COMPOSITIO MATHEMATICA

LOUIS PIGNO

A note on translates of bounded measures

Compositio Mathematica, tome 26, nº 3 (1973), p. 309-312
<http://www.numdam.org/item?id=CM_1973__26_3_309_0>

COMPOSITIO MATHEMATICA, Vol. 26, Fasc. 3, 1973, pag. 309–312
Noordhoff International Publishing
Printed in the Netherlands

A NOTE ON TRANSLATES OF BOUNDED MEASURES

by

Louis Pigno

In this paper G is a locally compact group and $M(G)$ the measure algebra of G. By $L^p(G)$ $(1 \leqq p \leqq \infty)$ we mean the usual Lebesgue space of index p formed with respect to left Haar measure λ on G.

For $x \in G$ the left translate ${}_x\mu$ of $\mu \in M(G)$ may be defined by

$$ {}_x\mu(E) = \mu(xE) $$

where E is any Borel subset of G. Many authors (see [3, p. 278], [6, p. 230], [7], and [8, pp. 91–93]) have characterized in terms of translation those measures $\mu \in M(G)$ which are absolutely continuous with respect to λ. In this paper we give a characterization of absolutely continuous measures in terms of translation which enables us to solve a multiplier problem of Doss.

For abelian groups satisfying the first axiom of countability, Edwards [2, p. 407] has proved the following interesting theorem:

THEOREM 1. *Let $\mu \in M(G)$ be such that for each relatively compact Borel subset V of G the function $x \to \mu(V+x)$ is equal locally a.e. to a continuous function on G. Then μ is absolutely continuous with respect to λ.*

Actually Edwards proves Theorem 1 for all Radon measures, but we need only consider finite measures in the sequel. The hypothesis that G be first countable is used to obtain a sequence for an approximate identity instead of the usual net.

Theorem 1 has applications to the problem of multipliers of Fourier transforms so it is natural to want to remove the hypothesis of first countable. Moreover, in the multiplier problem which we shall consider, the following situation obtains: For each relatively compact Borel subset V of G the function $x \to \mu(V+x)$ is equal a.e. to a function h which is continuous except possibly on a null set. The desired conclusion, namely that μ be absolutely continuous, does not now follow from Theorem 1 even if G be first countable.

The proof of the following theorem was suggested by a reading of Theorem (35.13) of [4] and the referee's report on [5].

THEOREM 2. *Let G be a locally compact group and $\mu \in M(G)$ such that* *for each relatively compact Borel subset V of G the function $x \to \mu(xV)$ is equal locally a.e. to a function h on G which is continuous except possibly on a locally null set. Then μ is absolutely continuous with respect to λ.*

PROOF. As in [5] the proof of the present theorem is obtained by modifying the proof of Theorem (35.13) of [4]. Our notation for the remainder of the proof is that of [4].

Consider the Lebesgue decomposition of μ with respect to λ: $\mu = \mu^a + \nu$. It suffices to show that ν is the zero measure. Suppose not; then we may assume without loss of generality that $\|\nu\| = 1$. Let B, F, α, U, V_0, and W_0 be as in Theorem (35.13) of [4, pp. 382–383]. We may also assume U has compact closure. Define

$$c = \max\left\{\frac{1}{\Delta(x)} : x \in U^-\right\} < \infty,$$

where Δ is the modular function on G. Let H_0 be the open subgroup generated by the compact set $U^- \cup W_0^- \cup V_0^-$.

For each positive integer n construct open sets V_n such that,

$$F \subset V_n \subset V_0 \text{ and } c\lambda(V_n) < (\tfrac{1}{2})^n \lambda(W_0).$$

As in Theorem (35.13) of [4, p. 383] there is a neighborhood W_n of the identity such that $W_n \subset W_0$ and $W_n F \subset V_n$. Clearly the sets F, V_n, and W_n are subsets of H_0. By Theorem (8.7) of [3, p. 71] there is a closed normal subgroup H_1 of H_0 such that

$$H_1 \subsetneq \bigcap_{n=1}^{\infty} W_n$$

and H_0/H_1 is a metric group satisfying the second axiom of countability. Letting H_0/H_1 play the role of G/G_0, we construct W^{-1} and V as in Theorem (35.13). W^{-1} is an open dense subset of W_0 such that

$$|\nu(xV)| \geqq \alpha \qquad (x \in W^{-1}). \tag{1}$$

By construction we also have that

$$c\lambda(V) < \alpha\lambda(W_0). \tag{2}$$

By hypothesis, there is a function h such that $h(x) = \nu(xV)$ l.a.e. with h continuous except perhaps on a locally null set A_1. The set $A_2 = \{x \in G : \nu(xV) \neq h(x)\}$ is by hypothesis locally null. Hence $A = A_1 \cup A_2$ is locally null. Since $W^{-1} \setminus A$ is dense in $W_0 \setminus A$ we have by (1)

$$|h(x)| \geqq \alpha \quad (x \in W_0 \setminus A).$$

Hence

$$\alpha\lambda(W_0 \setminus A) \leqq \int_{W_0 \setminus A} |h|\, d\lambda = \int_{W_0 \setminus A} |\nu(xV)|\, d\lambda(x).$$

Since

$$\int_{W_0 \setminus A} \nu(xV)\, d\lambda(x) \leqq \int_G |\nu(xV)|\, d\lambda(x)$$

and $\nu(xV) = (\nu * \xi_{V^{-1}})(x)$ a.e. we conclude that

$$\alpha\lambda(W_0 \setminus A) \leqq \|\nu\| \|\xi_{V^{-1}}\| = \lambda(V^{-1}).$$

Next we observe that

$$\lambda(V^{-1}) = \int_V \frac{1}{\Delta}\, d\lambda \leqq c\lambda(V)$$

hence $\alpha\lambda(W_0 \setminus A) \leqq c\lambda(V)$. Since $\lambda(W_0 \setminus A) = \lambda(W_0)$ we conclude by (2) that

$$\alpha\lambda(W_0) \leqq c\lambda(V) < \alpha\lambda(W_0)$$

which is a contradiction. The proof is now complete.

We now proceed to our main result. Let φ be a complex-valued function defined on the additive group of real numbers R. Put $\{2\} = L^1(R) \cap L^\infty(R)$ and let $\{3\}$ be the set of $f \in \{2\}$ which are Riemann integrable on every finite interval. Doss [1, p. 170] has posed the problem of determining the multipliers (2, 3). The function φ is said to be a multiplier of type (2, 3) if, given $f \in \{2\}$, there corresponds a $g \in \{3\}$ such that $\varphi\hat{f} = \hat{g}$, where $\wedge$ denotes the Fourier transformation. We prove the following theorem:

THEOREM 3. *The function φ is a multiplier of type* (2, 3) *if and only if $\varphi = \hat{f}$ for some $f \in L^1(R)$.*

PROOF. One half of the theorem is obvious. To prove the converse, we suppose φ is a multiplier of type (2, 3). This implies (see [5, p. 757]) that $\varphi = \hat{\mu}$ for some $\mu \in M(R)$.

Let V be any relatively compact Borel subset of R and ξ_{-V} the characteristic function of $-V$. By hypothesis the convolution $\mu * \xi_{-V}$ is a.e. equal to a function h which is continuous except perhaps on a set of Lebesgue measure zero. By Theorem 2 $d\mu(x) = f(x)dx$ for some $f \in L^1(R)$ and this concludes the proof.

REFERENCES

R. Doss

[1] 'On the multiplicators of some classes of Fourier transforms', Proc. London Math. Soc., (2) 50 (1949), 169–195.

R. E. Edwards
[2] 'Translates of L_∞ functions and of bounded measures', J. Austr. Math. Soc., IV (1964), 403–409.

E. Hewitt and K. A. Ross
[3] *Abstract Harmonic Analysis*, Vol. I, Springer-Verlag, Heidelberg and New York, 1963.

E. Hewitt and K. A. Ross
[4] *Abstract Harmonic Analysis*, Vol. II, Springer-Verlag, Heidelberg and New York, 1964.

L. Pigno
[5] 'A multiplier theorem', Pacific J. Math., 34 (1970), 755–757.

W. Rudin
[6] 'Measure algebras on abelian groups', Bull. Amer. Math. Soc., 65 (1959), 227–247.

D. A. Raikov
[7] 'On absolutely continuous set functions', Doklady Akad. Nauk. S.S.S.R. (N.S.), 34 (1942), 239–241.

S. Saks
[8] *Theory of the Integral*, Second Ed., Hafner, New York, 1937.

(Oblatum 4–IX–1972)

Kansas State University
Dept. of Mathematics
MANHATTAN, Kansas 66502
U.S.A.