram (10')). *Main problem*: do we have

$$(14) \qquad \Xi \equiv (\lambda^*)^{-1}(\theta^*) \text{ modulo algebraic equivalence?}$$

3.6. θ' on J' defines on $T_l(J') \times T_l(J')$ a Riemann form $e^{\theta'}$ ([10] p. 186 or [8] p. 189). Identifying $E_l(J') \cong H^1(\hat{\Delta})$, we have for $\xi, \eta \in T_l(J') \subset E_l(J')$:

$$(15) \qquad e^{\theta'}(\xi, \eta) = \xi \cup \eta.$$

This follows from the construction of the duality theorem for étale cohomology (for coefficients $\boldsymbol{Z}/l^n\boldsymbol{Z}$ and passing to the limit; see [14] 5.5.2 on page 198). There is a similar statement for θ^*.

3.7. Let $\phi : X' \to X$ be the morphism from 2.2; combined with the birational morphism $\lambda : X^* \to X$ we get a rational transformation as indicated below. Moreover since ϕ is generically 2–1, this rational transformation is generically 2–1

$$\begin{tikzcd} & X' \arrow[dl, "\phi"'] \arrow[d, dashed] \\ X & X^* \arrow[l, "\lambda"] \end{tikzcd}$$

By Abhyankar [1] we can construct for $\lambda^{-1} . \phi$ the following commutative diagram:

$$(16) \qquad \begin{tikzcd} & X' \arrow[dl, "\phi"'] & \\ X & & X'' = B_{\{\Delta_j\}}(X') \arrow[ul, "\psi"'] \arrow[dl, "\mu"] \\ & X^* \arrow[ul, "\lambda"] & \end{tikzcd}$$

X'' is obtained by a sequence of monoidal transformations from X', with as centres non-singular curves $\Delta_j (j > 0)$ and points. Putting $\hat{\Delta} = \Delta_0$ we can also say: obtained from V (see 2.2) by a sequence of monoidal transformations with centres $\Delta_j (j \geqq 0)$. Summarizing we have:

$$(16') \qquad \begin{cases} \lambda \text{ and } \psi \text{ are birational morphisms} \\ \phi \text{ and } \mu \text{ are morphisms, generically 2–1} \\ \lambda . \mu = \phi . \psi = \chi \text{ (for abbreviation)} \end{cases}$$

Beside the principally polarized abelian varieties (J', θ^1) and (J^*, θ^*) introduced before we have also to consider (J'', θ'') with

$$J'' = \prod_{j \geqq 0} J(\Delta_j)$$
$$\theta'' = \Sigma\, \theta_j''$$
$$\theta_j'' = J(\Delta_0) \times \cdots \times \theta_j \times \ldots (j^{th} \text{ place}) \quad (j \geqq 0),$$

where the θ_j are the canonical polarizations on the Jacobian varieties $J(\Delta_j)$. Again this is a principally polarized abelian variety and since $J' = J(\Delta_0)$, $\theta' = \theta_0$, we have (modulo algebraic equivalence)

$$\theta' \equiv \psi^{*-1}(\theta'').$$

Finally from (16) we get, using lemma 2, the following commutative diagram (see also (10′)):

$$\begin{tikzcd}
 & J' = A^2_{\mathrm{alg}}(X') \arrow[dr, hook, "\psi^*"] & \\
P(\hat{\Delta}/\Delta) = A^2_{\mathrm{alg}}(X) \arrow[ur, hook, "i = \phi^*"] \arrow[dr, hook, "\lambda^*"'] & & J'' = A^2_{\mathrm{alg}}(X''). \\
 & J^* = A^2_{\mathrm{alg}}(X^*) \arrow[ur, "\mu^*"'] &
\end{tikzcd} \tag{16''}$$

3.8. Starting with $\xi \in T_l(P(\hat{\Delta}/\Delta))$ there are two ways of associating with ξ a cohomology class in $H^3(X'')$, namely

(i) $\xi \mapsto h_1(\xi) \in H^3(X'')$ by applying the following homomorphisms:

$$T_l(P(\hat{\Delta}/\Delta)) \underset{\chi^*}{\rightarrow} T_l(J'') \rightarrow \prod_j H^1(\Delta_j) \underset{\beta}{\tilde{\rightarrow}} H^3(X'')$$

where χ is from (16′), β is from lemma 3 and the other map comes from the well-known identifications $T_l(J(\Delta_j)) \subset E_l(J(\Delta_j)) \cong H^1(\Delta_j)$.

(ii) $\xi \mapsto h_2(\xi) \in H^3(X^*)$ and next $h_2(\xi) \mapsto \mu^* h_2(\xi) \in H^3(X'')$ as follows:

$$T_l(P(\hat{\Delta}/\Delta) \underset{\lambda^*}{\rightarrow} T_l(J^*) \rightarrow \prod_i H^1(\Omega_i) \underset{\beta}{\tilde{\rightarrow}} H^3(X^*) \underset{\mu^*}{\rightarrow} H^3(X'').$$

with similar explanations.

3.9. Lemma 8: *For $\xi \in T_l(P(\hat{\Delta}/\Delta))$ we have $h_1(\xi) = \mu^* h_2(\xi)$.*

Proof: Consider a curve Ω_i (see 3.1) and a curve Δ_j (see 3.7; note $j \geqq 0$, i.e. $\hat{\Delta} = \Delta_0$ is included). Using $\mu : X'' \rightarrow X^*$ from (16) we get correspondences $\Sigma_{ji} \in A^1(\Omega_i \times \Delta_j)$ from the product of graphs

$$\Sigma_{ji} = {}^t\Gamma_{g_j} \,.\, \Gamma_{r_j} \,.\, \Gamma_\mu \,.\, {}^t\Gamma_{u_i} \,.\, \Gamma_{v_i} \tag{17}$$

where the maps are indicated in the following diagram (compare also with diagram (2) where the role of the Δ, Δ' is played by Y and Y' and similar for Ω, Ω'):

(18)
$$\begin{tikzcd} \Delta_j & \Delta_j \arrow[l, "g_j"'] \arrow[r, hook, "r_j"] & X'' \arrow[d, "\mu"] \\ \Omega_i & \Omega_i' \arrow[l, "v_i"] \arrow[r, hook, "u_i"'] & X^* \end{tikzcd}$$

The Σ_{ji} give rise to homomorphisms σ_{ji} and commutative diagrams:

(18′)
$$\begin{tikzcd} A^1(\Omega_i) \arrow[r, "u_* \cdot v^*"] \arrow[d, "\sigma_{ij}"'] & A^2(X^*) \arrow[d, "\mu^*"] \\ A^1(\Delta_j) & A^2(X'') \arrow[l, "g_* \cdot r^*"] \end{tikzcd}$$

Similar for cohomology:

(18″)
$$\begin{tikzcd} H^1(\Omega_i) \arrow[r, "u_* \cdot v^*"] \arrow[d, "\sigma_{ji}"'] & H^3(X^*) \arrow[d, "\mu^*"] \\ H^1(\Delta_j) & H^3(X'') \arrow[l, "g_* \cdot r^*"] \end{tikzcd}$$

The proof of lemma 8 follows from $\chi^* = \mu^* \,.\, \lambda^*$, from the description given in 3.8 and from the commutativity of the following diagram:

$$\begin{tikzcd} \lambda^*(\xi) \in T_l(J^*) \arrow[r, "\sim", "\alpha"'] \arrow[d, "\mu^*"'] \arrow[dr, phantom, "(*)"] & \prod_i T_l(J(\Omega_i)) \arrow[r] \arrow[d, "\sigma_{ji}"] \arrow[dr, phantom, "(**)"] & \prod_i H^1(\Omega_i) \arrow[r, "\sim", "\beta"'] \arrow[d, "\sigma_{ji}"] \arrow[dr, phantom, "(***)"] & H^3(X^*) \arrow[d, "\mu^*"] \\ \chi^*(\xi) \in T_l(J'') \arrow[r, "\sim", "\alpha"'] & \prod_j T_l(J(\Delta_j)) \arrow[r] & \prod_j H^1(\Delta_j) \arrow[r, "\sim", "\beta"'] & H^3(X'') \end{tikzcd}$$

Commutativity of (*): the map α is as in lemma 2. The commutativity follows from the description of the maps α and β in lemma 2, from $\alpha = \beta^{-1}$ and from the commutative diagram (18′).

Commutativity of (**): lemma 1

Commutativity of (***): as for (*) with lemma 2, (18′) and α replaced by lemma 3, (18″) and β respectively.

COROLLARY: *For* ξ, $\eta \in T_l(P(\hat{\Delta}/\Delta))$ *we have* $2(h_2(\xi) \cup h_2(\eta)) = h_1(\xi) \cup h_1(\eta)$.

REMARK: Recall the convention that we use . for the product in $H(-)$, but $\cup$ after the orientation map has been applied (see 1.3c).

PROOF: Follows from
(a) $h_1(\xi) = \mu^* h_2(\xi)$ and $h_1(\eta) = \mu^* h_2(\eta)$
(b) μ^* is a *ring* homomorphism

(c) the commutativity of the following diagram, where the horizontal maps are the orientation maps δ([7] 1.2):

$$\begin{tikzcd} H^6(X^*) \arrow[r, "\sim", "\delta_{X^*}"'] \arrow[d, "\mu^*"'] & \mathbb{Q}_l \arrow[d, "2"] \\ H^6(X'') \arrow[r, "\sim", "\delta_{X''}"'] & \mathbb{Q}_l \end{tikzcd}$$

and where the *right-hand vertical arrow is multiplication by* 2. This in turn comes from the fact that $\mu : X'' \to X^*$ is generically 2–1, hence $\mu_* \mu^* = 2$ and μ_* commutes with the orientation map.

3.10. Proposition 1: *With the notations of* 3.5, *one has*

$$\Xi \equiv (\lambda^*)^{-1}(\theta^*) \text{ modulo algebraic equivalence.}$$

Proof: Consider the two corresponding Riemann forms on $T_l(P(\hat{\Delta}/\Delta))$. Abbreviate

$$e_1(\xi, \eta) = e^{\Xi}(\xi, \eta)$$

and

$$e_2(\xi, \eta) = e^{(\lambda^*)^{-1}(\theta^*)}(\xi, \eta)$$

with $\xi, \eta \in T_l(P(\hat{\Delta}/\Delta))$.

Lemma 9: $e_1(\xi, \eta) = e_2(\xi, \eta)$ *for all* $\xi, \eta \in T_l(P(\hat{\Delta}/\Delta))$.

Proof: By linearity on the divisor we have $2e_1(\xi, \eta) = e^{2\Xi}(\xi, \eta)$. Next by the definition of Ξ (see (13')) and by ([10], page 187 (II) or [8], page 191 prop. 6) we have $e^{2\Xi}(\xi, \eta) = e^{\theta'}(\phi^*\xi, \phi^*\eta) = e^{\theta''}(\chi^*(\xi), \chi^*(\eta))$. Finally using 3.6 and lemma 4 (ii) and the definition of h_1 in 3.8 we have $e^{\theta''}(\chi^*(\xi), \chi^*(\eta)) = -h_1(\xi) \cup h_1(\eta)$, hence $2e_1(\xi, \eta) = -h_1(\xi) \cup h_1(\eta)$. Similarly

$$e_2(\xi, \eta) = e^{(\lambda^*)^{-1}(\theta^*)}(\xi, \eta) = -h_2(\xi) \cup h_2(\eta).$$

Hence by the corollary of lemma 8 $2e_1(\xi, \eta) = 2e_2(\xi, \eta)$ and hence $e_1(\xi, \eta) = e_2(\xi, \eta)$.

Lemma $\Rightarrow$ Proposition: Put as abbreviation $D = \Xi - (\lambda^*)^{-1}(\theta^*)$. From $e_1(-) = e_2(-)$ we get by using again the linearity of the symbol with respect to the divisor ([8], p. 189) $e^D(\xi, \eta) = 0$ for all $\xi, \eta \in T_l(P(\hat{\Delta}/\Delta))$. But then, using the notation of [8], p. 189, proposition 3 we have $e(\xi, D_\eta - D = 0$ for all points ξ and η on $P(\hat{\Delta}/\Delta)$ which are of order l^n (all n). Then by [8], p. 189 proposition 4 we have $D_\eta - D \sim 0$ for all points η on $P(\hat{\Delta}/\Delta)$ which are of order l^n (all n). However the points η for which $D_\eta(-D \sim 0$ (linear equivalence) form an algebraic subgroup of $P(\hat{\Delta}/\Delta)$; the above assertion implies that it is $P(\hat{\Delta}/\Delta)$ itself. Then D is algebraically equivalent to zero ([8], p. 100 cor. 3). This completes the proof of the proposition.

3.11. THEOREM: *Let char $(k) \neq 2$. Let X be a non-singular cubic threefold in $\mathbf{P}_4$, defined over k. If there exists a birational transformation between X and $\mathbf{P}_3$ then the canonically polarized Prym variety $(P(\hat{\Delta}/\Delta), \Xi)$ associated with X, is isomorphic, as polarized abelian variety, to a product of canonically polarized Jacobian varieties of curves (cf. with* [4] 3.26).

PROOF: Consider, as before in 3.5, the product $(J^*, \theta^*) = \prod_i (J(\Omega_i), \theta_i)$, where the $(J(\Omega_i), \theta_i)$ are the canonically polarized Jacobians of the curves Ω_i from 3.1. Now remark that the Jacobian of a curve is 'irreducible' as principally polarized abelian variety (i.e. does not split up in a product of principally polarized abelian varieties). The theorem follows now at once from the following three facts:
(a) λ^* in (10) is injective;
(b) $\Xi \equiv (\lambda^*)^{-1}(\theta^*)$, modulo algebraic equivalence, by the proposition in 3.10;
(c) the following well-known, general lemma on the decomposition of principally polarized abelian varieties (see [4] 3.23):

LEMMA 10: (i) *Let (A, θ) be a pair consisting of an abelian variety and a positive divisor θ defining a principal polarization on A. Let (A', θ') be another such pair and $i : A' \to A$ an injective homomorphism such that $i^{-1}(\theta) \equiv \theta'$. Then there exists a third pair (A'', θ'') with the same properties and an injection $j : A'' \to A$ such that $j^{-1}(\theta) = \theta''$. Furthermore, with the obvious map, $A' \times A'' \xrightarrow{\sim} A$ and $\theta \equiv \theta' \times A'' + A' \times \theta''$ (the equivalence is always algebraic equivalence).*

(ii) *A principally polarized abelian variety has a unique decomposition into a product of irreducible principally polarized abelian varieties.*

PROOF: (i) Without loss of generality we can assume $\theta \,.\, A' = \theta'$. Consider the homomorphism $f : A \to \hat{A}'$ (dual of A') defined by

$$a \mapsto \text{class}\ \{(\theta_a - \theta) \,.\, A'\}$$

where class is in the sense of linear equivalence. From the assumptions we have that $f \,.\, i : A' \to \hat{A}'$ is the morphism (cf. [8], p. 75):

$$a' \mapsto \phi_{\theta'}(a) = \text{class}\ \{\theta'_{a'} - \theta'\}.$$

This is an isomorphism by the assumption that θ' is principal. Therefore f is onto and $i^{-1}(\text{Ker}\ (f)) = 0$, i.e.

$$A' \cap \text{Ker}\ (f) \doteq \{0\} \tag{19}$$

as *group schemes* on A. Let A'' be the connected component of the zero element in Ker (f), then A'' is an abelian variety and $A' \cap A'' = \{0\}$ as group schemes; also we have dim A = dim A'+dim A''. Let $j : A'' \to A$ be the natural embedding and consider $\rho : A' \times A'' \to A$ given by $\rho(a', a'') = i(a')+j(a'')$. Using the fact that $A' \cap A'' = \{0\}$ as group schemes we get that ρ is injective in the sense of group schemes; next we see, by counting dimensions, that it is surjective. Therefore ρ is an isomorphism; in the following we identify $A' \times A'' \xrightarrow{\sim} A$.

From the relation

$$\theta_{(a', a'')} - \theta \sim (\theta_{(a', 0)} - \theta) + (\theta_{(0, a'')} - \theta)$$

we get $\theta_{(a', a'')} \, . \, A' \sim \theta'_{a'}$. Next consider on $A' \times A''$ the divisor $D = \theta - \theta' \times A''$, then

$$D(a'') = pr_{A'}[D \, . \, (A' \times a'')] = pr_{A'}[\theta \, . \, (A \times a'') - (\theta' \times A'') \, . \, (A' \times a'')].$$

From the remarks above we have

$$pr_{A'}[\theta \cdot (A' \times a'')] = pr_{A'}[\theta_{(0, -a'')} \cdot A'] \sim \theta'.$$

Hence $D(a'') \sim 0$ and hence by [8], theorem on page 241, we have $D \sim A' \times \theta''$ for some divisor θ'' on A''. Therefore we have

$$\theta \sim \theta' \times A'' + A' \times \theta''. \tag{20}$$

Applying the Riemann-Roch theorem for the principal divisor θ ([10], p. 150) we get, if we put n = dim A, n_1 = dim A', n_2 = dim A'' (and hence $n = n_1 + n_2$),

$$n! = \theta^{(n)} = \binom{n}{n_1} \theta'^{(n_1)} \cdot \theta''^{(n_2)}.$$

Using the fact that θ' is principal on A' this gives $\theta''^{(n_2)} = n_2!$, i.e. θ'' is principal on A''. Therefore we can assume θ'' to be positive and then we must have

$$\theta = \theta' \times A'' + A' \times \theta''.$$

From this we see that $j^{-1}(\theta) = \theta''$. This completes the proof of (i).

(ii) For the proof we refer to [4] to the proof of 3.20. The proof there works also for positive characteristic provided we read the set theoretical intersections in [4] as intersections of group schemes.

Appendix

In correspondence on this topic, Mumford raised the question whether the $(P(\hat{\Delta}/\Delta), \Xi)$ is canonically associated with the cubic; i.e. satisfies some universal property. As far as $P(\hat{\Delta}/\Delta)$ is concerned the answer is affirmative as will be shown in this appendix. For the pair $(P(\hat{\Delta}/\Delta), \Xi)$ we have not settled the question yet.

Consider homomorphisms $\lambda : A^2_{\text{alg}}(X) \to A$, where A is an abelian variety and where for every *algebraic* map $\psi : S \to A^2_{\text{alg}}(X)$, with S a non-singular variety, we have that $\lambda . \psi : S \to A$ is a morphism.

Using the splitting $A^2_{\text{alg}}(X) = P \oplus T$ of lemma 6, we have a homomorphism $\lambda_0 : A^2_{\text{alg}}(X) \to P(\hat{\Delta}/\Delta)$, and by [12], proposition (10.5) that λ_0 has the required property concerning composition with algebraic families.

PROPOSITION: *For every $\lambda : A^2_{\text{alg}}(X) \to A$ as above we have a unique homomorphism of abelian varieties $\bar{\lambda} : P(\hat{\Delta}/\Delta) \to A$ such that the following diagram is commutative*

$$\begin{tikzcd} A^2_{\text{alg}}(X) \arrow[rr, "\lambda"] \arrow[dr, "\lambda_0"'] & & A \\ & P(\hat{\Delta}/\Delta) \arrow[ur, "\bar{\lambda}"'] & \end{tikzcd}$$

PROOF: Group theoretically $\bar{\lambda}$ is obtained, using lemma 6, by the composition

$$P(\hat{\Delta}/\Delta) \xrightarrow{(\phi^*|P)^{-1}} P \oplus T \xrightarrow{\lambda} A.$$

In order to see that this is actually a homomorphism of abelian varieties we repeat the argument given in the proof of lemma 7 (i). This gives $\bar{\lambda} : P(\hat{\Delta}/\Delta) \to A$ and by construction it follows that $\lambda - \bar{\lambda} . \lambda_0 = 0$ on $P \subset A^2_{\text{alg}}(X)$. In order to complete the proof we must see that the composition

$$T \underset{j}{\hookrightarrow} A^2_{\text{alg}}(X) \xrightarrow{\lambda} A$$

is zero (where j is the natural inclusion $T \to T \oplus P$). For $t \in T$ there exists an algebraic map $\psi : S \to A^2_{\text{alg}}(X)$, with S a connected, non-singular curve and two points $s_1, s_0 \in S$ such that $\psi(s_0) = 0$ and $pr_T \psi(s_1) = t$. Then we have a morphism $\bar{\psi} : S \to A$ given by $\bar{\psi} = (\lambda - \bar{\lambda} . \lambda_0) . \psi$. Then $\bar{\psi}(S) \subset A_2$, $\bar{\psi}(s_0) = 0$, hence $\bar{\psi} = 0$, i.e. $\lambda . j(t) = (\lambda - \bar{\lambda} . \lambda_0) . \psi(s_1) = \bar{\psi}(s_1) = 0$.

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