JOAN RAND MOSCHOVAKIS

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A TOPOLOGICAL INTERPRETATION OF SECOND-ORDER INTUITIONISTIC ARITHMETIC

by

Joan Rand Moschovakis

In [7] Dana Scott developed a classical topological model for the first-order intuitionistic theory of the continuum as a dense partially ordered set, using continuous functions from e.q. Baire space into the classical reals to interpret the intuitionistic reals. Later (in [8]) he extended this model to the second-order theory, interpreting intuitionistic real functions by certain 'extensional' operators on the reals of his model, and verifying the continuity principle and a strong version of Kripke's Schema. The problem remained to adapt Scott's interpretation to the usual axiomatic theories of intuitionistic analysis, expressed in the language of second-order number theory (cf. [1], [2]).

Here we give such an adaptation, to a theory **I!** not much weaker than Kleene's **I**([1]), and thus obtain a classical consistency proof for a system of second-order intuitionistic number theory with Kripke's Schema. (It is well known that **I** itself is inconsistent with Kripke's Schema, cf. [3], pp. 173–174.)

Henceforth we assume familiarity with Chapter I of Kleene-Vesley [1]. Let **I!** be Kleene's system **I** of second-order intuitionistic arithmetic, but with Brouwer's Principle ^{x}27.1 replaced by

^{x}27.1! $$\forall\alpha\exists!\ \beta A(\alpha,\beta) \supset \exists\tau\forall\alpha\{\forall t\exists!\ y\tau(2^{t+1}*\bar{\alpha}(y))>0\ \&\ \forall\beta[\forall t\exists y\tau(2^{t+1}*\bar{\alpha}(y))=\beta(t)+1\supset A(\alpha,\beta)]\},$$

or by the equivalent (cf. [4], pp. 21–22)

^{x}27.2!. $$\forall\alpha\exists!bA(\alpha,b)\supset\exists\tau\forall\alpha\exists y\{\tau(\bar{\alpha}(y))>0\ \&\ \forall x[\tau(\bar{\alpha}(x))>0\supset y=x]\ \&\ A(\alpha,\tau(\bar{\alpha}(y))\dot{-}1)\}.$$

Let KS^s be the strong form of Kripke's Schema:

KS^s. $\exists\alpha[\exists x\alpha(x)\neq 0\sim A]$,

where α is not free in A. We shall give a topological model for $\mathbf{I!}+KS^s$.

One possible objection to this model – perhaps to any topological interpretation – is that it *may* fail to satisfy some open instances of

'Brouwer's Principle for numbers' $^{x}27.2$ (like $^{x}27.2!$ but without the '!'). Thus the consistency of e.g. Myhill's system [5] is still in doubt. However, $^{x}27.2!$ ought to be strong enough for an intuitionist, who – if he asserts 'for each α there is a b' – should have in mind some specific way of assigning to each α a (unique) b. Moreover, in § 3 we show that all instances of $^{x}27.2$ without free function variables *are* valid in the model, whence so are all the standard examples of classically false consequences of Brouwer's Principle discussed in §§ 7.10–7.14 of [1].

A more serious objection is that, although the objects of our model can be interpreted intuitionistically, for several of the postulates we can give only classical verifications. Specifically, we use classical bar induction (in verifying $^{x}2.1$ and RDC-F), the law of double negation (in the proof of Lemma 4 and in the verifications of $^{x}26.3c$, $^{x}27.2!$, and $^{x}27.2$ without free function variables), definition by classical cases (in the proof of Lemma 6 and the verification of KS^{s}), and countable dependent choices (in the proof of Lemma 4 and the verification of $^{x}26.3c$).

§ 1. Description of the Model and Statement of Results

Let $N = \{0, 1, 2, 3, \cdots\}$, let N^N be Baire space, and let $\mathscr{IPS}$ be the set of all continuous operators on N^N. With each formula $\mathrm{A}(\alpha_1, \cdots, \alpha_k, \mathrm{x}_1, \cdots, \mathrm{x}_k)$ containing free at most the distinct i.p.s. (infinitely proceeding sequence, or function) variables $\alpha_1, \cdots, \alpha_k$ and number variables $\mathrm{x}_1, \ldots, \mathrm{x}_n$, and with each assignment of values $\xi_1, \cdots, \xi_k \in \mathscr{IPS}$ to $\alpha_1, \cdots, \alpha_k$ and each assignment of values $x_1, \cdots, x_n \in N$ to $\mathrm{x}_1, \cdots, \mathrm{x}_n$, we associate a topological value $[\![\mathrm{A}(\xi_1, \cdots, \xi_k, x_1, \cdots, x_n)]\!]$(an open subset of N^N) as follows.

First consider a prime formula $\mathrm{s}(\alpha_1, \cdots, \alpha_k, \mathrm{x}_1, \cdots, \mathrm{x}_n) = \mathrm{t}(\alpha_1, \cdots, \alpha_k, \mathrm{x}_1, \cdots, \mathrm{x}_n)$ where s, t are terms expressing the (primitive recursive, hence continuous) number-valued functions $s(\alpha_1, \cdots, \alpha_k, x_1, \cdots, x_n)$, $t(\alpha_1, \cdots, \alpha_k, x_1, \cdots, x_n)$ respectively. Define

$$[\![\mathrm{s}(\xi_1, \cdots, \xi_k, x_1, \cdots, x_n) = \mathrm{t}(\xi_1, \cdots, \xi_k, x_1, \cdots, x_n)]\!] = \\ = \{\beta \in N^N : s(\xi_1(\beta), \cdots, \xi_k(\beta), x_1, \cdots, x_n) = t(\xi_1(\beta), \cdots, \xi_k(\beta), x_1, \cdots, x_n)\}.$$

(This is open since s, t, $\xi_1, \cdots, \xi_k$ are continuous.)

Values of composite formulas are determined inductively by

$$[\![\mathrm{A} \,\&\, \mathrm{B}]\!] = [\![\mathrm{A}]\!] \cap [\![\mathrm{B}]\!]$$
$$[\![\mathrm{A} \vee \mathrm{B}]\!] = [\![\mathrm{A}]\!] \cup [\![\mathrm{B}]\!]$$
$$[\![\neg \mathrm{A}]\!] = In\,(N^N - [\![\mathrm{A}]\!])$$
$$[\![\mathrm{A} \supset \mathrm{B}]\!] = In\,[(N^N - [\![\mathrm{A}]\!]) \cup [\![\mathrm{B}]\!]]$$

$$[\![\exists \mathrm{x} \mathrm{A}(\mathrm{x})]\!] = \bigcup_{x \in N} [\![\mathrm{A}(x)]\!]$$

$$[\![\forall \mathrm{x} \mathrm{A}(\mathrm{x})]\!] = In \bigcap_{x \in N} [\![\mathrm{A}(x)]\!]$$

$$[\![\exists \alpha \mathrm{A}(\alpha)]\!] = \bigcup_{\xi \in \mathscr{IPS}} [\![\mathrm{A}(\xi)]\!]$$

$$[\![\forall \alpha \mathrm{A}(\alpha)]\!] = In \bigcap_{\xi \in \mathscr{IPS}} [\![\mathrm{A}(\xi)]\!]$$

Call the formula E *valid* if $[\![\mathrm{E}]\!] = N^N$ for every choice of values of the free variables of E. We shall prove (classically) the

THEOREM. *If* $\vdash_{\mathbf{I}!+\mathrm{KS}^s}\mathrm{E}$ *then* E *is valid.*

Since $1 = 0$ is not valid (in fact, $[\![1 = 0]\!] = \emptyset$), we obtain the immediate

COROLLARY. $\mathbf{I}!+\mathrm{KS}^s$ *is consistent.*

§ 2. Proof of the Theorem

We must show that every axiom of $\mathbf{I}!+\mathrm{KS}^s$ is valid, and that the rules of inference preserve validity. For the postulates of two-sorted intuitionistic predicate calculus (Group A of [1], p. 13) we refer to [6], except that for the quantifier schemata ION–IIN, IOF–IIF we need a lemma on evaluating terms and functors (cf. [1] p. 10), as follows.

If $\mathrm{s}(\alpha_1, \cdots, \alpha_k, \mathrm{x}_1, \cdots, \mathrm{x}_n)$ is a term (containing free at most the distinct variables shown) expressing the primitive recursive, hence continuous, function $s(\alpha_1, \cdots, \alpha_k, x_1, \cdots, x_n)$ and if $\xi_1, \cdots, \xi_k \in \mathscr{IPS}$, $x_1, \cdots, x_n \in N$, and $\beta \in N^N$, define

$$x^s(\xi_1, \cdots, \xi_k, x_1, \cdots, x_n, \beta) = s(\xi_1(\beta), \cdots, \xi_k(\beta), x_1, \cdots, x_n).$$

If $\mathrm{u}[\alpha_1, \cdots, \alpha_k, \mathrm{x}_1, \cdots, \mathrm{x}_n]$ is a functor expressing the functional

$$\lambda x u[\alpha_1, \cdots, \alpha_k, x_1, \cdots, x_n](x),$$

and if

$$\xi_1, \cdots, \xi_k \in \mathscr{IPS}, x_1, \cdots, x_n \in N,$$

define

$$\xi^u[\xi_1, \cdots, \xi_k, \ x_1, \ \cdots, x_n] = \lambda\beta\lambda x u[\xi_1(\beta), \cdots, \xi_k(\beta), x_1, \ \cdots, x_n](x).$$

Since u is primitive recursive and $\xi_1, \cdots, \xi_k$ are continuous,

$$\xi^u[\xi_1, \cdots, \xi_k, x_1, \cdots, x_n] \in \mathscr{IPS}.$$

We now state the lemma, omitting for simplicity the free variables other than x, α which may occur in A, s, or u. The proof, which is an exercise for the reader, is by a straightforward but messy induction on the logical form of A.

LEMMA 1. (a) *If* s *is any term free for* x *in* A(x), *then (for each choice of values for the free variables of* A(s) *and) for each* $\beta \in N^N : \beta \in [\![A(s)]\!]$ *iff* $\beta \in [\![A(x^s(\beta))]\!]$.
(b) *Similarly, if* u *is a functor free for* α *in* A(α), *then for each* $\beta \in N^N$:

$$\beta \in [\![A(u)]\!] \text{ iff } \beta \in [\![A(\xi^u)]\!].$$

Consider now an axiom by the schema ION: $\forall x A(x) \supset A(s)$, where s is a term free for x in A(x). Assume

$$\beta \in [\![\forall x A(x)]\!] = In \bigcap_{x \in N} [\![A(x)]\!];$$

then in particular $\beta \in [\![A(x^s(\beta))]\!]$, whence by Lemma 1(a) $\beta \in [\![A(s)]\!]$. So the axiom is valid. Similarly for IIN, IOF, IIF.

We observe that if A contains no ips variable, then $[\![A]\!]$ is N^N or $\emptyset$ according as A is true or false; hence the (classical) first-order theory of our model is just classical number theory. It follows immediately that all the postulates of intuitionistic number theory (Group B of [1], p. 14), except possibly the induction schema 13, are valid. For 13, if $\beta \in [\![A(0)\ \&\ \forall x(A(x) \supset A(x'))]\!]$, then $\beta \in [\![A(x)]\!]$ by induction on $x \in N$, using Lemma 1(a).

The postulates $^x0.1.$ $\{\lambda x r(x)\}(t) = r(t)$ and $^x1.1.$ $a = b \supset \alpha(a) = \alpha(b)$ of [1] Group C are trivially valid, but the axiom of choice $^x2.1$ needs the following lemma.

LEMMA 2. *If* $\beta \in N^N$, A(α) *is any formula,* $\xi, \eta \in \mathscr{I}\mathscr{P}\mathscr{S}$ *and*

$$\beta \in [\![\xi = \eta]\!] = [\![\forall x \xi(x) = \eta(x)]\!],$$

then $\beta \in [\![A(\xi)]\!]$ *if and only if* $\beta \in [\![A(\eta)]\!]$.

PROOF. If $\beta \in [\![\xi = \eta]\!]$ then for some $z_0 \in N$ and all $\gamma \in N^N$ with

$$\bar{\gamma}(z_0) = \bar{\beta}(z_0): \qquad \gamma \in \bigcap_{x \in N} [\![\xi(x) = \eta(x)]\!],$$

whence $(x)([\xi(\gamma)](x) = [\eta(\gamma)](x))$, i.e. $\xi(\gamma) = \eta(\gamma)$. So ξ, η agree on a neighborhood of β, whence (by induction on the logical form of A(α))

$$\beta \in [\![A(\xi)]\!] \text{ iff } \beta \in [\![A(\eta)]\!].$$

VERIFICATION OF $^x2.1.$ $\forall x \exists \alpha A(x, \alpha) \supset \exists \alpha \forall x A(x, \lambda y \alpha(2^x \cdot 3^y))$. If $\beta \in [\![\forall x \exists \alpha A(x, \alpha)]\!]$, then there is a neighborhood $V_0 = \{\gamma \in N^N : \bar{\gamma}(z_0) = \bar{\beta}(z_0)\}$ of β such that, for each $x \in N$,

$$V_0 \subset \bigcup_{\xi \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![A(x, \xi)]\!].$$

Since the $[\![A(x, \xi)]\!]$ are open, by a classical bar induction we can partition V_0 into countably many disjoint (clopen) neighborhoods V_1^x, V_2^x, V_3^x, $\cdots$ with associated elements ξ_1^x, ξ_2^x, ξ_3^x, $\cdots$ of $\mathscr{I}\mathscr{P}\mathscr{S}$ such that, if $\gamma \in V_j^x$, then $\gamma \in [\![A(x, \xi_j^x)]\!]$. Now define, for each $\gamma \in N^N$ and each $n \in N$:

$$[\zeta(\gamma)](n) = \begin{cases} [\xi_j^x(\gamma)](y) \text{ if } n = 2^x \cdot 3^y \text{ and } \gamma \in V_j^x; \\ 0 \text{ otherwise.} \end{cases}$$

Clearly $\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}$ and, for each $x \in N$ and $\gamma \in V_0$,

$$\lambda y[\zeta(\gamma)](2^x \cdot 3^y) = \xi_j^x(\gamma) \text{ where } \gamma \in V_j^x,$$

whence $\gamma \in [\![A(x, \lambda\gamma\lambda y[\zeta(\gamma)](2^x \cdot 3^y))]\!]$ by Lemma 2, so $\gamma \in [\![A(x, \lambda y\zeta(2^x \cdot 3^y))]\!]$ by Lemma 1(b).

To establish the validity of the Bar Theorem ^{x}26.3c we require two more lemmas.

LEMMA 3. *If $\beta \in N^N$, A(α) is any formula, $\xi, \eta \in \mathscr{I}\mathscr{P}\mathscr{S}$, and $\beta \in Cl\,[\![\xi = \eta]\!]$, and if $[\![A(\xi)]\!]$ and $[\![A(\eta)]\!]$ are clopen, then $\beta \in [\![A(\xi)]\!]$ if and only if $\beta \in [\![A(\eta)]\!]$.*

PROOF. Let $\beta_1, \beta_2, \beta_3, \cdots$ be a sequence of elements of $[\![\xi = \eta]\!]$ converging to β. By Lemma 2, $\beta_i \in [\![A(\xi)]\!]$ if and only if $\beta_i \in [\![A(\eta)]\!]$. Since $[\![A(\xi)]\!]$, $[\![A(\eta)]\!]$ are clopen, the conclusion follows.

LEMMA 4. *Suppose $\beta \in [\![\forall\alpha\exists!xR(\bar{\alpha}(x))\ \&\ \forall w[\mathrm{Seq}\,(w)\ \&\ R(w) \supset A(w)]\ \&\ \forall w[\mathrm{Seq}\,(w)\ \&\ \forall sA(w*2^{s+1}) \supset A(w)]]\!]$. Suppose w is a sequence number* ([1] p. 46) *and $\beta \notin [\![A(w)]\!]$ and*

$$(z)_{z<lh(w)}\,\beta \notin [\![R(\overline{(\lambda t(w)_t \dot- 1)(z)})]\!].$$

Then there must be some sequence number u for which

$$(z)_{z<lh(w*u)}\,\beta \notin [\![R(\overline{(\lambda t(w*u)_t \dot- 1)(z)})]\!]$$

and

$$\beta \in \bigcap_{s\in N} [\![A(w*u*2^{s+1})]\!]$$

but

$$\beta \notin In \bigcap_{s\in N} [\![A(w*u*2^{s+1})]\!].$$

PROOF. Assume the hypotheses, and assume for contradiction that if u is any sequence number such that

$$(z)_{z<lh(w*u)}\,\beta \notin [\![R(\overline{(\lambda t(w*u)_t \dot- 1)(z)})]\!]$$

and

$$\beta \in \bigcap_{s\in N} [\![A(w*u*2^{s+1})]\!]$$

then

$$\beta \in In \bigcap_{s \in N} [\![A(w * u * 2^{s+1})]\!].$$

Then

$$\beta \notin \bigcap_{s \in N} [\![A(w * 2^{s+1})]\!];$$

otherwise the assumptions, with $u = 1$ (the number of the empty sequence), imply

$$\beta \in In \bigcap_{s \in N} [\![A(w * 2^{s+1})]\!]$$

so $\beta \in [\![A(w)]\!]$, a contradiction. So there is some $s_1 \in N$ for which

$$\beta \notin [\![A(w * 2^{s_1+1})]\!], \text{ whence } \beta \notin [\![R(w * 2^{s_1+1})]\!].$$

In general, given $s_1, \cdots, s_n$ such that

$$(z)_{z < lh(w)+n}\beta \notin [\![R(\overline{(\lambda t(w * 2^{s_1+1} * \cdots * 2^{s_n+1})_t \dot{-} 1)(z)})]\!]$$

and

$$\beta \notin [\![A(w * 2^{s_1+1} * \cdots * 2^{s_n+1})]\!],$$

choose $s_{n+1} \in N$ so that

$$\beta \notin [\![A(w * 2^{s_1+1} * \cdots * 2^{s_n+1} * 2^{s(n+1)+1})]\!],$$

whence

$$\beta \notin [\![R(w * 2^{s_1+1} * \cdots * 2^{s_n+1} * 2^{s(n+1)+1})]\!].$$

Now define $\gamma_0 \in N^N$ by

$$\gamma_0(n) = \begin{cases} (w)_n \dot{-} 1 \text{ if } n < lh(w), \\ s_{n+1-lh(w)} \text{ if } n \geqq lh(w). \end{cases}$$

If $\xi_0 \notin \mathscr{I}\mathscr{P}\mathscr{S}$ is defined by

$$\xi_0(\alpha) = \gamma_0 \text{ for all } \alpha \in N^N,$$

then $(x)\beta \notin [\![R(\overline{\xi_0}(x))]\!]$ (using Lemmas 1 and 2), contradicting $\beta \in [\![\forall\alpha\exists! xR(\overline{\alpha}(x))]\!]$ and establishing the lemma.

VERIFICATION OF ^{x}26.3c. $\forall\alpha\exists!xR(\overline{\alpha}(x))$ & $\forall$w[Seq (w) & R(w) $\supset$ A(w)] & $\forall$w[Seq (w) & $\forall$sA(w$*2^{s+1}$) $\supset$ A(w)] $\supset$ A(1). Assume $\beta_0 \in N^N$ and V_0 is a neighborhood of β_0 such that $V_0 \subseteq [\![\forall\alpha\exists!\, xR(\overline{\alpha}(x))]\!] \cap [\![\forall w [\text{Seq}(w) \& R(w) \supset A(w)]]\!] \cap [\![\forall w[\text{Seq}(w) \& \forall sA(w*2^{s+1}) \supset A(w)]]\!]$ but $\beta_0 \notin [\![A(1)]\!]$. Then $\beta_0 \notin [\![R(1)]\!]$, so by Lemma 4 there is a sequence number u_1 for which

$$(z)_{z < lh(u_1)}\beta_0 \notin [\![R(\overline{(\lambda t(u_1)_t \dot{-} 1)(z)})]\!]$$

and

$$\beta_0 \in \bigcap_{s \in N} [\![A(u_1 * 2^{s+1})]\!]$$

but

$$\beta_0 \notin In \bigcap_{s \in N} [\![A(u_1 * 2^{s+1})]\!].$$

Since $[\![R((\overline{\lambda t(u_1)_t \dot{-} 1})(z))]\!]$ is *clopen* in V_0 (since $V_0 \subseteq [\![\exists!xR(\overline{\lambda t(u_1)_t \dot{-} 1})$ (x))]] by hypothesis), there is a neighborhood $V_1 \subseteq V_0$ of β_0 such that

$$(z)_{z<lh(u_1)}(V_1 \cap [\![R((\overline{\lambda t(u_1)_t \dot{-} 1})(z))]\!] = \emptyset),$$

So there are $s_1 \in N$ and $\beta_1 \in N^N$ such that

($\mathbf{a}_1$) $\beta_1 \in V_1 \subseteq V_0$,
($\mathbf{b}_1$) $\beta_1 \notin [\![A(u_1 * 2^{s_1+1})]\!]$,
($\mathbf{c}_1$) $\bar{\beta}_1(1) = \bar{\beta}_0(1)$.

By (a_1) and (b_1),

$$(z)_{z<lh(u_1*2^{s_1+1})}\beta_1 \notin [\![R(\overline{\lambda t(u_1 * 2^{s_1+1})_t \dot{-} 1})(z))]\!].$$

In general, suppose $s_1, \cdots s_n \in N$, $u_1, \cdots, u_n$ are sequence numbers, and $\beta_n \in V_0$, such that (putting $w_n = u_1 * 2^{s_1+1} * \cdots * u_n * 2^{s_n+1}$) $\beta_n \notin$ $[\![A(w_n)]\!]$
and

$$(z)_{z<lh(w_n)}\beta_n \notin [\![R((\overline{\lambda t(w_n)_t \dot{-} 1})(z))]\!].$$

Then by Lemma 4 there is a sequence number u_{n+1} such that

$$(z)_{z<lh(w_n*u_{n+1})}\beta_n \notin [\![R((\overline{\lambda t(w_n * u_{n+1}) \dot{-} 1})(z))]\!]$$

and

$$\beta_n \in \bigcap_{s \in N} [\![A(w_n * u_{n+1} * 2^{s+1})]\!]$$

but

$$\beta_n \notin In \bigcap_{s \in N} [\![A(w_n * u_{n+1} * 2^{s+1})]\!].$$

Again, there is a neighborhood $V_{n+1} \subseteq V_0$ of β_n such that

$$(z)_{z<lh(w_n*u_{n+1})}(V_{n+1} \cap [\![R((\overline{\lambda t(w_n * u_{n+1})_t \dot{-} 1})(z))]\!] = \emptyset).$$

So there are $s_{n+1} \in N$ and $\beta_{n+1} \in N^N$ such that

($\mathbf{a}_{n+1}$) $\beta_{n+1} \in V_{n+1} \subseteq V_0$
($\mathbf{b}_{n+1}$) $\beta_{n+1} \notin [\![A(w_n * u_{n+1} * 2^{s(n+1)+1})]\!]$,
($\mathbf{c}_{n+1}$) $\bar{\beta}_{n+1}(n+1) = \bar{\beta}_n(n+1)$.

By (a_{n+1}) and (b_{n+1}),

$$(z)_{z<lh(w_{n+1})}\beta_{n+1} \notin [\![R((\overline{\lambda t(w_{n+1})_t \dot{-} 1})(z))]\!]$$

(where we write w_{n+1} for $w_n * u_{n+1} * 2^{s(n+1)+1}$). By conditions (c), the sequence $\beta_0, \beta_1, \beta_2, \cdots$ of elements of N^N converges to some $\beta \in N^N$; in fact, since V_0 is clopen, $\beta \in V_0$ by conditions (a), so $\beta \in [\![\forall\alpha\exists!x\mathrm{R}(\overline{\alpha}(x))]\!]$. However, consider the element ξ of $\mathscr{I}\mathscr{P}\mathscr{S}$ defined by $[\xi(\gamma)](n) = (w_{n+1})_n \dot{-} 1$ for all $\gamma \in N^N, n \in N$. Since $\beta \in [\![\exists!x\mathrm{R}(\overline{\xi}(x))]\!]$, there is some $n \in N$ and a neighborhood U of β such that $U \subseteq [\![\mathrm{R}(\overline{\xi}(n))]\!]$; so for some $k \in N$ and all $m \geqq k$, $\beta_m \in [\![\mathrm{R}(\overline{\xi}(n))]\!]$. But for m sufficiently large and for all $\gamma \in N^N$, $\overline{[\xi(\gamma)]}(n) = \overline{(\lambda t(w_m)_t \dot{-} 1)}(n)$, whence by Lemmas 1 and 2 $\beta_m \in [\![\mathrm{R}(\overline{(\lambda t(w_m)_t \dot{-} 1)}(n))]\!]$ for all m sufficiently large, in particular for some $m > n$, contradicting

$$(z)_{z<lh(w_m)}\beta_m \notin [\![\mathrm{R}(\overline{(\lambda t(w_m)_t \dot{-} 1)}(z))]\!]$$

since $m \leqq lh(w_m)$. So the Bar Theorem is valid.

The next lemma is needed in verifying our version ^{x}27.2! of Brouwer's Principle.

Lemma 5. *If* $\beta \in N^N$, $\mathrm{A}(\alpha)$ *is any formula,* $\xi, \eta \in \mathscr{I}\mathscr{P}\mathscr{S}$, *and* $\xi(\beta) = \eta(\beta)$, *and if* $[\![\mathrm{A}(\zeta)]\!]$ *is clopen for every* $\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}$, *then* $\beta \in [\![\mathrm{A}(\xi)]\!]$ *if and only if* $\beta \in [\![\mathrm{A}(\eta)]\!]$.

Proof. By Lemma 3 it will suffice to exhibit some $\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}$ such that $\beta \in Cl[\![\xi = \zeta]\!]$ and $\beta \in Cl[\![\zeta = \eta]\!]$. For all $\gamma \in N^N$ and $n \in N$, let

$$[\zeta(\gamma)](n) = \begin{cases} [\xi(\gamma)](n) = [\eta(\gamma)](n) \text{ if } \overline{\gamma}(n) = \overline{\beta}(n), \\ [\xi(\gamma)](n) \text{ if } (Ez)_{z<n}[\overline{\gamma}(z) = \overline{\beta}(z) \;\&\; \gamma(z) \neq \beta(z) \;\&\; 2|z], \\ [\eta(\gamma)](n) \text{ if } (Ez)_{z<n}[\overline{\gamma}(z) = \overline{\beta}(z) \;\&\; \gamma(z) \neq \beta(z) \;\&\; 2|(z+1)]. \end{cases}$$

Then ζ has the desired properties.

Lemma 6. *Suppose* $\mathrm{A}(\alpha, x)$ *is a formula with the property that for each* $\beta, \delta \in N^N$ *there exists* $b, z \in N$ *such that whenever* $\gamma \in N^N$ *and* $\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}$ *with*

$$\overline{\gamma}(z) = \overline{\beta}(z) \text{ and } \overline{[\zeta(\gamma)]}(z) = \overline{\delta}(z)$$

then $\gamma \in [\![\mathrm{A}(\zeta, b)]\!]$. *Then there is a* $\tau \in \mathscr{I}\mathscr{P}\mathscr{S}$ *for which* $[\![\forall\alpha\exists y(\tau(\overline{\alpha}(y)) > 0 \;\&\; \forall x(\tau(\overline{\alpha}(x)) > 0 \supset y = x) \;\&\; \mathrm{A}(\alpha, \tau(\overline{\alpha}(y)) \dot{-} 1))]\!] = N^N$.

Proof. For each $\beta \in N^N$ and $n \in N$, let

$$[\tau(\beta)](n) = \begin{cases} b+1 \text{ if } B(\beta, n, b) \;\&\; (y)_{y<b}\overline{B}(\beta, n, y) \;\&\; \\ \qquad (y)_{y<lh(n)}(z)\overline{B}(\beta, \prod_{i<y} p_i^{(n)_i}, z), \\ 0 \text{ if } (y)\overline{B}(\beta, n, y) \vee (Ey)_{y<lh(n)}(Ez)B(\beta, \prod_{i<y} p_i^{(n)_i}, z), \end{cases}$$

where $B(\beta, n, y)$ is $Seq\,(n) \;\&\; (\gamma)(\zeta)[\gamma \in N^N \;\&\; \overline{\gamma}(lh(n)) = \overline{\beta}(lh(n)) \;\&\; \zeta \in \mathscr{I}\mathscr{P}\mathscr{S} \;\&\; \overline{[\zeta(\gamma)]}(lh(n)) = n \rightarrow \gamma \in [\![\mathrm{A}(\zeta, y)]\!]]$.

VERIFICATION OF ^{x}27.2! Suppose $\beta_0 \in N^N$ and V_0 is a neighborhood of β_0 such that $V_0 \subset [\![\forall\alpha\exists!bA(\alpha, b)]\!]$. Hence for each $\xi \in \mathscr{I}\mathscr{P}\mathscr{S}$ and each $b \in N$, the set $[\![A(\xi, b)]\!]$ is clopen relative to V_0; and for each $\xi \in \mathscr{I}\mathscr{P}\mathscr{S}$, the sets $[\![A(\xi, b)]\!] \cap V_0$ (for $b = 0, 1, 2, \cdots$) form a partition of V_0. By (the relativization to V_0 of) Lemma 5, if $\gamma \in V_0$ and $\xi(\gamma) = \alpha$, then (putting $\xi_\alpha(\delta) = \alpha$ for all $\delta \in N^N$) $\gamma \in [\![A(\xi, b)]\!]$ iff $\gamma \in [\![A(\xi_\alpha, b)]\!]$. Now define $\varphi : N^N \times N^N \to N$ by

$$\varphi(\gamma, \alpha) = \begin{cases} b \text{ if } \gamma \varepsilon V_0 \ \& \ \gamma \in [\![A(\xi_\alpha, b)]\!], \\ 0 \text{ if } \gamma \notin V_0. \end{cases}$$

Claim: φ is continuous. Since V_0 is clopen, it suffices to show φ is continuous on $V_0 \times N^N$. Suppose not; then there are *distinct* points γ^*, $\gamma_1, \gamma_2, \cdots \in V_0$ and points $\alpha^*, \alpha_1, \alpha_2, \cdots \in N^N$ such that $\{\gamma_n\} \to \gamma^*$, $\{\alpha_n\} \to \alpha^*$, and $\varphi(\gamma_n, \alpha_n) \neq \varphi(\gamma^*, \alpha^*)$ for $n = 1, 2, \ldots$. Choose some $\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}$ such that

$$\zeta(\gamma_n) = \alpha_n \text{ for } n = 1, 2, \cdots, \text{ and}$$
$$\zeta(\gamma^*) = \alpha^*.$$

Let $b^* = \varphi(\gamma^*, \alpha^*)$; then $\gamma^* \in [\![A(\zeta, b^*)]\!]$ by Lemma 5, as above, so for n sufficiently large $\gamma_n \in [\![A(\zeta, b^*)]\!]$. But $\gamma_n \varepsilon [\![A(\zeta, b_n)]\!]$ for each n, where $b_n = \varphi(\gamma_n, \alpha_n)$, and $[\![A(\zeta, b_n)]\!] \cap [\![A(\zeta, b^*)]\!] \cap V_0 = \emptyset$ since $b_n \neq b^*$, so we have reached a contradiction and established the continuity of φ. Hence by (the relativizations to V_0 of) Lemmas 5 and 6, $V_0 \subset [\![\exists\tau\forall\alpha \exists y(\tau(\bar{\alpha}(y)) > 0 \ \& \ \forall x(\tau(\bar{\alpha}(x)) > 0 \supset y = x) \ \& \ A(\alpha, \tau(\bar{\alpha}(y)) \dot{-} 1))]\!]$.

VERIFICATION OF KS^s. Given any formula A in which α is not free, and any interpretation of the free variables of A (determining a value $[\![A]\!] \subseteq N^N$), define $\xi \in \mathscr{I}\mathscr{P}\mathscr{S}$ as follows:

$$[\xi(\beta)](x) = \begin{cases} 1 \text{ if } (\gamma)[\bar{\gamma}(x) = \bar{\beta}(x) \to \gamma \varepsilon [\![A]\!]], \\ 0 \text{ otherwise.} \end{cases}$$

Then for each $\beta \in N^N$:

$$\beta \in [\![\exists x\xi(x) \neq 0]\!] \text{ if and only if } \beta \in [\![A]\!].$$

Hence $[\![\exists x\xi(x) \neq 0 \sim A]\!] = N^N$, and KS^s is valid.

This completes the proof, and establishes the classical consistency of **I**! + KS^s.

§ 3. Further Considerations

While it is an open problem whether Brouwer's Principle for numbers ^{x}27.2 holds in full generality in the model, it is not hard to see that all instances of $^{x}27.2$ (hence also of Brouwer's principle for decisions $^{x}27.3$ ([1], p. 74)) not containing free function variables are valid. It follows that all Kleene's particular examples (in §§ 7.10–7.14 of [1]) of classically false consequences of Brouwer's Principle are true in the model. In this section we give a classical proof of this result, and establish the validity of certain additional axioms which have been considered by Kreisel and Troelstra [2], [9].

LEMMA 7. *Let* $\mathrm{E}(\alpha)$ *be a formula with no free function variables other than* α. *Let* $\alpha, \beta \in N^N$ *and* $\xi, \xi \in \mathscr{I}\mathscr{P}\mathscr{S}$, *and suppose there is an isomorphism* $\psi \in \mathscr{I}\mathscr{P}\mathscr{S}$ (*of* N^N *onto* N^N) *such that*

$$\psi(\alpha) = \beta \textit{ and } \zeta = \zeta \circ \psi.$$

Then $\alpha \in [\![\mathrm{E}(\xi)]\!]$ *if and only if* $\beta \in [\![\mathrm{E}(\zeta)]\!]$.

PROOF by induction on the logical form of $\mathrm{E}(\alpha)$. We give three representative cases.

Case 4. $\mathrm{A}(\alpha) \supset \mathrm{B}(\alpha)$. Assume $\alpha \in [\![\mathrm{A}(\xi) \supset \mathrm{B}(\xi)]\!]$, so there is some neighborhood U_0 of α such that $U_0 \subset [(N^N - [\![\mathrm{A}(\xi)]\!]) \cup [\![\mathrm{B}(\xi)]\!]]$, so by the ind. hyp. $\psi(U_0) \subset [(N^N - [\![\mathrm{A}(\zeta)]\!]) \cup [\![\mathrm{B}(\zeta)]\!]]$. But $\beta \in \psi(U_0)$, and $\psi(U_0)$ is open, so $\beta \in [\![\mathrm{A}(\zeta) \supset \mathrm{B}(\zeta)]\!]$. The converse is similar, using ψ^{-1}.

Case 8. $\forall\gamma \mathrm{A}(\alpha, \gamma)$. Assume $\alpha \in [\![\forall\gamma\mathrm{A}(\xi, \gamma)]\!]$, so there is some neighborhood U_0 of α with

$$U_0 \subset \bigcap_{\eta \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![\mathrm{A}(\xi, \eta)]\!].$$

Given $\eta \in \mathscr{I}\mathscr{P}\mathscr{S}$, it follows that $U_0 \subset [\![\mathrm{A}(\xi, \eta \circ \psi)]\!]$, so by the ind. hyp. $\psi(U_0) \subset [\![\mathrm{A}(\zeta, \eta)]\!]$. Since η was arbitrary, $\beta = \psi(\alpha) \in [\![\forall\gamma\mathrm{A}(\zeta, \gamma)]\!]$. The converse is similar.

Case 9. $\exists\gamma\mathrm{A}(\alpha, \gamma)$. Assume $\alpha \in [\![\exists\gamma\mathrm{A}(\xi, \gamma)]\!]$, so for some $\eta \in \mathscr{I}\mathscr{P}\mathscr{S}$ $\alpha \in [\![\mathrm{A}(\xi, \eta)]\!]$. Then by the ind. hyp. $\beta \in [\![\mathrm{A}(\zeta, \eta \circ \psi^{-1})]\!]$, where $\eta \circ \psi^{-1} \in \mathscr{I}\mathscr{P}\mathscr{S}$. So $\beta \in [\![\exists\gamma\mathrm{A}(\zeta, \gamma)]\!]$. The converse is similar.

VERIFICATION OF $^{x}27.2$ WITHOUT FREE FUNCTION VARIABLES. Assume $\beta_0 \in N^N$ and U_0 is a neighborhood of β_0 such that $U_0 \subset [\![\forall\alpha\exists\mathrm{b}\mathrm{A}(\alpha, \mathrm{b})]\!]$, where $\mathrm{A}(\alpha,\mathrm{b})$ contains no free function variables except α, and assume for contradiction that $U_0 \not\subset [\![\exists\tau\forall\alpha\exists\mathrm{y}\{\tau(\overline{\alpha}(y)) > 0 \;\&\; \forall\mathrm{x}[\tau(\overline{\alpha}(\mathrm{x})) > 0 \supset \mathrm{y} = \mathrm{x}] \;\&\; \mathrm{A}(\alpha, \tau(\overline{\alpha}(\mathrm{y})) \dot{-} 1)\}]\!]$. By (the relativization to U_0 of the classical contrapositive of) Lemma 6, there are $\beta \in U_0$, $\delta \in N^N$ such that for each $x \in N$ there are

$$\beta_1^x, \beta_2^x, \cdots \in U_0 \text{ and } \zeta_1^x, \zeta_2^x, \cdots \in \mathscr{I}\mathscr{P}\mathscr{S}$$

such that for each z

$$\bar{\beta}_z^x(z) = \bar{\beta}(z) \text{ and } \overline{[\zeta_z^x(\beta_z^x)]}(z) = \bar{\delta}(z)$$

but

$$\beta_z^x \notin [\![A(\zeta_z^x, x)]\!].$$

For each $x, z \in N$ let $\gamma_z^x \in N^N$ be determined by

$$\gamma_z^x(n) = \begin{cases} \beta(n) \text{ if } n < 2^x \cdot 3^z, \\ \beta(n)+1 \text{ if } n = 2^x \cdot 3^z, \\ 0 \text{ if } n > 2^x \cdot 3^z. \end{cases}$$

Clearly $\{\gamma_z^x\} \underset{z}{\rightarrow} \beta$ for each $x \in N$. Now let

$$U_z^x = \{\gamma \in \mathscr{I}\mathscr{P}\mathscr{S} : \bar{\gamma}(2^x \cdot 3^z+1) = \bar{\gamma}_z^x(2^x \cdot 3^z+1) = \bar{\beta}(2^x \cdot 3^z) * * 2^{\beta(2^x.3^z)+2}\},$$

and let $\psi_z^x \in \mathscr{I}\mathscr{P}\mathscr{S}$ be an isomorphism of N^N onto N^N such that

$$\psi_z^x(\gamma_z^x) = \beta_z^x$$

and, if $\gamma \in U_z^x$, then ($\psi_z^x(\gamma)$ and β_z^x agree far enough so that)

$$\overline{[\zeta_z^x(\psi_z^x(\gamma))]}(z) = \bar{\delta}(z).$$

Then, since the U_z^x are disjoint and miss β, there is an $\eta \in \mathscr{I}\mathscr{P}\mathscr{S}$ so that

$$\eta(\gamma) = \begin{cases} \zeta_z^x(\psi_z^x(\gamma)) \text{ if } \gamma \in U_z^x, \\ \delta \text{ if } \gamma = \beta. \end{cases}$$

(In fact, we might as well put

$$\eta(\gamma) = \delta \text{ if } \gamma \notin \bigcup_{x, z \in N} U_z^x.$$

Assume for contradiction that $\beta \in [\![A(\eta, x)]\!]$ for some x. Then for all z sufficiently large, $\gamma_z^x \in [\![A(\eta, x)]\!]$, whence by the relativization to U_z^x of Lemma 7, $\beta_z^x \in [\![A(\zeta_z^x, x)]\!]$, a contradiction. But then $\beta \notin [\![\exists bA(\eta, b)]\!]$, contradicting our original hypothesis $\beta \in U_0 \subset [\![\forall\alpha\exists bA(\alpha, \beta)]\!]$. So each instance of ^{x}27.2 without free function variables is valid.

The difficulty in extending this proof to the case of ^{x}27.2 with free function variables is that Lemma 7 fails for E(α) having free function variables other than α.

In § 2.7 of [2] Kreisel and Troelstra discuss a number of choice schemata (including Kleene's ^{x}2.1) for intuitionistic analysis, all of which are valid in the present topological model. We consider only the principle of 'relative dependent choices for functions'

$$\begin{aligned} \text{RDC-F: } & \forall\alpha[A(\alpha) \supset \exists\beta(B(\alpha, \beta) \,\&\, A(\beta))] \supset \\ & \forall\alpha[A(\alpha) \supset \exists\gamma(\lambda y\gamma(\langle 0, y\rangle) = \alpha \,\&\, \\ & \forall x B(\lambda y\gamma(\langle x, y\rangle), \lambda y\gamma(\langle x+1, y\rangle))))], \end{aligned}$$

since it entails all the others (cf. [2] § 2.7.2).

VERIFICATION OF RDC-F. Assume $\beta_0 \in N^N$ and U_0 is a neighborhood of β_0 such that

$$U_0 \subset \bigcap_{\xi \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![A(\xi) \supset \exists\beta(B(\xi, \beta) \,\&\, A(\beta))]\!].$$

Consider any $\xi \in \mathscr{I}\mathscr{P}\mathscr{S}$. Then

$$U_0 \cap [\![A(\xi)]\!] \subset \bigcup_{\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![B(\xi, \zeta) \,\&\, A(\zeta)]\!].$$

If $\beta \in U_0 \cap [\![A(\xi)]\!]$, then there is a neighborhood V_0 of β with $V_0 \subset U_0 \cap [\![A(\xi)]\!]$. By a classical bar induction, we can partition V_0 into countably many disjoint neighborhoods V_1^1, V_2^1, $V_3^1, \cdots$ with associated $\zeta_1^1, \zeta_2^1, \zeta_3^1, \cdots \in \mathscr{I}\mathscr{P}\mathscr{S}$ such that, for $j = 1, 2, 3, \cdots$,

$$V_j^1 \subset [\![B(\xi, \zeta_j^1)]\!] \cap [\![A(\zeta_j^1)]\!]$$

so by hypothesis

$$V_j^1 \cap [\![A(\zeta_j^1)]\!] = V_j^1 \subset \bigcup_{\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![B(\zeta_j^1, \zeta) \,\&\, A(\zeta)]\!].$$

In general, partition

$$V_{j_1 \cdots j_i}^i \subset \bigcap_{\xi \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![B(\zeta_{j_1 \cdots j_i}^i, \zeta) \,\&\, A(\zeta)]\!]$$

into countably many disjoint neighborhoods

$$V_{j_1 \cdots j_i k}^{i+1} \text{ with associated } \zeta_{j_1 \cdots j_i k}^{i+1} \in \mathscr{I}\mathscr{P}\mathscr{S}$$

so that

$$V_{j_1 \cdots j_i k}^{i+1} \subset [\![B(\zeta_{j_1 \cdots j_i}^i, \zeta_{j_1 \cdots j_i k}^{i+1})]\!] \cap [\![A(\zeta_{j_1 \cdots j_i k}^{i+1})]\!],$$

whence

$$V_{j_1 \cdots j_i k}^{i+1} \subset \bigcup_{\zeta \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![B(\zeta_{j_1 \cdots j_i k}^{i+1}, \zeta) \,\&\, A(\zeta)]\!].$$

Now define $\eta \in \mathscr{I}\mathscr{P}\mathscr{S}$ by

$$[\eta(\delta)](n) = \begin{cases} 0 \text{ if } n \neq \langle (n)_0, (n)_1 \rangle \text{ or } \delta \notin V_0, \\ [\xi(\delta)]((n)_1) \text{ if } n = \langle 0, (n)_1 \rangle \,\&\, \delta \in V_0, \\ [\zeta_{j_1 \cdots j_i}^i(\delta)]((n)_1) \text{ if } n = \langle i, (n)_1 \rangle \,\&\, i > 0 \,\&\, \delta \in V_{j_1 \cdots j_i}^i. \end{cases}$$

Then

$$V_0 \subset [\![\lambda y \eta(\langle 0, y \rangle) = \xi]\!] \cap \bigcap_{x \in N} [\![B(\lambda y \eta(\langle x, y \rangle), \lambda y \eta(\langle x+1, y \rangle))]\!],$$

using Lemma 2. It follows that $U_0 \cap [\![A(\xi)]\!] \subset [\![\exists\gamma(\lambda y \gamma(\langle 0, y \rangle) = \xi \,\&\, \forall x B(\lambda y \gamma(\langle x, y \rangle), \lambda y \gamma(\langle x+1, y \rangle)))]\!]$ for every $\xi \in \mathscr{I}\mathscr{P}\mathscr{S}$, whence

$$U_0 \subset \bigcap_{\xi \in \mathscr{I}\mathscr{P}\mathscr{S}} [\![A(\xi) \supset \exists\gamma(\lambda y\gamma(\langle 0, y\rangle) = \xi \;\&\; \forall x B(\lambda y\gamma(\langle x, y\rangle), \lambda y\gamma(\langle x+1, y\rangle)))]\!].$$

Thus RDC-F is valid.

Kreisel and Troelstra's 'special bar continuity' schema ([2] § 5.5.9, [9]), which can be expressed in our system as

$$\text{SBC: } \forall\alpha\exists x A(\bar{\alpha}(x)) \supset \exists\tau\forall\alpha\exists y[\tau(\bar{\alpha}(y)) > 0 \;\& \\ \forall x(\tau(\bar{\alpha}(x)) > 0 \supset y = x) \;\&\; A(\bar{\alpha}(\tau(\bar{\alpha}(y)) \dot{-} 1))],$$

is easily verified using Lemma 6 with Lemma 1 (a). From Lemma 6 we also see that the 'weak continuity' schema ([2] § 5.5.7)

$$\text{WC-N: } \forall\alpha\exists b A(\alpha, b) \supset \forall\alpha\exists x\exists b\forall\beta(\bar{\beta}(x) = \bar{\alpha}(x) \supset A(\beta, b))$$

is valid for exactly those A(α, b) for which $^{x}27.2$ is valid, although $^{x}27.2$ does not seem to be derivable from WC-N (cf. [1] § 7.7).

§ 4. Relation to Scott's Model of Intuitionistic Analysis.

In developing the present model, we had in mind that it should agree with Scott's interpretation [7] via Vesley's representation ([1], Chapter III) of the intuitionistic theory of the continuum within Kleene's **I**. Recall that in Scott's axiomatization of the theory of order in the intuitionistic continuum, the basic relation is the 'measurable natural ordering' $<$; the coincidence relation is defined in terms of $<$ by

$$x = y \leftrightarrow \neg\, x < y \;\&\; \neg\, y < x$$

(where, of course, the variables x, y range now over intuitionistic reals), and other relations are defined similarly. Then if φ, ψ are continuous functions from N^N into the classical reals, the formulas $x < y$, $x = y$ take the topological values

$$[\![\varphi < \psi]\!] = \{\beta \in N^N : \varphi(\beta) < \psi(\beta)\},$$
$$[\![\varphi = \psi]\!] = \mathit{In}\,\{\beta \in N^N : \varphi(\beta) = \psi(\beta)\}$$

when x, y are interpreted by φ, ψ respectively.

In Vesley's [1] Chapter III, the class R′ of 'canonical real number generators' (c.r.n.g.) is defined as a subclass of the i.p.s. by (*RO.4 of [1])

$$\alpha \in R' \sim \forall x|2\alpha(x) - \alpha(x')| \leqq 1.$$

The measurable natural ordering $< \circ$ and the coincidence predicate $\doteq$ for c.r.n.g. are then given by (*R.6.1, *R.1.1 of [1], respectively)

$$\alpha < \circ\, \beta \sim \exists k\exists x\forall p 2^k(\beta(x+p) \dot{-} \alpha(x+p)) \geqq 2^{x+p},$$
$$\alpha \doteq \beta \sim \forall k\exists x\forall p 2^k|\alpha(x+p) - \beta(x+p)| < 2^{x+p}.$$

In **I**! we have the theorem

$$\alpha, \beta \in R' \supset (\alpha \mathrel{\underline{\underline{\circ}}} \beta \sim \neg\, \alpha < \circ \beta \;\&\; \neg\, \beta < \circ \alpha)$$

(cf. *R6.4, *R6.5 of [1]), which verifies the correspondence between $< \circ$, $\mathrel{\underline{\underline{\circ}}}$ (for c.r.n.g.) and Scott's $<$, $=$ (for intuitionistic reals).

Under this correspondence, our model agrees nicely with Scott's. If $\xi \in \mathscr{IPS}$ and $\beta \in [\![\xi \in R']\!]$, then $\xi(\beta)$ is a c.r.n.g. representing a classical real number which we shall call $\rho(\xi(\beta))$. Further, if $\xi, \eta \in \mathscr{IPS}$ then

$$[\![\xi, \eta \in R' \;\&\; \xi < \circ\, \eta]\!] = \{\beta \in N^N : \beta \in [\![\xi, \eta \in R']\!] \;\&\; \rho(\xi(\beta)) < \rho(\eta(\beta))\},$$

$$[\![\xi, \eta \in R' \;\&\; \xi \mathrel{\underline{\underline{\circ}}} \eta]\!] = \mathit{In}\,\{\beta \in N^N : \beta \in [\![\xi, \eta \in R']\!] \;\&\; \rho(\xi(\beta)) = \rho(\eta(\beta))\}.$$

In particular, if $[\![\xi, \eta \in R']\!] = N^N$, we have

$$[\![\xi < \circ\, \eta]\!] = \{\beta \in N^N : \rho(\xi(\beta)) < \rho(\eta(\beta))\},$$
$$[\![\xi \mathrel{\underline{\underline{\circ}}} \eta]\!] = \mathit{In}\,\{\beta \in N^N : \rho(\xi(\beta)) = \rho(\eta(\beta))\}.$$

Since every continuous function φ from N^N into the classical reals is $\rho \circ \xi$ for some $\xi \in \mathscr{IPS}$ with $[\![\xi \in R']\!] = N^N$, (the first-order part of) Scott's model is included in ours.

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Dept. of Maths.
Occidental College
Los Angeles
Cal. 90041
U.S.A.