COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 27, nº 1 (1973), p. 1-23
<http://www.numdam.org/item?id=CM_1973__27_1_1_0>

COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 1, 1973, pag. 1-23
Noordhoff International Publishing
Printed in the Netherlands

THE WHITEHEAD THEOREM IN THE PROPER CATEGORY

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A classical theorem of J. H. C. Whitehead [2, 8] states that a continuous map between CW-complexes is a homotopy equivalence iff it induces an isomorphism of fundamental groups and an isomorphism on the homology of the universal covering spaces. This paper deals with the problem of finding an algebraic criterion for a proper map between locally finite CW-complexes to be a proper homotopy equivalence. The results are an extension of those announced in [1].

The first section treats generalities about the category of locally finite CW-complexes and proper maps and the second section outlines the homology and homotopy theories needed to work within this category. In § 3 we prove the 'Proper Whitehead Theorem'. Section 4 presents some special cases of this general theorem where the criteria that a map be a proper homotopy equivalence are more algebraic than in § 3. For example, any proper, degree one map between open n-manifolds which are simply connected and simply connected at infinity is a proper homotopy equivalence iff the map induces an isomorphism on homology and on cohomology with compact supports.

1. Preliminary remarks about proper maps

The term map means a continuous function. A *proper map* $f: X \to Y$ is a map such that $f^{-1}(C)$ is compact whenever C is a compact subset of Y. A *proper homotopy* from X to Y is a homotopy, that is a map $h: X \times I \to Y$, which is a proper map. Here $I = [0, 1]$. Let $\mathscr{S}$ be a collection of subsets of a topological space X; $\mathscr{S}$ is said to be *locally finite* iff every point in X has a neighborhood which meets only finitely many members of $\mathscr{S}$. If X happens to be locally compact then one easily deduces that $\mathscr{S}$ is locally finite iff each compact subset of X meets only finitely many members of $\mathscr{S}$. Recall that a CW-complex K is *locally finite* iff the collection consisting of all the closed cells of K is

[1] Partially supported by NSF grant GP 29697.
[2] N.S.F. graduate fellow
[3] Partially supported by NSF grant GP 29073.

locally finite. This condition is equivalent to K being locally compact. Also K is locally finite iff each point of K has some neighborhood which is a finite subcomplex. We will say that K is *strongly locally finite* iff K can be covered in a locally finite way by finite subcomplexes. Clearly, a strongly locally finite CW-complex is locally compact and hence locally finite. In this section we have three objectives: first, to prove that a finite dimensional, locally finite CW-complex is strongly locally finite; second, to prove a version of the homotopy extension theorem for proper homotopies when the domain space is strongly locally finite and the range space is arbitrary; and third, to prove that a proper map between strongly locally finite complexes is properly homotopic to a cellular map.

Note that one can also define the notion of an indexed family of sets being locally finite.

LEMMA 1.1. *Let $f: X \to Y$ be a proper map and let $\mathscr{S}$ be a locally finite collection of subsets of X. Also, assume that Y is locally compact. Then, the indexed family $f(\mathscr{S}) = \{f(S) | S \in \mathscr{S}\}$ of subsets of Y is locally finite. Here $f(\mathscr{S})$ is indexed by $\mathscr{S}$.*

The proof of (1.1) is left to the reader.

If S is a subset of a CW-complex K, then $K(S)$ denotes the carrier of S in K.

LEMMA 1.2. *Let K be a locally finite CW-complex, $\mathscr{C} = \{K(e) | e$ is a cell in $K\}$, and $\mathscr{S}$ be an arbitrary locally finite collection of compact subsets of K. If $\mathscr{C}$ is a locally finite collection, then $K(\mathscr{S}) = \{K(S) | S \in \mathscr{S}\}$ is a locally finite collection indexed by $\mathscr{S}$.*

PROOF. Since $\mathscr{C}$ is locally finite, K is locally compact. Let C be a compact subset of K. Denote the cells of K whose carriers meet C by $e_1, e_2, \cdots, e_n$. Let $C' = \bar{e}_1 \cup \cdots \cup \bar{e}_n$ and denote the members of $\mathscr{S}$ which meet C' by $S_1, \ldots, S_m$. Suppose that $K(S)$ meets C. Recall that $K(S)$ equals the union of the carriers of all the cells which meet S. See [8, p. 97]. Therefore, there exists a cell e of K which meets S and whose carrier meets C. Therefore, e is one of the cells $e_1, e_2, \cdots, e_n$; and hence S meets C' and is one of the finite collection $S_1, \cdots, S_m$.

COROLLARY 1.3. *If K is a strongly locally finite CW-complex, and $\mathscr{S}$ is a locally finite collection of compact subsets of K, then $K(\mathscr{S})$ is a locally finite collection indexed by $\mathscr{S}$.*

PROOF. Since K has a locally finite cover by finite subcomplexes, the collection $\mathscr{C} = \{K(e) | e$ a cell of $K\}$ is locally finite. Now apply (1.2).

THEOREM 1.4. *Every locally finite, finite dimensional CW-complex K is strongly locally finite.*

PROOF. We proceed by induction on the dimension of K. When $\dim K = 0$, K is discrete and hence strongly locally finite. Let us next suppose that (1.4) has been proved for all locally finite CW-complexes whose dimension is less than $n = \dim K$. Denote the $(n-1)$-skeleton of K by K_{n-1}. Let $\mathscr{C} = \{K(e) | e \text{ a cell of } K\}$; $\mathscr{C}' = \{K(e) | e \text{ an } n\text{-dimensional cell of } K\}$; and $\mathscr{C}'' = \{K(e) | e \text{ a cell of } K_{n-1}\}$. By the inductive assumption and (1.3), $\mathscr{C}''$ is locally finite; and hence once we show that $\mathscr{C}'$ is locally finite we will have proved (1.4) because $\mathscr{C} = \mathscr{C}' \cup \mathscr{C}''$. If e is an n-dimensional cell of L then $K(e) = e \cup K_{n-1}(\partial e)$ where $\partial e = \bar{e} \cap K_{n-1}$. By our inductive assumption and (1.3), we see that $\{K(\partial e) | e \text{ an } n\text{-dim cell of } K\}$ is a locally finite collection indexed by $\mathscr{C}'$. Since the set of all n-dimensional cells of K is also locally finite, we see that $\mathscr{C}'$ is locally finite.

ADDENDUM. Every locally finite simplicial complex (whether finite dimensional or not) is strongly locally finite since the collection of closed simplexes is a locally finite cover by finite subcomplexes.

LEMMA 1.5. *Let K be a connected, strongly locally finite CW-complex. Then K has a locally finite cover of the form $\{A_n, B_n | n = 0, 1, 2, \cdots\}$ where each A_i and B_i is a finite subcomplex of K and $A_i \cap A_j = \emptyset = B_i \cap B_j$ whenever $i \neq j$.*

PROOF. Note that K has only a countable number of cells and hence only a countable number of finite subcomplexes. See [8]. Hence K has a countable locally finite cover by finite subcomplexes; say

$$\mathscr{S} = \{C_n | n = 1, 2, 3, \cdots\}.$$

Inductively, we define a strictly increasing sequence of integers n_i as follows: $n_1 = 1$; and if n_{j-1} has been defined, let n_j be the first integer larger than n_{j-1} such that for all $m \geqq n_j$ we have

$$C_m \cap (C_1 \cup C_2 \cup \cdots \cup C_{n_{j-1}}) = \emptyset.$$

Define

$$A_0 = C_1,$$
$$A_i = C_{n_{2i}+1} \cup \cdots \cup C_{n_{2i+1}} \qquad \text{for} \quad i \neq 1,$$
$$B_i = C_{n_{2i+1}+1} \cup \cdots \cup C_{n_{2i+2}}$$

REMARK. Each connected component of a CW-complex is an open subcomplex.

THEOREM 1.6. (Homotopy Extension Theorem). *Let K be a strongly locally finite CW-complex, L a subcomplex of K, and Y an arbitrary* *topological space. Let $h : L \times [0, 1] \cup K \times 0 \to Y$ be a proper map. Then there exists a proper map $\hat{h} : K \times [0, 1] \to Y$ which extends h.*

PROOF. To prove (1.6), it is necessary and sufficient to prove that the hat on L, i.e. $L \times [0, 1] \cup K \times 0$, is a proper retract of $K \times [0, 1]$. By the remark above, it is sufficient to consider the case when K is connected. Via (1.5), we can express K as the union of two subcomplexes A and B where

$$A = \bigcup_{i=0}^{\infty} A_i, \qquad B = \bigcup_{i=0}^{\infty} B_i,$$

and each A_i and B_i is a finite subcomplex of K with $A_i \cap A_j = \emptyset = B_i \cap B_j$ when $i \neq j$. In order to show that the hat on L is a proper retract of $K \times [0, 1]$, we first show that the hat on $L \cup A$ in K is a proper retract of $K \times [0, 1]$ and then that the hat on L in K is a proper retract of the hat on $L \cup A$ in K. To define a proper retract of $K \times [0, 1]$ onto $K \times 0 \cup A \times [0, 1] \cup L \times [0, 1]$, we express $K \times [0, 1]$ as the union of the two closed subspaces $A \times [0, 1]$ and $B \times [0, 1]$. On $A \times [0, 1]$ define a map r to be the identity. Since $B \times [0, 1]$ intersected with the hat on $A \cup L$ in K is the hat on $(A \cup L) \cap B$ in B, and since the inclusion map of $(A \cup L) \cap B$ into B is a cofibration, (see [6, p. 402]) there exists a retraction s of $B \times [0, 1]$ onto the hat on $(A \cup L) \cap B$ in B. Since B is a disjoint union of finite subcomplexes, we see by a connectivity argument that s is a proper map. Both r and s agree where their domains overlap; and hence we can piece them together to obtain a proper retraction of $K \times [0, 1]$ onto the hat on $L \cup A$ in K. The hat on $L \cup A$ in K can be expressed on the union of the two closed subsets $A \times [0, 1]$ and the hat on L in K. On the hat on L in K define the identity map. Since $A \times [0, 1]$ intersected with the hat on L in K is the hat on $L \cap A$ in A, and the inclusion map of $L \cap A$ into A is a cofibration, there exists a retraction map s of $A \times [0, 1]$ onto the hat on $L \cap A$ in A. By connectivity considerations s is a proper map. Piecing this map together with the identity map on the hat on L in K, we obtain a proper retraction of the hat on $L \cup A$ in K onto the hat on L in K.

THEOREM 1.7 (Cellular 'Approximation' Theorem). *Let K and M be strongly locally finite CW-complexes, L a subcomplex of K, and $f : K \to M$ a proper map with $f|L$ cellular; then f is properly homotopic to a cellular map g through a homotopy fixed on L.*

PROOF. Applying (1.5) to each connected component of K, we decompose K as the union of two subcomplexes A and B where both A and B can be expressed as the disjoint union of finite subcomplexes; i.e.

$$A = \bigcup_{\alpha \in \mathscr{S}} \alpha, \qquad B = \bigcup_{\beta \in \mathscr{S}'} \beta$$

where each $\alpha \in \mathscr{S}$ and $\beta \in \mathscr{S}'$ is a finite subcomplex and if $\alpha_1, \alpha_2 \in \mathscr{S}$ and $\alpha_1 \neq \alpha_2$ then $\alpha_1 \cap \alpha_2 = \emptyset$; and likewise if $\beta_1, \beta_2 \in \mathscr{S}'$ and $\beta_1 \neq \beta_2$ then $\beta_1 \cap \beta_2 = \emptyset$. By (1.1) $f(\mathscr{S})$ is a locally finite collection of compact subsets of M indexed by $\mathscr{S}$. By (1.3), $M(f(\mathscr{S}))$ is a locally finite collection indexed by $f(\mathscr{S})$, and hence by $\mathscr{S}$. For each $\alpha \in \mathscr{S}$, there exists a homotopy of $f|\alpha : \alpha \to M(f(\alpha))$ to a cellular map which is fixed on $L \cap \alpha$. See [6, p. 404]. Piecing all of these homotopies together, we obtain a proper homotopy of $f|A : A \to M$ to a cellular map where the homotopy is fixed on $L \cap A$. We can extend this homotopy to a proper homotopy of $f|A \cup L : A \cup L \to M$ where this homotopy is fixed on L. Then we use (1.6) to extend this homotopy to a homotopy of $f : K \to M$. Hence we have properly deformed $f : K \to M$ to a map $f_1 : K \to M$ which is cellular on $A \cup L$ through a homotopy fixed on L. Now $f_1|B$ is cellular on $(A \cup L) \cap B$. By an argument similar to the one given above, $f_1|B$ is properly homotopic to a cellular map g_1 through a homotopy fixed on $(A \cup L) \cap B$. Hence $f_1 : K \to M$ is properly homotopic (through a homotopy fixed on $L \cup A$) to a cellular map g where $g|A = f_1$ and $g|B = g_1$. If we string together the proper homotopy from f to f_1 with the proper homotopy from f_1 to g, then we obtain a proper homotopy of f to a cellular map g. This homotopy is fixed on L.

REMARK 1.8. Theorem 1.7 is not true in general if the condition 'strongly locally finite' is dropped as the following example shows: Let K be an infinite set of points $\{p_1, p_2, \cdots\}$ with the discrete topology. Let $M = e^0 \cup e^1 \cup e^2 \cup \cdots$ where e^n is attached to

$$M_{n-1} = e^0 \cup e^1 \cup \cdots \cup e^{n-1}$$

by collapsing all of ∂e^n to a point $q_n \in \operatorname{int} e_{n-1}$. M is not strongly locally finite because any subcomplex of M must contain e^0. The proper map $f : K \to M$ given by $f(p_i) = q_i$ can certainly not be deformed (properly or otherwise) to a proper cellular map.

REMARK 1.9. Define a pair (K, L) of locally finite CW-complexes to be *strongly locally finite* if $K = L \cup K_\alpha$ where $\{K_\alpha\}$ is a locally finite collection of finite subcomplexes of K. If $\dim (K - L) < \infty$, then the pair (K, L) is strongly locally finite. Furthermore (1.6) and (1.7) still hold under the weakened hypothesis that the pair (K, L) is strongly locally finite.

Finally we say that a proper map $f : X \to Y$ is a *proper homotopy equivalence* provided that there exists a proper map $g : Y \to X$ such that both $g \circ f$ and $f \circ g$ are properly homotopic to the identity map.

2. Δ-homology and Δ-homotopy

In this section we review from [7] the definitions and results concerning Δ-homology and Δ-homotopy, which take the place in the proper category of ordinary homology and homotopy in the categories of spaces and all continuous maps.

Let $\mathscr{S}$ denote the category of pointed sets and base point preserving functions. Let $S = \{S_\alpha\}$ denote a collection of objects in $\mathscr{S}$ where α runs through some indexing set. Define $\mu(S)$ to be $\prod_\alpha S_\alpha$ modulo the equivalence relation which identifies $\{s_\alpha\} \in \prod_\alpha S_\alpha$ with $\{s'_\alpha\} \in \prod_\alpha S_\alpha$ provided that $s_\alpha = s'_\alpha$ except for possibly finitely many values of α. If each S_α is actually a group (with base point the identity element), then $\mu(S)$ is a group and in fact

$$\mu(S) = \prod_\alpha S_\alpha \bmod \bigoplus_\alpha S_\alpha .$$

If $f = \{f_\alpha : S_\alpha \to T_\alpha\}$ is a collection of morphisms, there is a natural morphism $\mu(f) : \mu(S) \to \mu(T)$ induced by $\prod_\alpha f_\alpha : \prod_\alpha S_\alpha \to \prod_\alpha T_\alpha$. If each f_α is a homomorphism, so is $\mu(f)$. Sometimes we shall denote $\mu(S)$ by $\mu_\alpha(S)$ to indicate what indexing set is being used.

Let X be a locally finite CW-complex. A locally finite collection $\{p\}$ of points in X will be called a *set of base points* for X provided that (a) for any compact set $K \subset X$ each infinite component of $X - K$ (i.e. each component not contained in a compact subset of X) contains an element of $\{p\}$ and (b) any subset of $\{p\}$ satisfying condition (a) has the same cardinality as $\{p\}$. Note that if X is compact and connected any set of base points consists of just one point.

Now suppose G is a functor from the category Υ_0 of based topological spaces and base point preserving maps to the category $\mathscr{S}$. Let $\{p\}$ be a set of base points for X. We define pointed sets $\varepsilon(X, \{p\}; G)$ and $\Delta(X, \{p\}; G)$ as follows: For each compact set $C \subset X$ and base point $p \in \{p\}$ let

$$(2.1) \qquad G(C, p) = \begin{cases} G(X-C, p), & \text{if } p \in X-C \\ pt, & \text{if } p \notin X-C. \end{cases}$$

Then form $\mu_p(G(C, p))$ and note that if $C \subset D$ there is a morphism

$$\mu_p(G(D, p)) \to \mu_p(G(C, p))$$

Define

$$(2.2) \qquad \varepsilon(X, \{p\}; G) = \varinjlim_C \mu_p(G(C, p))$$

Define Δ as the pull-back of the diagram

$$\begin{array}{ccc} \Delta(X, \{p\}; G) & \dashrightarrow & \prod_p G(\emptyset, p) \\ \vdots\downarrow & & \downarrow \\ \varepsilon(X, \{p\}; G) & \longrightarrow & \mu_p(G(\emptyset, p)) \end{array} \tag{2.3}$$

Note that if G is actually a functor into the category of groups and homomorphisms, then ε and Δ are groups. Also, if $\{C_\alpha\}$ is any cofinal collection of compacta in X (i.e. given a compactum $D \subset X$ there is some C_α containing D), one can just as well take the inverse limit in (2.2) over the sets C_α to calculate ε and then Δ.

REMARK. In all of the examples of interest in this paper the isomorphism class of ε or Δ is independent of the choice of the set of base points $\{p\}$. See [7]. Any natural transformation of functors $G \to H$ induces a morphism $\Delta(X, \{p\}; G) \to \Delta(X, \{p\}; H)$.

For any space X set

$$H_e^*(X) = \varinjlim_C H^*(X-C)$$

where the limit runs through the compact subsets of X. Any proper map $f: X \to Y$ will be called *properly 0-connected* provided X and Y are connected and $H_e^0(Y) \to H_e^0(X)$ is an isomorphism. The map f is properly 0-connected iff the inclusion $X \hookrightarrow M_f$ of X into the mapping cylinder M_f of f is properly 0-connected.

Now let $f: X \to Y$ be properly 0-connected and let $\{p\}$ denote a set of base points for X. Then $\{f(p)\}$ is a set of base points for Y and by refining $\{p\}$ we can insure that $f: \{p\} \to \{f(p)\}$ is a bijection. We define the induced map

$$\varepsilon f_* : \varepsilon(X, \{p\}; G) \to \varepsilon(Y, \{f(p)\}; G)$$

as follows: The set of compacta $f^{-1}(C)$ where C is a compactum of Y is cofinal and hence the $G(f^{-1}(C), p)$ can be used to compute $\Delta(X, \{p\}; G)$. The map εf_* is the one induced by the collection of maps $G(f^{-1}(C), p) \to G(C, f(p))$. The map εf_* induces a map

$$\Delta f_* : \Delta(X, \{p\}; G) \to \Delta(X, \{f(p)\}; G)$$

The ε and Δ constructions can also be described by a direct limit process (see [7]): Let Q denote the set of collections $\{g_p\}$ where $g_p \in G(C_p, p)$, p runs through the set of base points $\{p\}$, and $\{C_p\}$ is a collection of compacta such that $p \in X - C_p$ and such that any compactum of X is contained in C_p for all but finitely many p. Two such collections $\{g_p'\}$ and $\{g_p''\}$ are *ε-equivalent*, written $\{g_p'\} \equiv_\varepsilon \{g_p''\}$, iff there is a collection $\{g_p\}$ such that for all but at most finitely many p's

(i) $C_p \subset C'_p$ and $C_p \subset C''_p$, and

(ii) under the maps $G(C'_p, p) \to G(C_p, p)$ and

$$G(C''_p, p) \to G(C_p, p) \quad \text{both} \quad g'_p \quad \text{and} \quad g''_p \quad \text{go to} \quad g_p.$$

We say that $\{g'_p\}$ and $\{g''_p\}$ are *Δ-equivalent*, written $\{g'_p\} \equiv_\Delta \{g''_p\}$, iff conditions (i) and (ii) hold for all $p \in \{p\}$. Then

$$\varepsilon(X, \{p\}; G) = Q \text{ modulo } ''\equiv_\varepsilon'' \tag{2.2'}$$

and

$$\Delta(X, \{p\}; G) = Q \text{ modulo } ''\equiv_\Delta'' \tag{2.3'}$$

If G is a functor to the category of groups both ε and Δ are groups. The multiplication is given by

$$\{g'_p\} \cdot \{g''_p\} = \{h'_p \cdot h''_p\}$$

where h'_p and h''_p are the images of g'_p and g''_p under the maps

$$G(C'_p, p) \to G(C'_p \cap C''_p, p) \quad \text{and} \quad G(C''_p, p) \to G(C'_p \cap C''_p, p).$$

In what follows the functor G will, with one or two exceptions (see 2.7 and 2.10) always be a functor to the category of groups and, more often then not, to the category of abelian groups. Thus the general theorems concerning exact sequences, conditions insuring triviality of ε, etc. will be stated in the category of groups for the sake of economy. We leave it to the reader to interpret things in the category of pointed sets when necessary.

THEOREM 2.4. $\varepsilon(X, \{p\}; G) = 0$ *iff for any compact set C there is a compact set D containing C such that for all base points $p \in X - D$*

$$G(D, p) \to G(C, p)$$

is the zero homomorphism. Furthermore $\Delta(X, \{p\}; G) = 0$ iff $\varepsilon = 0$ and $G(\emptyset, p) = 0$ for all $p \in \{p\}$.

REMARK. The direct limit description of ε also implies that whenever each morphism $G(f^{-1}(C_\alpha), p) \to G(C_\alpha, f(p))$ is surjective for a cofinal collection $\{C_\alpha\}$ of compacta $C_\alpha \subset Y$ then the map

$$\varepsilon(X, \{p\}; G) \to \varepsilon(Y, \{f(p)\}; G)$$

is also surjective.

As usual, let $\{p\}$ be a set of base points for X. *A covering functor*, denoted by $\sim$, assigns to each pair (C, p), where C is a compact subset of X and $p \in \{p\}$ lies in $X - C$, a subgroup $\pi(C, p) \subset \pi_1(X - C, p)$ such that $i_*(\pi(D, p)) \subset \pi(C, p)$ whenever $C \subset D$. Here $i_* : \pi(D, p) \to$

$\pi_1(X-C, p)$ is the homomorphism induced by the inclusion $X-D \subset X-C$. For each compact $C \subset X$ and base point $p \in \{p\}$ lying in $X-C$, let $\rho : \widetilde{X-C} \to X-C$ denote the covering space of the connected component of $X-C$ containing p corresponding to the subgroup $\pi(C, p) \subset \pi_1(X-C, p)$. Choose a lifting $\hat{p} \in \widetilde{X-C}$ of $p \in X-C$ such that $\rho_*(\pi_1(\widetilde{X-C}, \hat{p})) = \pi(C, p)$. Now let $C \subset D$. Since $i_*(\pi(D, p)) \subset \pi(C, p)$ there is a unique map $\tilde{\imath} : \widetilde{X-D} \to \widetilde{X-C}$ covering the inclusion $i : X-D \hookrightarrow X-C$ such that the base point $\hat{p} \in \widetilde{X-D}$ goes to the base point $\hat{p} \in \widetilde{X-C}$. Thus each choice of liftings $\{\hat{p}\}$ determines a functor from Υ_0 to itself.

Now let $\sim$ be a covering functor of X and choose a set of liftings $\{\hat{p}\}$ as above. Define $\varepsilon(X, \{p\}; G, \sim)$ and $\Delta(X, \{p\}; G, \sim)$ by the ε and Δ constructions using the pointed sets (or groups) $G(C, p) = G(\widetilde{X-C}, \hat{p})$. Then $\varepsilon(X, \{p\}; G, \sim)$ and $\Delta(X, \{p\}; G, \sim)$ are independent, up to isomorphism, of the choice of liftings $\{\hat{p}\}$ of $\{p\}$. See [7].

Two covering functors $\sim'$ and $\sim''$ of X are *pre-equivalent* provided there is a cofinal collection of compacta $\{C_\alpha\}$ in X such that $\phi \in \{C_\alpha\}$ and such that if $C_\alpha \in \{C_\alpha\}$ and $p \in \{p\}$ then $\pi'(C_\alpha, p) = \pi''(C_\alpha, p)$. *Equivalence* of covering functors is the equivalence relation generated by 'pre-equivalence'. If $\sim'$ and $\sim''$ are equivalent there are natural isomorphisms

$$\varepsilon(X, \{p\}; G, \sim') \cong \varepsilon(X, \{p\}; G, \sim'')$$

and

$$\Delta(X, \{p\}; G, \sim') \cong \Delta(X, \{p\}; G, \sim'').$$

EXAMPLES.

(i) $\pi(C, p) = \pi_1(X-C, p)$. Then $\widetilde{X-C} = X-C$ and $\hat{p} = p$.

(ii) $\pi(C, p) =$ trivial group. Then $\widetilde{X-C}$ is the universal cover of the component of $X-C$ containing p. In this case we let 'univ' denote the covering functor.

(iii) $\pi(C, p) = \ker [\pi_1(X-C, p) \to \pi_1(X, p)]$. Let $Y_p \subset X-C$ be the component containing p and let $\pi : U \to X$ be the universal cover. Then $\widetilde{X-C}$ is just one of the components of $\pi^{-1}(Y_p)$ and the lifting of p to $\hat{p}$ picks out which component it is.

(iv) Let $f : X \to Y$ be a proper map. Let $\{p\}$ be a set of base points for X such that $\{f(p)\}$ is a set of base points for Y and $f : \{p\} \to \{f(p)\}$ is a bijection. Let $\sim$ be a covering functor for Y. We shall define the *induced* covering functor $f^*(\sim)$ on X. Although the construction of $f^*(\sim)$ involves some ambiguity, the equivalence class of $f^*(\sim)$ is well determined and it is in this sense we speak of 'the' induced covering functor.

Choose a cofinal sequence $\phi = D_0 \subset D_1 \subset \cdots$ of compacta in Y. Given $p \in \{p\}$ and any compactum C in X let

$$\pi^*(C, p) = f_{\#}^{-1}[\pi(D_k, f(p))]$$

where k is the largest integer such that $f(X-C) \subset Y-D_k$. The $\pi^*(C, p)$ are the subgroups which define $f^*(\sim)$.

From now on we shall not distinguish between equivalent covering functors.

The induced morphisms.

Let $f: X \to Y$ and $\{p\}$ be as in (iv) above. The induced morphisms

$$\varepsilon f_* : \varepsilon(X, \{p\}; G, f^*(\sim)) \to \varepsilon(Y, \{f(p)\}; G, \sim)$$

and

$$\Delta f_* : \Delta(X, \{p\}; G, f^*(\sim)) \to \Delta(Y, \{f(p)\}; G, \sim)$$

are induced by the morphisms

$$G(\widetilde{X-f^{-1}(D_k)}, \hat{p}) \to G(\widetilde{Y-D_k}, \widehat{f(p)})$$

where the D_k are as in (iv).

The rest of this section lists the examples of the Δ-construction used in this paper and states some of their general properties.

The absolute groups.

In each of the following examples it suffices to define the $G(C, p)$. The morphism $G(D, p) \to G(C, p)$ is induced by $\widetilde{X-D} \to \widetilde{X-C}$. We adopt the convention that whenever $p \notin X-C$ then $G(C, p) =$ trivial group. For the Δ-group in each example use the notation as indicated.

(2.5) $\Delta(X, \{p\}; \pi_k, \sim)$ for $k \geqq 1$.

Let $G(C, p) = \pi_k(\widetilde{X-C}, \hat{p})$

(2.6) $\Delta(X, \{p\}; H_k, \sim)$ for $k \geqq 1$.

Let $G(C, p) = H_k(\widetilde{X-C})$

The relative groups.

Let $A \hookrightarrow X$ be a properly 0-connected inclusion. This allows one to choose a set of base points $\{p\}$ for A which is also a set of base points for X. Let $\sim$ be a covering functor on X and $\pi_c : \widetilde{X-C} \to X-C$ be the associated covering maps (strictly speaking the range of π_c is a connected component Y_p of $X-C$ containing a base point $p \in \{p\}$ and π_c depends on p). Let $\overline{A-C}$ denote $\pi_c^{-1}(Y_p \cap (A-C))$.

(2.7) $$\Delta(X, A; \{p\}; \pi_k, \sim) \quad \text{for} \quad k \geqq 1.$$

Let $G(C, p) = \pi_k(\widetilde{X-C}, \overline{A-C}; \hat{p})$

(2.8) $$\Delta(X, A; \{p\}; H_k, \sim) \quad \textit{for} \quad k \geqq 1.$$

Let $G(C, p) = H_k(\widetilde{X-C}, \overline{A-C})$

(2.9) $$\Delta(X, A; \{p\}; \pi'_k \sim) \quad \text{for} \quad k \geqq 2.$$

Let $G(C, p) = \pi'_k(\widetilde{X-C}, \overline{A-C}; \hat{p})$ where as in [6] π'_k is $\pi_k(\widetilde{X-C}, \overline{A-C}; \hat{p})$ modulo the action of $\pi_1(\overline{A-C}, \hat{p})$.

(2.10) $$\Delta(X, A; \{p\}; \bigoplus_{k \geqq 1} \pi_k, \sim)$$

Let
$$G(C, p) = \bigoplus_{k \geqq 1} \pi_k(\widetilde{X-C}, \overline{A-C}; \hat{p}).$$

Note that in example (2.7) the Δ construction gives a pointed set for $k = 1$ and groups for $k \geqq 2$. Example (2.10) gives a pointed set.

The following general properties of the Δ groups are analogous to those which hold in ordinary homology and homotopy.

THEOREM 2.11. *Let $f: X \to Y$ be a proper homotopy equivalence. Let $\sim$ and $-$ be covering functors of X and Y respectively such that $\sim$ is equivalent to $f^*(-)$. Let $G = \pi_k$ or H_k. Then εf_* and Δf_* are isomorphisms.*

An inclusion $A \hookrightarrow X$ is *properly n-connected* provided it is properly 0-connected and $\Delta(X, A; \{p\}; \pi_k, \text{no cov}) = 0$ for $1 \leqq k \leqq n$. Here 'no cov' is the covering functor in example (i).

In (2.12) below we set

$$\Delta_k^\pi(X) = \Delta(X, \{p\}; \pi_k, \sim), \qquad \Delta_k^\pi(X, A) = \Delta(X, A; \{p\}; \pi_k, \sim), \quad \text{etc.}$$

THEOREM 2.12. *Let (X, A) be properly 1-connected and let be a covering functor on X. Form $\Delta_k^\pi(A)$ with respect to the covering functor on A induced by the inclusion $A \hookrightarrow X$. Then there is a long exact sequence*

$$\cdots \to \Delta_k^\pi(A) \to \Delta_k^\pi(X) \to \Delta_k^\pi(X, A) \to \Delta_{k-1}^\pi(A) \to \Delta_{k-1}^\pi(X) \to \cdots$$
$$\cdots \to \Delta_1^\pi(A) \to \Delta_1^\pi(X) \to 1.$$

Similarly there is a long exact sequence using the Δ-homology groups of (2.6).

REMARK. For simplicity we have assumed that (X, A) is properly 1-connected. When this is not the case it is possible to define Δ functors which extend the sequence to the right several terms. Also (2.12) and (2.11) hold for any homotopy functor. See [7].

In the next theorem set $\Delta_k^{\pi'}(X, A) = \Delta(X, A; \{p\}; \pi'_k \sim)$ and $\Delta_k^h(X, A) = \Delta(X, A; \{p\} H_k, \sim)$. The ordinary Hurewicz homomorphism induces a Hurewicz homomorphism

$$(2.13) \qquad H_k : \Delta_k^{\pi'}(X, A) \to \Delta_k^h(X, A)$$

THEOREM 2.14. (Proper Hurewicz Theorem. c.f. [6, p. 397]). *Suppose (X, A) is properly n-connected ($n \geqq 1$). Let $\sim$ be any covering functor. Then $\Delta_k^h(X, A) = 0$ for $1 \leqq k \leqq n$ and*

$$H_{n+1} : \Delta_{n+1}^{\pi'}(X, A) \overset{\to}{\simeq} \Delta_{n+1}^h(X, A)$$

is an isomorphism. Conversely, if $A \hookrightarrow X$ is properly 0-connected, A and X are properly 1-connected (i.e. $\Delta(A, \{p\}; \pi_1,$ no cov$) = 0$ and similarly for X), and there is an $n \geqq 1$ with $\Delta_k^h(X, A) = 0$ for $1 \leqq k \leqq n$, then (X, A) is properly n-connected and there is an isomorphism

$$H_{n+1} : \Delta_{n+1}^{\pi'}(X, A) \overset{\to}{\simeq} \Delta_{n+1}^h(X, A)$$

THEOREM 2.15. *Suppose $\Delta(A, \{p\}; \pi_1, -) = 0$ where '$-$' is the covering functor on A induced by a covering functor $\sim$ on X. Then for $k \geqq 2$ there is an isomorphism*

$$\Delta(X, A; \{p\}; \pi_k, \sim) \overset{\to}{\simeq} \Delta(X, A; \{p\}; \pi'_k, \sim).$$

EXAMPLE 2.16. Suppose that the composite map

$$(*) \qquad \Delta(A, \{p\}; \pi_1, \text{no cov}) \to \prod_p \pi_1(A, p) \to \prod_p \pi_1(X, p)$$

is a monomorphism. Let $\sim$ be the covering functor of X obtained from the universal covering space of X as in (iii). Then $\Delta(A, \{p\}; \pi_1, -) = 0$, so Theorem 2.15 applies. For example, suppose A and X each have one stable end (c.f. [4]) with fundamental groups $\pi_1 \varepsilon_a$ and $\pi_1 \varepsilon_x$ respectively. Suppose that $\pi_1 \varepsilon_a \to \pi_1 \varepsilon_x$ and $\pi_1 A \to \pi_1 X$ are isomorphisms and that $\pi_1 \varepsilon_a \to \pi_1 A$ is a monomorphism. Then (*) holds.

EXAMPLE 2.17. Let $\sim$ = universal covering functor of X as in example (ii). Suppose that $\Delta(A; \{p\}; \pi_1, \text{no cov}) \to \Delta(X, \{p\}; \pi_1, \text{no cov})$ is a monomorphism. Then $\Delta(A, \{p\}; \pi_1, -) = 0$ because $\Delta(X, \{p\}; \pi_1,$ univ cov$) = 0$.

THEOREM 2.18. *Suppose that*

$$\Delta(A, \{p\}; \pi_1, \text{no cov}) \to \Delta(X, \{p\}; \pi_1, \text{no cov})$$

is an isomorphism. Let $\sim$ be any covering functor of X. Then the natural homomorphism

$$\Delta(X, A; \{p\}; \pi_k, \sim) \to \Delta(X, A; \{p\}; \pi_k, \text{no cov})$$

is an isomorphism whenever $k \geqq 2$.

The following theorem will be useful for applications in § 4.

THEOREM 2.19. *Let* (X, A) *be a properly* 0*-connected pair. Suppose*
(a) $\Delta(A, \{p\}; \pi_1, \text{no cov}) \to \Delta(X, \{p\}; \pi_1, \text{no cov})$ *is an isomorphism*
(b) $\Delta(A, \{p\}; \pi_1, \text{no cov}) \to \prod_P \pi_1(A, p)$ *is a monomorphism*
(c) $\Delta(X, A; \{p\}; H_k, \sim) = 0$ *where* $\sim$ *is as in example* (iii). *Then* (X, A) *is properly n-connected for all* $n \geqq 1$.

Condition (a) implies (X, A) is properly 1-connected (c.f. [7]). When $n > 1$ the proof of (2.19) is an induction argument along classical lines using (2.14), (2.15), (2.16), and (2.18).

3. The Proper Whitehead Theorem

Let $f : X \to Y$ be any proper map between locally finite CW-complexes. Then f is a proper homotopy equivalence iff the inclusion $X \subset M_g$ is a proper homotopy equivalence, where M_g is the mapping cylinder of any cellular map $g : X \to Y$ properly homotopic to f. Therefore in this section we shall always work with inclusions $A \subset X$ where A is a subcomplex of X. We shall also assume $A \subset X$ is properly 0-connected.

THEOREM 3.1. (Proper Whitehead Theorem). *Suppose* $\dim (X-A) = n < \infty$. *Then* A *is a proper deformation retract of* X *if*
(a) $\Delta(X, A; \{p\}; \pi_k, \text{no cov}) = 0$ for $1 \leqq k \leqq n$ *or*
(b) $\Delta(A, \{p\}; \pi_1, \text{no cov}) \to \Delta(X, \{p\}; \pi_1, \text{no cov})$ *is an isomorphism and* $\Delta(X, A; \{p\}; H_k, \text{univ}) = 0$ *for* $1 \leqq k \leqq n$.

Conditions (a) and (b) in (3.1) are certainly necessary conditions.

First we shall show that (a) implies A is a proper deformation retract of X and then we show that (b) implies (a). Part (a) is a special case of

PROPOSITION 3.2. (c.f. p. 402 of [6]). *Let* (X, A) *be a properly* n*-connected pair. Let* $f : (K, L) \to (X, A)$ *be any proper map. Suppose* L *is a subcomplex of* K *and that* $\dim (K-L) \leqq n < \infty$. *Then* f *can be properly deformed* rel L *to a map into* A.

The following will be needed in the proof of (3.2):

LEMMA 3.3. *Let* $K = L \cup (P_1 \cup P_2 \cup \cdots)$ *where* $\{P_i\}$ *is a locally finite collection of finite subcomplexes of* K *and* L *is a subcomplex of* K. *Suppose that the* $P_i - (L \cap P_i)$ *are all disjoint. Let* (X, A) *be a pair of locally finite, countable* CW*-complexes and let* $f : (K, L) \to (X, A)$ *be a proper map such that*
(i) *For each* $i \geqq 1$, $f : (P_i, L \cap P_i) \to (X, A)$ *deforms* rel $L \cap P_i$ *down into* A.
(ii) *Given any compact set* $C \subset X$ *there is a compact set* $D \subset X$

containing C and there is a positive integer m such that f maps $(P_i, L \cap P_i)$ into $(X-D, A-D)$ whenever $i \geqq m$ and such that the map

$$f: (P_i, L \cap P_i) \to (X-C, A-C)$$

deforms rel $L \cap P_i$ *down into $A-C$ within $X-C$. Then $f: (K, L) \to (X, A)$ deforms properly* rel L *to a map into A.*

PROOF. Let $n_0 = 1$ and choose a compactum $C_1 \subset X$ so that $f: P_1 \to X$ can be deformed into A rel $L \cap P_1$ within C_1. Choose a compactum C_2 containing C_1 and an $n_1 > n_0$ such that for $i \geqq n$, $f(P_i) \subset X-C_2$ and $f: P_i \to X-C_1$ can be deformed rel $L \cap P_i$ into A within $X-C_1$. Choose a compactum C_3 containing C_2 so that $P_1, \cdots P_{n_1-1}$ can be deformed into A rel $L \cap P_i$ within C_3. Choose a compactum C_4 containing C_3 and an integer $n_2 > n_1$ such that for $i \geqq n_2$, $f(P_i) \subset X-C_4$ and $f: P_i \to X-C_3$ can be deformed rel $L \cap P_i$ into A within $X-C_3$. Choose a compactum C_5 so that $f: P_i \to X-C_1$ can be deformed rel $L \cap P_i$ into A within C_5-C_1 for $n_1 \leqq i \leqq n_2-1$. Continuing in this way we can construct a cofinal sequence of compacta

$$C_1 \subset C_2 \subset C_3 \subset \cdots$$

and an increasing sequence

$$1 = n_0 < n_1 < n_2 < \cdots$$

such that P_{n_k} through $P_{n_{k+1}-1}$ deform rel L into A within $C_{2k+3}-C_{2k-1}$. These deformations clearly give a proper deformation of $f: K \to X$ into A keeping f fixed on L.

PROOF OF 3.2. *Part* (*a*). Write $K = L \cup P \cup Q$ where both P and Q are the disjoint union of finite complexes $P = P_1 \cup P_2 \cup \cdots$ and $Q = Q_1 \cup Q_2 \cup \cdots$ such that $\dim P_i \leqq n$ and $\dim Q_i \leqq n$. We show that $f|L \cup P$ can be properly deformed rel L down into A. The proper homotopy extension theorem then implies that $f: K \to X$ can be properly deformed rel L to a map $g: K \to X$ with $g(L \cup P) \subset A$. Then a repeat of the first half of the argument shows that g can be properly deformed rel $L \cup P$ to a map of K into A, which completes the proof.

To show that $f|L \cup P$ can be pulled down into A it suffices to show that the conditions of Lemma 3.3 are satisfied. For short let

$$\Delta_k = \Delta(X, A; \{p\}; \pi_k, \text{no cov}).$$

Since $\Delta_k = 0$ for $1 \leqq k \leqq n$, $\pi_k(X, A; p) = 0$ for $1 \leqq k \leqq n$ and hence standard obstruction theory says that (i) of (3.3) holds.

To get (ii) let $C \subset X$ be any compactum. Use (2.4) to get a sequence of compacta $C = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_n$ such that for all p in any infinite component of $A - C_i$ the morphism

$$\pi_{i+1}(X - C_{n-i}, A - C_{n-i}; p) \to \pi_{i+1}(X - C_{n-i-1}, A - C_{n-i-1}; p)$$

is the zero map for $0 \leqq i \leqq n-1$. Choose $D = C_n$. Choose the m of (ii) to be an integer so large that whenever $j \geqq m$, $f(P_j)$ is contained in an infinite component of $X - D$. Standard obstruction theory shows that this choice of D and m satisfies (ii) of (3.3).

Proof that $(b) \Rightarrow (a)$ *in* (3.1).

Since $\Delta(A, \{p\}; \pi_1, \text{no cov}) \to \Delta(X, \{p\}; \pi_1, \text{no cov})$ is an isomorphism the pair (X, A) is properly 1-connected. That (b) $\Rightarrow$ (a) now follows inductively as in ordinary homotopy theory using (2.14), (2.15), (2.17), and (2.18).

This completes the proof of (3.1).

Let $f : X \to Y$ be a properly 0-connected map between locally finite, countable CW-complexes. We say f is a *proper n-equivalence* iff the pair (M_f, X) is properly n-connected where M_f is the mapping cylinder of f. If K is a space let $[K, X]_P$ denote the set of proper homotopy classes of proper maps of K into X. The argument proving Corollary 23 on p. 405 in [6] can be mimiced in the proper category using (3.2) to show

THEOREM 3.4. *Let K be a connected, locally finite CW-complex of dimension $k < \infty$. Let $f : X \to Y$ be a proper n-equivalence. The natural map*

$$[K, X]_P \to [K, Y]_P$$

is surjective if $k \leqq n$ and injective if $k < n$.

Here is another proper Whitehead theorem.

THEOREM 3.5. *Suppose (X, A) is strongly locally finite. Then A is a proper deformation retract of X iff*

$$\Delta(X, A; \{p\}; \bigoplus_{k \geqslant 1} \pi_k, \text{no cov}) = 0.$$

The proof is similar to that of (3.1) and is left to the reader. In practice (3.1) is more useful than (3.5).

A result similar to (3.1) and to (3.5) when $\dim (X - A) < \infty$ has been obtained by L. C. Siebenmann in [5].

4. Applications

In this section we show that when the fundamental group system at infinity maps monomorphically into the fundamental group of the whole space it is possible to prove a proper Whitehead theorem with hypotheses more algebraic than those of (3.1).

For any space X, let $H_c^*(X)$ denote the singular cohomology with compact supports.

The main theorem of this section is

THEOREM 4.1. *Let $f : X \to Y$ be a properly 0-connected map between locally finite, finite dimensional CW-complexes. Suppose that*

$$\Delta(X, \{p\}; \pi_1, \text{no cov}) \to \prod_P \pi_1(X, p)$$

is a monomorphism. Then f is a proper homotopy equivalence iff

(1) *The map* $f_\# : \Delta(X, \{p\}; \pi_1, \text{no cov}) \to \Delta(Y, \{f(p)\}; \pi_1, \text{no cov})$ *is an isomorphism*

(2) $f_* : H_*(\tilde{X}) \to H_*(\tilde{Y})$ *and* $f^* : H_c^*(\tilde{Y}) \to H_c^*(\tilde{X})$ *are isomorphisms where $\tilde{X}$ and $\tilde{Y}$ are the universal covering spaces of X and Y.*

The proof of (4.1) is given after a sequence of propositions and lemmas (4.2 through 4.7).

Let R be a ring with identity and M a countably based free R module whose basis is denoted by B. By 'countable', we mean either countably infinite or finite. A submodule U of M is called a *neighborhood of infinity* if U contains all but a finite number of basis elements from B. Let $f : M \to N$ be a R-module homomorphism between countably based R-modules M and N. We say that f is *locally finite* if for every neighborhood U of infinity in N there exists a neighborhood V of infinity in M such that $f(V) \subseteq U$. By $\mathrm{Hom}^f(M, N)$, we denote the set of all locally finite homomorphisms from M to N.

In particular, we denote $\mathrm{Hom}(M, R)$ by M^* and $\mathrm{Hom}^f(M, R)$ by $\bar{M}$. (Since R is a ring with identity, say 1, R has a natural countable basis namely $\{1\}$.) Note that $\bar{M}$ consists of those R-homomorphisms from M to R which vanish on all but finitely many basis elements in B. $\bar{M}$ is again naturally countably based with basis $\bar{B} = \{\bar{b} | b \in B\}$ where $\bar{b}(b) = 1$ and $\bar{b}(b') = 0$ for $b' \in B, b' \neq b$. Although M^* may not be countably based, we still define neighborhoods of infinity for M^* as follows: A submodule U of M^* is a *neighborhood of infinity* if there exists a finite subset of B such that U contains all elements of M^* which annihilate this subset. Note that the intersection of a neighborhood of infinity in M^* with $\bar{M}$ is a neighborhood of infinity in $\bar{M}$; and that any neighborhood of infinity in $\bar{M}$ can be so expressed. Also note that if $f : M \to N$ is an R-module homomorphism then $f^* : N^* \to M^*$ is a locally finite R-module homomorphism. If in addition f is locally finite, then $f^*(\bar{N}) \subseteq \bar{M}$; and hence we can then define $\bar{f} : \bar{N} \to \bar{M}$ as f^* restricted to $\bar{N}$. Note that $\bar{f}$ is a locally finite R-module homomorphism. A chain complex is said to be *countably based, locally finite* if each chain group is countably based and the boundary maps are locally finite. If C is a countably based, locally finite chain complex, we define a countably based, locally finite cochain complex $\bar{C}$ as follows: $(\bar{C})^n = \overline{(C_n)}$ and $\partial^n : (\bar{C})^{n-1} \to (\bar{C})^n$ is $\overline{\partial^n}$ where $\partial_n : C_n \to C_{n-1}$. There exists a natural isomorphism $\alpha : M \to \overline{\overline{M}}$ if M is a countably based R-module; α is natural in the sense that if $f : M \to N$ is locally finite, then $\alpha \circ f = \bar{\bar{f}} \circ \alpha$.

THEOREM 4.2. *Let C be a finite dimensional, countably based, locally finite chain complex; and assume that both C and $\bar{C}$ are acyclic (i.e. $H_*(C) = H_*(\bar{C}) = 0$). Then C is locally finitely contractible (i.e. there exist locally finite maps $s_n : C_n \to C_{n+1}$ such that $\partial_{n+1} s_n + s_{n-1}\partial_n = \mathrm{id}$).*

In order to prove (4.2) it is sufficient to show that $\bar{\bar{C}}$ is locally finitely contractible. The proof of this fact depends on the following lemma.

LEMMA 4.3. *Let C be a finite dimensional, countably based, locally finite chain complex such that both C and $\bar{C}$ are acyclic; then given any neighborhood U of infinity in $\overline{\overline{C_{n+1}}}$ there exists a neighborhood V of infinity in $\bar{\bar{C}}_n$ which has the property that for any cycle $x \in V$ there exists a chain $y \in U$ with $\bar{\bar{\partial}}_{n+1} y = x$.*

This lemma and its proof were motivated by a result of S. P. Novikov. See [3], proof of Theorem 2.

PROOF OF 4.3. Let $t_i : \bar{C}_i \to \bar{C}_{i-1}$ be contraction operators which exist since $\bar{C}$ is finite dimensional and acyclic. These maps satisfy the equations $\bar{\partial}_i t_i + t_{i+1}\bar{\partial}_{i+1} = \mathrm{id}$. Let U' be a neighborhood of infinity for $\bar{C}^*_{n+1}$ such that $\bar{\bar{C}}_{n+1} \cap U' = U$. Since t^*_{n+1} is locally finite, there exists a neighborhood V' of infinity in $\bar{C}^*_n$ such that $t^*_{n+1}(V') \subseteq U'$. Let $V = V' \cap \bar{\bar{C}}_n$. From the equation

$$\bar{\partial}^*_{n+1} t^*_{n+1} + t^*_n \bar{\partial}^*_n = \mathrm{id},$$

we see that if x is a cycle in V then there exists a chain $z \in \bar{C}_{n+1}$ (namely $z = t^*_{n+1} x$) such that $z \in U'$ and $\bar{\partial}^* z = x$. Since $\bar{\bar{C}}$ is acyclic, there exists a chain y' from $\bar{\bar{C}}_{n+1}$ such that $\bar{\bar{\partial}} y' = x$; and hence $\bar{\partial}^* y' = x$. Therefore, $z - y'$ is a cycle in $\bar{C}^*_{n+1}$. Since $\bar{C}$ is contractible, $\bar{C}^*$ is acyclic; and hence there exists $w \in \bar{C}^*_{n+2}$ with $\bar{\partial}^* w = z - y'$. Since $\bar{\partial}^*$ is locally finite, we can write w as the sum of two chains w_1 and w_2 such that $w_1 \in \bar{\bar{C}}_{n+2}$ and $\bar{\partial}^* w_2 \in U'$. Let $y = y' + \bar{\bar{\partial}} w_1$. Then $y \in \bar{\bar{C}}_{n+1}$ and $\bar{\bar{\partial}} y = x$. We need only show that $y \in U$ to complete the proof of (4.3). To see this, consider the following equation:

$$\begin{aligned} y = y' + \bar{\bar{\partial}} w_1 = y' + \bar{\partial}^*(w - w_2) &= y' + \bar{\partial}^* w_1 - \bar{\partial}^* w_2 \\ &= y' + (z - y') - \bar{\partial}^* w_2 = z - \bar{\partial}^* w_2 \in U' \cap \bar{\bar{C}}_{n+1}. \end{aligned}$$

But $U = U' \cap \bar{\bar{C}}_{n+1}$, so $y \in U$.

In order to prove (4.2), use (4.3) to inductively define the locally finite s_n. Compare [6, p. 164]. We leave the details of this construction to the reader.

Let X be a connected, finite dimensional CW-complex and A a subcomplex of X. Let $\tilde{p} : X \to X$ denote a universal cover of X and $\bar{A}$ the induced cover over A. (i.e. $\bar{A} = p^{-1}(A)$). Define a chain complex $C(X, A)$ as follows: $C_n(X, A) = H_n(\overline{X_n \cup A}, \overline{X_{n-1} \cup A})$. Here, we are using singular homology. The boundary map from $C_n(X, A)$ to $C_{n-1}(X, A)$ is defined as the composite of the two maps

$$\partial : H_n(\overline{X_n \cup A}, \overline{X_{n-1} \cup A}) \to H_{n-1}(\overline{X_{n-1} \cup A})$$

and

$$j : H_{n-1}(\overline{X_{n-1} \cup A}) \to H_{n-1}(\overline{X_{n-1} \cup A}, \overline{X_{n-2} \cup A})$$

where ∂ and j appear in the exact homology sequences for the pairs $(\overline{X_n \cup A}, \overline{X_{n-1} \cup A})$, and $(\overline{X_{n-1} \cup A}, \overline{X_{n-2} \cup A})$ respectively. For each n cell e^n of $X - A$, pick a characteristic map $f : (E^n, S^{n-1}) \to (e^n, \dot{e}^n)$ and a lifting $\tilde{f}$ of f to $\tilde{X}$; then $\tilde{f}_* : H_n(E^n, S^{n-1}) \to C_n(X, A)$ maps the generator for $H_n(E^n, S^{n-1})$ to an element of $C_n(X, A)$. In this fashion we have constructed a countable basis for $C_n(X, A)$ as a $Z[\pi_1 X]$-module. Note that we have made choices in constructing this basis; but that the neighborhood system of infinity for $C_n(X, A)$ is independent of the choices made. In this way, $C(X, A)$ becomes a finite dimensional, countably based, locally finite chain complex.

Lemma 4.4. *If $C(X, A)$ is locally finitely contractible, then for any compact subset K of X there exists a bigger compact subset L such that the map induced by inclusion from $H_*(\overline{X-L}, \overline{A-L})$ to $H_*(\overline{X-K}, \overline{A-K})$ is the zero map.*

Proof. Our technique of proof is to construct two subcomplexes Y and Z of X such that both omit only a finite number of cells from X, $Z \subset Y$, $Y \cap K = \emptyset$, and the map from $H_*(\bar{Z}, \overline{Z \cap A})$ to $H_*(\bar{Y}, \overline{Y \cap A})$ induced by inclusion is zero. If we can do this, we have clearly proved (4.4). Since the carriers of cells in X form a locally finite collection, we define Y to be the union of the carriers of all cells whose carriers do not meet K. Recall that the n-cells of $X - A$ are in one to one correspondence with the basis elements of $C_n(X, A)$; let U_n be the submodule of $C_n(X, A)$; let U_n be the submodule of $C_n(X, A)$ generated by all the basis elements corresponding to n-cells of $Y - A$. Then U_n is a neighborhood of infinity for $C_n(X, A)$. Let $s_n : C_n(X, A) \to C_{n+1}(X, A)$ be the locally finite contraction operators. There exist neighborhoods of infinity U'_n for $C_n(X, A)$ such that $s_n(U'_n) \subseteq U_{n+1}$. Since U'_n is a neighborhood of infinity for $C_n(X, A)$, there exist a finite set of n-cells of X which we denote by S_n such that U'_n contains all the basis elements of $C_n(X, A)$ except those corresponding to the cells in S_n. Let L denote the union of the closures of all the cells from all the S'_n s. Let V be the union of all the carriers of all the cells whose carriers do not meet L; and let $Z = V \cap Y$. Define the chain complex $C^x(Y, Y \cap A)$ as follows:

$$C^x_n(Y, Y \cap A) = H_n(\overline{Y_n \cup (Y \cap A)}, \quad \overline{Y_{n-1} \cup (Y \cap A)})$$

The boundary maps are defined in the usual way. Likewise define $C^x(Z, Z \cap A)$. Then we see that $C^x(Z, Z \cap A)$ is a based chain subcomplex of $C^x(Y, Y \cap A)$ which in turn is a based chain subcomplex of $C(X, A)$. Since $S_n(C^x_n(Z, Z \cap A)) \subset C^x_{n+1}(Y, Y \cap A)$, we see that the inclusion map from $C^x(Z, Z \cap A)$ to $C^x(Y, Y \cap A)$ induces the zero map on homology. By a standard argument, this shows that the map induced by inclusion from $H_*(\bar{Z}, \overline{Z \cap A})$ to $H_*(\bar{Y}, \overline{Y \cap A})$ is zero.

If M is a countably based $Z[G]$-module and G is a countable group, then M is a countably based Z-module with basis elements of the form $g \cdot e$ where $g \in G$ and e is a member of the basis for M. If N is also a countably based $Z[G]$-module and $h : M \to N$ is locally finite $Z[G]$-homomorphism, then it is also a locally finite Z-homomorphism. $\mathrm{Hom}^f_Z(M, Z)$ can be made into a $Z[G]$-module by defining, for each $\phi \in \mathrm{Hom}^f_Z(M, Z)$ and $g \in G$, the homomorphism $(\phi)g : M \to Z$ to be the function that sends $m \in M$ to $\phi(g \cdot m)$. If $h : M \to N$ is a locally finite $Z[G]$-module homomorphism, then

$$\bar{h} : \mathrm{Hom}^f_Z(N, Z) \to \mathrm{Hom}^f_Z(M, Z)$$

is a $Z[G]$-module homomorphism. Hence, $\mathrm{Hom}^f_Z(\ \ , Z)$ is a functor from the category of countably based $Z[G]$-modules and locally finite $Z[G]$ homomorphisms to the category of $Z[G]$-modules. We can consider $\mathrm{Hom}^f_{Z[G]}(\ \ , Z[G])$ as another such functor.

LEMMA 4.5. *The two functors* $\mathrm{Hom}^f_{Z[G]}(\ \ , Z[G])$ *and* $\mathrm{Hom}^f_Z(\ \ , Z)$ *are naturally isomorphic.*

PROOF. Define $h : Z[G] \to Z$ to be the Z-linear map such that $h(1) = 1$ and $h(g) = 0$ if $g \in G$, $g \neq 1$. Define the natural isomorphism from $\mathrm{Hom}^f_{Z[G]}(M, Z[G])$ to $\mathrm{Hom}^f_Z(M, Z)$ to be Hom (id, h).

COROLLARY 4.6. *If C is a countably based, locally finite chain complex over $Z[G]$ where G is countable, then $\bar{C}$ is isomorphic (as a cochain complex) to the cochain complex* $\mathrm{Hom}^f_Z(C, Z)$.

LEMMA 4.7. *Let X be a connected, finite dimensional, locally finite CW-complex and A a subcomplex of X. Assume that the homology and* *cohomology with compact supports of the pair* $(\tilde{X}, \bar{A})$ *vanish; then* $C(X, A)$ *is locally finitely contractible.*

PROOF. In view of (4.2) and (4.6), we need only show that the chain complex $C(X, A)$ and the cochain complex $\mathrm{Hom}^f_Z(C(X, A), Z)$ are both acyclic. A standard argument shows that $C(X, A)$ is acyclic and that the cochain complex $D(X, A)$ is acyclic where

$$D^n(X, A) = H^n_c(\overline{X_n \cup A}, \overline{X_{n-1} \cup A})$$

and the coboundary maps are defined in the obvious way. But one can show, using the universal coefficient theorem relating homology to cohomology together with the expression for cohomology with compact supports as a direct limit of the cohomology of appropriate spaces, that $D(X, A)$ is isomorphic $\mathrm{Hom}^f_Z(C(X, A), Z)$.

Finally, here is the proof of (4.1):

The necessity of (1) and (2) is clear. We prove they are sufficient conditions. First deform $f: X \to Y$ to a cellular map $g: X \to Y$. We show that the hypothesis (a), (b), and (c) of (2.19) are satisfied for the pair (M_g, X) where M_g is the mapping cyclinder of g. Since X and M_g are finite dimensional, (3.1) says that X is a proper deformation retract of M_g. Hence f is a proper homotopy equivalence.

Now (a) and (b) of (2.19) are satisfied by assumption. Condition (2) implies by (4.4) and (4.7) that for any compact set C of M_g there is a compact subset D containing C such that

$$H_*(\overline{M_g - D}, \overline{X - D}) \to H_*(\overline{M_g - C}, \overline{X - C})$$

is the zero map. This says that $\varepsilon(M_g, X; \{p\}; H_k \sim) = 0$ where $\sim$ is as in example (iii) of § 2. Condition (1) implies $\pi_1 X \to \pi_1 M_g$ is an isomorphism. Since $f_*: H_*(\tilde{X}) \to H_*(\tilde{Y})$ is an isomorphism by (2) the ordinary Whitehead theorem says that f is a homotopy equivalence. Hence $\Delta(M_g, X; \{p\}; H_k, \sim) = 0$, which is just ($c$).

Q.E.D.

COROLLARY 4.8. *Let X and Y be connected, finite dimensional, locally finite CW-complexes and assume that each has a finite number of stable ends (see* [4]*). Let $\varepsilon_1, \cdots, \varepsilon_n$ denote the ends for X and $e_1, \cdots, e_m$ denote the ends of Y. Suppose that $n = m$ and that $\pi_1 \varepsilon_i$ maps monomorphically into $\pi_1(X)$ for each integer i, $1 \leqq i \leqq n$. Let f be a proper map from X to Y such that f takes the end ε_i and to the end e_i for each i, $1 \leqq i \leqq n$. Then f is a proper homotopy equivalence iff* (1) *f induces isomorphisms from $\pi_1 X$ to $\pi_1 Y$ and from $\pi_1 \varepsilon_i$ to $\pi_1 e_i$ for each i, $1 \leqq i \leqq n$;* (2) *f induces an isomorphism from $H_*(\tilde{X})$ to $H_*(\tilde{Y})$ and from $H^*_c(\tilde{Y})$ to $H^*_c(\tilde{X})$.*

COROLLARY 4.9. *Let X and Y be finite dimensional, locally finite CW-complexes which are simply connected and simply connected at infinity. Let f be a proper map from X to Y. Then f is a proper homotopy equivalence iff f induces isomorphisms from $H_*(X)$ to $H_*(Y)$ and from $H_c^*(Y)$ to $H_c^*(X)$.*

The derivation of Corollaries (4.8) and (4.9) from Theorem (4.1) will be left to the reader.

Let X^n be an open, smooth or piecewise linear, connected manifold of dimension n. Denote by $H_*^{l.f.}(X; Z^t)$ the homology groups based on locally finite, infinite simplicial chains (with respect to a given triangulation) with twisted integral coefficients Z^t. Then $H_n^{l.f.}(X; Z^t) = Z$. The first Stiefel-Whitney class of X determines the orientation homomorphism $\omega_x : \pi_1 X \to \{\pm 1\}$. Let $f : X^n \to Y^n$ be a properly 0-connected map between two open connected n-manifolds X and Y. Suppose there is a commutative diagram

$$\begin{array}{ccc} \pi_1 X & \xrightarrow{f\#} & \pi_1 Y \\ & {\scriptstyle \omega_x} \searrow \quad \swarrow {\scriptstyle \omega_y} & \\ & \{\pm 1\} & \end{array}$$

Then f induces a map

$$f_* : H_*^{l.f.}(X; Z^t) \to H_*^{l.f.}(Y; Z^t)$$

and the *degree of f*, written $\deg f$, is the integer determined by

$$f_n : H_n^{l.f.}(X; Z^t) \to H_n^{l.f.}(Y; Z^t)$$

in the usual way.

COROLLARY 4.10. *Let $f : X^n \to Y^n$ be a properly 0-connected map between open smooth or p.l. manifolds of dimension n. Suppose $\theta : \Delta(X, \{p\}; \pi_1, \text{no cov}) \to \prod_p \pi_1(X, p)$ is a monomorphism. Then f is a proper homotopy equivalence iff f is a homotopy equivalence and* $\deg f = 1$.

To prove (4.10) we must show that conditions (1) and (2) of (4.1) hold. Condition (2) follows by Poincaré duality from the fact that f is a homotopy equivalence. It remains to get condition (1). Since f is a homotopy equivalence and θ is a monomorphism, to show that (1) holds it is sufficient to show that

$$\gamma : \varepsilon(X, \{p\}; \pi_1, \text{no cov}) \to \varepsilon(Y, \{f(p)\}; \pi_1, \text{no cov})$$

is surjective.

It is no loss of generality to assume that n is large. This is because

$f: X \to Y$ is a proper homotopy equivalence iff $f \times \mathrm{id} : X \times S^k \to Y \times S^k$ is a proper homotopy equivalence for any large integer k. Now write Y as an increasing union $Y = \cup L_i$ where $L_0 \subset \mathrm{int}\, L_1 \subset \mathrm{int}\, L_2 \subset \cdots$ and L_i is a compact, connected n-submanifold of Y. Since f has degree $= +1$, an easy surgery argument shows f can be deformed so that there is a sequence $K_0 \subset \mathrm{int}\, K_1 \subset \cdots$ of connected, compact n-submanifolds K_i of X satisfying

(a) $f^{-1}(L_i) = K_i$ and $f^{-1}(Y - L_i) = X - K_i$

(b) $f: X - K_i \to Y - L_i$ maps each infinite component X_α of $X - K_i$ onto exactly one infinite component Y_α of $Y - L_i$.

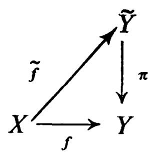
Now in view of the remark following (2.4), to show that γ is onto it suffices to show that for any i and any pair (X_α, Y_α) as in (b) above we have

(4.11) $$f_\# : \pi_1(X_\alpha) \to \pi_1(Y_\alpha) \text{ is surjective.}$$

To show this first note

LEMMA 4.12. *Suppose $f: X^n \to Y^n$ is a map of degree one. Then $f_\# : \pi_1 X \to \pi_1 Y$ is surjective.*

PROOF OF 4.12. Consider the commutative diagram



where $\pi : \tilde{Y} \to Y$ is the covering space of Y corresponding to the subgroup $f_\#(\pi_1 X) \subset \pi_1 Y$. Let $H_*^\pi(\tilde{Y}; Z^t)$ be the homology with twisted integral coefficients based on locally finite, infinite simplicial chains c on $\tilde{Y}$ with the property that for each k-simplex σ in Y there are at most finitely many k-simplices lying over σ in $\tilde{Y}$ which appear with a non-zero coefficient in c. The group $H_n^\pi(\tilde{Y}; Z^t)$ is 0 or Z and is Z iff π is a finite sheeted cover. The map

$$f_* : H_n^{l.f.}(X; Z^t) \to H_n^{l.f.}(Y; Z^t)$$

factors as $f_* = \pi_* \circ \tilde{f}_*$ where

$$\pi_* : H_n^\pi(\tilde{Y}; Z^t) \to H_n^{l.f.}(Y; Z^t)$$

and

$$\tilde{f}_* : H_n^{l.f.}(X; Z^t) \to H_n^\pi(\tilde{Y}; Z^t).$$

Since $\deg f = 1 \neq 0$, π must be a finite sheeted cover. Since $\deg f_* =$

$\deg \pi_* \circ \deg f_* = 1$ we see that π must be a one-sheeted cover. Hence $f_{\#}(\pi_1 X) = \pi_1 Y$.

Now (4.11) follows from the lemma because whenever $f: X \to Y$ has degree one so does $f: X_\alpha \to Y_\alpha$.

Corollary (4.10) applies, for example, when both X and Y are simply connected and simply connected at infinity.

REFERENCES

F. T. Farrell and J. B. Wagoner

[1] An algebraic criterion for a map to be a proper homotopy equivalence, Notices A. M. S., Nov. 1969.

P. J. Hilton

[2] *An introduction to homotopy theory*, Cambridge University Press, Cambridge Tracts in Mathematics and Mathematical Physics. No. 43.

S. P. Novikov

[3] Homotopic and topological invariance of certain rational classes of Pontryagin, Doklady 1965, Tom 162, No. 6.

L. C. Siebenmann

[4] Thesis, Princeton University (1965).

L. C. Siebenmann

[5] Infinite simple homotopy types, Indag. Nath. 32, No. 5, 1970.

E. H. Spanier

[6] *Algebraic Topology*, McGraw-Hill (1966).

L. R. Taylor

[7] Surgery on paracompact manifolds, thesis, University of California at Berkeley, 1971.

J. H. C. Whitehead

[8] Combinatorial Homotopy I and II, Bull. Amer. Math. Soc. 55, 3 and 5 (1949).

(Oblatum 4–X–71)

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