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SYMMETRIC SUBSPACES RELATED TO CERTAIN EIGENVALUE PROBLEMS

by

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0. Introduction

Let $F \subset G$ and H be linear spaces and $\mathrm{F} : F \to H$, $\mathrm{G} : G \to H$ be linear mappings. We shall be concerned with the eigenvalue problem

$$\mathrm{F}y = \lambda \mathrm{G}y, \ y \in F. \tag{0.1}$$

This eigenvalue problem was considered by Schäfke and Schneider [6] under conditions, such that they could apply a theorem of Wielandt. The abstract results were then applied to eigenvalue problems for systems of differential equations on a compact interval, cf. [6], [7], [8].

In [4] Niessen gave a theory for singular differential systems of the form (0.1), thereby generalizing the results of Schneider [9], [10] for certain real systems. Both authors wished to connect a self-adjoint operator in a Hilbert space with their problems, in order to apply the spectral theory for such operators.

Several considerations in the papers of Niessen and Schneider are not restricted to systems of differential equations. Indeed, we shall show that under certain conditions one may associate a self-adjoint operator with (0.1). Our main assumption is that there exists a linear operator B_λ : $G \to F (\lambda \in \{i, -i\})$ such that $y = \mathrm{B}_\lambda z$ is a solution of

$$(\mathrm{F} - \lambda \mathrm{G})y = \mathrm{G}z, \ y \in F, \ z \in G, \ \lambda \in \{i, -i\}, \tag{0.2}$$

and such that $(\mathrm{B}_{\bar{\lambda}} y, z) = (y, \mathrm{B}_{\bar{\lambda}} z)$ for all $y, z \in G$. This assumption then leads to results analogous to those found by Kodaira [3], Kimura and Takahasi [2], Schneider [9] a.o. Since G in (0.1) need not be injective and since in general G will only have a semi-inner product, it is not possible to define a maximal and a minimal operator associated with our eigenvalue problem. However (0.2) and results in Coddington's paper [1] enable us to replace these notions by maximal and minimal subspaces (closed linear manifolds) in a certain Hilbert space. Having done this we may apply Coddington's extension theory of symmetric subspaces in order to determine all self-adjoint subspace extensions (if there are any) of the minimal subspace associated with (0.1), which is symmetric. With such a self-adjoint subspace extension one finds a self-adjoint operator in a smaller Hilbert space. Under an additional condition the domain of this operator is mapped bijectively onto a manifold in G.

We would like to point out that in contrast to Schäfke and Schneider [6, cf. (2.3)] we do not restrict ourselves to so called S-hermitian eigenvalue problems, that is problems in which F and G in (0.1) are chosen in such a way that one immediately obtains a self-adjoint subspace (without having to extend the minimal subspace: minimal and maximal subspace are equal). It was this restriction that enabled Schäfke and Schneider in [6], [7], [8] and Niessen in [4, cf. [4.15)] to consider systems of differential equations with boundary values depending linearly on the eigenvalue parameter.

Many discussions are similar to those of Kodaira [3] and Niessen [4]. However, it seems desirable to have a theory available for eigenvalue problems of the general form (0.1), so that it can be applied to differential equations and to systems of differential equations alike. Also one may apply the theory to pairs of symmetric operators F and G in a Hilbert space (cf. Coddington [1], Pleijel [5]) for which the inhomogeneous equation (0.2) can be solved under our assumptions.

In section 1 we collect certain elementary facts, which will be needed in the sequel. Section 2 gives the main assumption and some immediate consequences. With this we define in section 3 minimal and maximal subspaces associated with (0.1) and we show that they are adjoint. Self-adjoint subspace extensions of the minimal subspace are investigated in section 4. In section 5 we consider self-adjoint subspaces associated with (0.1), determined by boundary value operators.

1. Some algebraic properties

Let G be a linear space with semi-inner product (,). We assume G to be complete. H is a linear space with inner product [,]. H is not necessarily complete. G is a linear mapping from G into H.
Let G_0 be a linear manifold in G. We assume S is a linear mapping from G_0 into H, such that for all $u \in G$, $v \in G_0$

$$[\mathrm{G}u, \mathrm{S}v] = (u, v).$$

Hence for all $u, v \in G_0$ it follows that

$$[\mathrm{G}u, \mathrm{S}v] = [\mathrm{S}u, \mathrm{G}v],$$
$$[\mathrm{G}u, \mathrm{S}u] \geqq 0.$$

Let F' be a linear manifold in G_0, on which the linear mapping F from F'

into H is defined. The linear manifold $F \subset F'$ is defined by

$$F := \mathrm{F}^{-1}\mathrm{G}G.$$

For all $\lambda \in \boldsymbol{C}$ we define the linear manifold $E_\lambda \subset F$ by

$$E_\lambda := \{y \in F / \mathrm{F}y = \lambda \mathrm{G}y\}.$$

On F' we define the sesqui-linear form $\langle , \rangle$ by

$$\langle u, v \rangle := [\mathrm{F}u, Sv] - [\mathrm{S}u, \mathrm{F}v], \; u, v \in F'.$$

(1.1) Lemma. *If* $y_1, y_2 \in F$, $z_1, z_2 \in G$, $\lambda_1, \lambda_2 \in \boldsymbol{C}$ *and*

$$(\mathrm{F} - \lambda_1 \mathrm{G})y_1 = \mathrm{G}z_1, \; (\mathrm{F} - \lambda_2 \mathrm{G})y_2 = \mathrm{G}z_2,$$

then

$$\langle y_1, y_2 \rangle = (z_1, y_2) - (y_1, z_2) + (\lambda_1 - \bar{\lambda}_2)(y_1, y_2).$$

Proof. Since $\mathrm{F}y_i = \mathrm{G}(z_i + \lambda_i y_i) (i = 1, 2)$ it follows that

$$[\mathrm{F}y_1, \mathrm{S}y_2] - [\mathrm{S}y_1, \mathrm{F}y_2] = [\mathrm{G}(z_1 + \lambda_1 y_1), \mathrm{S}y_2] - [\mathrm{S}y_1, \mathrm{G}(z_2 + \lambda_2 y_2)].$$

(1.2) Corollary.

(a) *If* $y \in E_\lambda$, $z \in E_{\bar{\lambda}}$, *then* $\langle y, z \rangle = 0$.
(b) *If* $y, z \in E_\lambda$, *then* $\langle y, z \rangle = (2i \operatorname{Im} \lambda)(y, z)$.
(c) *If* $y_1, y_2 \in F$, $z_1, z_2 \in G$ *and* $\mathrm{F}y_1 = \mathrm{G}z_1$, $\mathrm{F}y_2 = \mathrm{G}z_2$, *then*

$$(z_1, y_2) - (y_1, z_2) = \langle y_1, y_2 \rangle.$$

We define the linear manifold $N \subset G$ by

$$N := \{y \in G / (y, y) = 0\}.$$

(1.3) Theorem. *Let* $\mathrm{G}N = \{0\}$. *If* $E_{\lambda_0} \cap N = \{0\}$ *for some* $\lambda_0 \in \boldsymbol{C}$, *then*

$$E_\lambda \cap N = \{0\} \textit{ for all } \lambda \in \boldsymbol{C}.$$

Proof. Let $y \in N \cap E_\lambda$, then $\mathrm{G}y = 0$ and

$$(\mathrm{F} - \lambda_0 \mathrm{G})y = (\mathrm{F} - \lambda \mathrm{G})y + (\lambda - \lambda_0)\mathrm{G}y = 0.$$

This implies $y \in E_{\lambda_0}$. Hence $y = 0$.

(1.4) Assumption. For all $\lambda \in \boldsymbol{C}$: $N \cap E_\lambda = \{0\}$. [1]) The linear manifold F_0 is defined by

$$F_0 := \{y \in F / \langle y, z \rangle = 0 \text{ for all } z \in F\}.$$

In general F_0 will be properly included in F. Let $\tilde{F}$ be a linear manifold in F such that $\langle y, z \rangle = 0$ for all $y, z \in \tilde{F}$, i.e. $(\tilde{F})_0 = \tilde{F}$. In this case we define $\tilde{E}_\lambda := E_\lambda \cap \tilde{F}$.

[1] In the following sections it suffices to assume this only holds for $\lambda \in \{i, -i\}$.

(1.5) THEOREM. *If* $y \in \tilde{E}_\lambda (y \neq 0)$ *then* $\lambda \in \boldsymbol{R}$. *If* $y_i \in \tilde{E}_{\lambda_i} (i = 1, 2)$ *and* $\lambda_1 \neq \lambda_2$ *then* $(y_1, y_2) = 0$.

PROOF. Let $y \in \tilde{E}_\lambda$, then $\mathrm{F}y = \lambda \mathrm{G}y$, thus

$$[\mathrm{F}y, \mathrm{S}y] = \lambda [\mathrm{G}y, \mathrm{S}y]$$

and the left side of this equality is real. Also by (1.4) $[\mathrm{G}y, \mathrm{S}y] > 0$ and hence $\lambda \in \boldsymbol{R}$. The remaining part of (1.5) follows from (1.1) with $z_1 = z_2 = 0$.

(1.6) COROLLARY. *Let* $F_0 = F$. *If* $y \in E_\lambda (y \neq 0)$ *then* $\lambda \in \boldsymbol{R}$. *If* $y_i \in E_{\lambda_i} (i = 1, 2)$ *and* $\lambda_1 \neq \lambda_2$ *then* $(y_1, y_2) = 0$.

(1.7) COROLLARY. *If* $y \in \tilde{F}$, $z \in G$ *and for some* $\lambda \in \boldsymbol{C}$

$$(\mathrm{F} - \lambda \mathrm{G}) y = \mathrm{G} z,$$

then

$$(z, w) = 0$$

for all $w \in \tilde{E}_\lambda$.

PROOF. For $w \in \tilde{E}_\lambda$ we have $[\mathrm{F}y, \mathrm{S}w] - \lambda [\mathrm{G}y, \mathrm{S}w] = [\mathrm{G}z, \mathrm{S}w]$. Also $[\mathrm{F}y, \mathrm{S}w] = [\mathrm{S}y, \mathrm{F}w] = \lambda [\mathrm{G}y, \mathrm{S}w]$. Hence $[\mathrm{G}z, \mathrm{S}w] = 0$ for all $w \in \tilde{E}_\lambda$.

2. Main assumption

(2.1) ASSUMPTION. For $\lambda \in \{i, -i\}$ we assume the existence of a linear mapping $\mathrm{B}_\lambda : G \to F$ such that

(a) $$(\mathrm{F} - \lambda \mathrm{G}) \mathrm{B}_\lambda y = \mathrm{G} y \text{ for all } y \in G,$$

(b) $$(\mathrm{B}_\lambda y, z) = (y, \mathrm{B}_{\bar{\lambda}} z) \text{ for all } y, z \in G.$$

(2.2) REMARK. $\mathrm{B}_\lambda N \subset N$.

(2.3) REMARK. If $\mathrm{B}_\lambda : G \to F$ only satisfies (2.1) (a) then by (1.1)

$$\langle \mathrm{B}_\lambda y, \mathrm{B}_{\bar{\lambda}} z \rangle = (y, \mathrm{B}_{\bar{\lambda}} z) - (\mathrm{B}_\lambda y, z),\ y, z \in G,\ \lambda \in \{i, -i\}.$$

Hence (2.1) (b) is equivalent to:

$$\langle \mathrm{B}_\lambda y, \mathrm{B}_{\bar{\lambda}} z \rangle = 0 \text{ for all } y, z \in G.$$

Note that if $F_0 = F$ then $\langle \mathrm{B}_\lambda y, \mathrm{B}_{\bar{\lambda}} z \rangle = 0$ for all $y, z \in G$, $\lambda \in \{i, -i\}$ and therefore (2.1) (b) is satisfied.

(2.4) LEMMA. *Let* $y \in G$, $\lambda \in \{i, -i\}$. *Then* $\mathrm{B}_\lambda y = 0$ *if and only if* $\mathrm{G}y = 0$.

PROOF. If $\mathrm{B}_\lambda y = 0$ then $\mathrm{G}y = 0$ by (2.1)(a). Now assume $\mathrm{G}y = 0$, then

$B_\lambda y \in E_\lambda$ by (2.1) (a) and by (2.1) (b)

$$(B_\lambda y, z) = (y, B_{\bar\lambda} z) = [Gy, SB_{\bar\lambda} z] = 0$$

for all $z \in G$. Hence $B_\lambda y \in N \cap E_\lambda$ and $B_\lambda y = 0$ by (1.4).

(2.5) THEOREM. $F = B_\lambda G \dotplus E_\lambda, \;\; \lambda \in \{i, -i\}$.

PROOF. If $y \in F$ then there exists $z \in G$, $Fy = Gz$. Define $g := z - \lambda y$ then $(F - \lambda G)y = Gz - \lambda Gy = Gg$. Therefore

$$y = B_\lambda g + y_\lambda, \, y_\lambda \in E_\lambda.$$

Conversely, if $g \in G$ and $y_\lambda \in E_\lambda$, then $B_\lambda g + y_\lambda \in F$. Assume $y \in B_\lambda G \cap E_\lambda$, then $y = B_\lambda g$, $g \in G$ and

$$Gg = (F - \lambda G) B_\lambda g = 0.$$

Therefore $y = B_\lambda g = 0$ by (2.4).

(2.6) THEOREM. $F_0 = B_\lambda(E_{\bar\lambda}^\perp), \; \lambda \in \{i, -i\}$.

PROOF. From (1.1) we deduce

$$\langle B_\lambda y, z_{\bar\lambda} \rangle = (y, z_{\bar\lambda}), \, y \in G, \, z_{\bar\lambda} \in E_{\bar\lambda}. \tag{2.7}$$

According to theorem (2.5):

$$y \in F \text{ if and only if } y = B_\lambda y' + y_\lambda, \, y' \in G, \, y_\lambda \in E_\lambda,$$

$$z \in F \text{ if and only if } z = B_{\bar\lambda} z' + z_{\bar\lambda}, \, z' \in G, \, z_{\bar\lambda} \in E_{\bar\lambda}.$$

Hence by (1.2) (a), (2.3) and (2.7) we find

$$\langle y, z \rangle = (y_\lambda, z') + (y', z_{\bar\lambda}).$$

So $y \in F_0$ if and only if $y_\lambda = 0$, $y' \perp E_{\bar\lambda}$, i.e. $F_0 = B_\lambda(E_{\bar\lambda}^\perp)$.

(2.8) ASSUMPTION. Either E_i or E_{-i} is complete in G. [1])

(2.9) THEOREM. $F = F_0 \dotplus E_\lambda \dotplus E_{\bar\lambda}, \;\; \lambda \in \{i, -i\}$.

PROOF. It is clear that $F_0 + E_\lambda + E_{\bar\lambda} \subset F$. Conversely let $y \in F$. Then $y = B_{-i} g + y_{-i}$, $g \in G$, $y_{-i} \in E_{-i}$ on account of (2.5). Now by (2.8)

$$g = g_1 + y_i, \, y_i \in E_i, \, g_1 \in E_i^\perp.$$

Then

$$y = B_{-i} g + y_{-i} = B_{-i} g_1 + (B_{-i} y_i + y_{-i} + \tfrac{1}{2} i y_i) - \tfrac{1}{2} i y_i,$$
$$-\tfrac{1}{2} i y_i \in E_i,$$
$$(F + iG)(B_{-i} y_i + y_{-i} + \tfrac{1}{2} i y_i) = Gy_i - Gy_i = 0,$$

[1] In the sequel we shall suppose E_i is complete.

thus $B_{-i}y_i+y_{-i}+\frac{1}{2}iy_i \in E_{-i}$. Also according to theorem (2.6) $B_{-i}g_1 \in F_0$. Hence $y \in F_0+E_\lambda+E_{\bar\lambda}$.

In order to show that the sum is direct we let $y_\lambda \in E_\lambda$, $\lambda \in \{i, -i\}$ and $y_\lambda+y_{\bar\lambda} \in F_0$. Then the definition of F_0 and corollary (1.2) imply

$$0 = \langle y_\lambda+y_{\bar\lambda}, y_\lambda\rangle = \langle y_\lambda, y_\lambda\rangle = (2i\,\mathrm{Im}\,\lambda)(y_\lambda, y_\lambda).$$

By (1.4) we obtain $y_\lambda = 0$, $\lambda \in \{i, -i\}$.

3. Linear manifolds

Let ϕ be the canonical mapping from G onto $\mathrm{H} := G/N$. On H we define an inner product (,) by

$$(\phi(y), \phi(z)) := (y, z), \qquad \phi(y), \phi(z) \in \mathrm{H}.$$

Then H is a Hilbert space, in which we define the linear manifolds D and D_0 by $D = \phi(F)$ and $D_0 = \phi(F_0)$. In the space $\mathrm{H}^2 = \mathrm{H}\times\mathrm{H}$ we define the linear manifolds L and L_0 by

$$\mathrm{L} := \{\{\phi(y), \phi(z)\} \in \mathrm{H}^2/y \in F,\ z \in \mathrm{G}^{-1}\mathrm{F}y\},$$

$$\mathrm{L}_0 := \{\{\phi(y), \phi(z)\} \in \mathrm{H}^2/y \in F_0,\ z \in \mathrm{G}^{-1}\mathrm{F}y\}.$$

It is clear that $\mathrm{L}_0 \subset \mathrm{L}$ and that D_0 and D are the domains of L_0 and L respectively.

(3.1) THEOREM. L *is a subspace* (*closed linear manifold*) *in* H^2.

PROOF. Let $\{\mathbf{y}_n, \mathbf{z}_n\}$ be a Cauchy sequence in L and let $\{\mathbf{y}, \mathbf{z}\}$ be its limit in H^2. Let $\{y_n, z_n\}$ be such that $\mathrm{F}y_n = \mathrm{G}z_n$ and $\phi(y_n) = \mathbf{y}_n$, $\phi(z_n) = \mathbf{z}_n$. Now (2.1) and $(\mathrm{F}-i\mathrm{G})y_n = \mathrm{G}(z_n-iy_n)$ imply that

$$y_n = \mathrm{B}_i(z_n-iy_n)+y_i^n,\ y_i^n \in E_i.$$

Let $\{y, z\}$ be such that $\phi(y) = \mathbf{y}$ and $\phi(z) = \mathbf{z}$. Then $y_n \to y$ and $z_n \to z$ in G as $n \to \infty$. By (2.1) (b) B_i is weakly continuous and hence

$$\mathrm{B}_i(z_n-iy_n) \rightharpoonup \mathrm{B}_i(z-iy),\ n \to \infty.$$

Since E_i is complete, $\phi(E_i)$ is closed in L and hence there exists $\mathbf{y}_i \in \phi(E_i)$ such that $\phi(y_i^n) \rightharpoonup \mathbf{y}_i$ in H and

$$\phi(y) = \phi(\mathrm{B}_i(z-iy))+\mathbf{y}_i.$$

Consequently there exists $y_i \in E_i$ and $m \in N$ such that

$$y_i = y+m-\mathrm{B}_i(z-iy).$$

We now put $y' = y+m$ and $z' = z+im$. Then

$$y' = \mathrm{B}_i(z'-iy')+y_i.$$

Consequently $y' \in F$ and $(\mathrm{F}-i\mathrm{G})y' = \mathrm{G}(z'-iy')$ or $\mathrm{F}y' = \mathrm{G}z'$. This, together with $\phi(y') = \boldsymbol{y}$ and $\phi(z') = \boldsymbol{z}$ imply $\{\boldsymbol{y}, \boldsymbol{z}\} \in \mathrm{L}$.

The adjoint of L in H^2 is defined (see [1]) by

$$\mathrm{L}^* := \{\{\boldsymbol{f}, \boldsymbol{g}\} \in \mathrm{H}^2/(\boldsymbol{z}, \boldsymbol{f}) = (\boldsymbol{y}, \boldsymbol{g}) \text{ for all } \{\boldsymbol{y}, \boldsymbol{z}\} \in \mathrm{L}\}.$$

Then L^* is a subspace of H^2.

(3.2) THEOREM. $\mathrm{L}^* = \mathrm{L}_0$.

PROOF. Let $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathrm{L}_0$. There exist $u \in F_0$, $v \in G$ such that $\mathrm{F}u = \mathrm{G}v$, $\phi(u) = \boldsymbol{u}$ and $\phi(v) = \boldsymbol{v}$. Let $\{\boldsymbol{y}, \boldsymbol{z}\} \in \mathrm{L}$. Then there exist $y \in F$, $z \in G$ such that $\mathrm{F}y = \mathrm{G}z$, $\phi(y) = \boldsymbol{y}$ and $\phi(z) = \boldsymbol{z}$. From corollary (1.2) and the definition of F_0 we find

$$\begin{aligned}(\boldsymbol{z}, \boldsymbol{u}) = (z, u) &= (y, v) + \langle y, u\rangle \\ &= (y, v) = (\boldsymbol{y}, \boldsymbol{v})\end{aligned}$$

for all $\{\boldsymbol{y}, \boldsymbol{z}\} \in \mathrm{L}$. Hence $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathrm{L}^*$ and $\mathrm{L}_0 \subset \mathrm{L}^*$.

Conversely let $\{\boldsymbol{f}, \boldsymbol{g}\} \in \mathrm{L}^*$. We must show that there exist $f' \in F_0$, $g' \in G$ such that $\mathrm{F}f' = \mathrm{G}g'$, $\phi(f') = \boldsymbol{f}$ and $\phi(g') = \boldsymbol{g}$. For each $y \in F$ and each $z \in G$ with $\mathrm{F}y = \mathrm{G}z$ and for all $f \in \phi^{-1}(\boldsymbol{f})$, $g \in \phi^{-1}(\boldsymbol{g})$ we have

$$(z, f) = (y, g) \text{ or equivalently}$$

$$(z-\lambda y, f) = (y, g-\bar{\lambda} f),\ \lambda \in \{i, -i\}.$$

Let $u \in G$ be arbitrary and put $y = \mathrm{B}_\lambda u$, $z = u + \lambda \mathrm{B}_\lambda u$, then $y \in F$, $z \in G$, $\mathrm{F}y = \mathrm{G}z$ and

$$(u, f) = (\mathrm{B}_\lambda u, g-\bar{\lambda} f) = (u, \mathrm{B}_{\bar{\lambda}}(g-\bar{\lambda} f)),\ \lambda \in \{i, -i\}.$$

Hence

$$f+n = \mathrm{B}_{\bar{\lambda}}(g-\bar{\lambda} f) = \mathrm{B}_{\bar{\lambda}}(g+\bar{\lambda} n-\bar{\lambda}(f+n))$$

for some $n \in N$. Put $f' = f+n$, $g' = g+\bar{\lambda} n$. Then f', g' satisfy the conditions $\phi(f') = \boldsymbol{f}$, $\phi(g') = \boldsymbol{g}$, $f' \in F$, $\mathrm{F}f' = \mathrm{G}g'$ and thus $\{\boldsymbol{f}, \boldsymbol{g}\} \in \mathrm{L}$. From $\mathrm{F}y = \mathrm{G}z$ and $\mathrm{F}f' = \mathrm{G}g'$ it follows that

$$\langle f', y\rangle = (g', y) - (f', z) = (g, y) - (f, z) = 0.$$

Since $y \in F$ was arbitrary $f' \in F_0$, and hence $\{\boldsymbol{f}, \boldsymbol{g}\} \in \mathrm{L}_0$ or $\mathrm{L}^* \subset \mathrm{L}_0$.

(3.3) COROLLARY. *L_0 is a symmetric subspace.*

PROOF. From theorems (3.1) and (3.2) it follows that

$$\mathrm{L}_0 \subset \mathrm{L} = \mathrm{L}_0^*.$$

4. Self-adjoint subspace extensions

We investigate the self-adjoint subspace extensions of L_0 in H^2 (if there are any). Define M by $M := \mathrm{L} \ominus \mathrm{L}_0$, then $M = M_i \oplus M_{-i}$, where

$$M_{\pm i} = \{\{y, z\} \in \mathrm{L}/z = \pm iy\}.$$

For $\lambda = \pm i$ we define the mapping $\phi_\lambda : E_\lambda \to M_\lambda$ in the following way. If $y \in E_\lambda$ then $\phi_\lambda y := \{\phi(y), \lambda\phi(y)\}$. It is easily seen that the mapping ϕ_λ is bijective, cf. (1.4).

An application of Coddington's theorem [1, theorem 15 and corollary] gives the following results.

(4.1) THEOREM. L_0 *has self-adjoint subspace extensions if and only if*

$$\dim E_i = \dim E_{-i}.$$

(4.2) THEOREM. *If* $\dim E_i = \dim E_{-i}$, *then all self-adjoint subspace extensions* $\tilde{\mathrm{L}}$ *of* L_0 *in* H^2 *have the form*

$$\tilde{\mathrm{L}} = \mathrm{L}_0 \oplus (I - V)M_i,$$

where the isometry V *from* M_i *onto* M_{-i} *is given by*

$$V = \phi_{-i} U \phi_i^{-1}.$$

Here U *is an isometri from* E_i *onto* E_{-i}.

Let $\tilde{D}$ be the domain in H o fthe self-adjoint subspace extension $\tilde{\mathrm{L}}$, $\mathrm{L}_0 \subset \tilde{\mathrm{L}} \subset \mathrm{L}$. Then

$$D_0 \subset \tilde{D} \subset D,$$

and

$$\tilde{D} = \{y_0 + y_i - y_{-i}/y_0 \in D_0,\ y_\lambda \in \phi(E_\lambda),\ \{y_{-i}, -iy_{-i}\} = V\{y_i, iy_i\}\}.$$

Introducing $\tilde{F}$ by

$$\tilde{F} := \{y_0 + \phi_i^{-1}(\alpha) - \phi_{-i}^{-1}(V\alpha)/y_0 \in F_0,\ \alpha \in M_i\}$$

we observe $\phi(\tilde{F}) = \tilde{D}$ and

$$F_0 \subset \tilde{F} \subset F.$$

On account of the definition of F_0 and (1.2)(a) and (b) one may verify:

(4.3) $$\langle y_1, y_2 \rangle = 0 \text{ for all } y_1, y_2 \in \tilde{F}.$$

According to Coddington [1] the space $\tilde{\mathrm{L}}$ can be written as

$$\tilde{\mathrm{L}} = \tilde{\mathrm{L}}_s \oplus \tilde{\mathrm{L}}_\infty,$$

i.e. as a direct sum of a single-valued part $\tilde{\mathrm{L}}_s$ and a multi-valued part $\tilde{\mathrm{L}}_\infty$. Then $\tilde{\mathrm{L}}_s$ generates a densely defined self-adjoint operator $\tilde{A}$ in $\tilde{\mathrm{L}}(0)^\perp$ with domain $\tilde{D}$.

We define the linear manifold $\tilde{V}$ by

$$\tilde{V} := \mathrm{G}^{-1}\mathrm{F}(N \cap \tilde{F}).$$

Then $\tilde{\mathrm{L}}(0) = \phi(\tilde{V})$ and $\tilde{\mathrm{L}}(0)^{\perp} = \phi(\tilde{V})^{\perp} = \phi(\tilde{V}^{\perp})$. Since $\phi(\tilde{F}) = \tilde{D} \perp \phi(\tilde{V})$ it follows that

$$\tilde{F} \subset \tilde{V}^{\perp}. \tag{4.4}$$

We define the linear manifold $\tilde{F}_1$ by

$$\tilde{F}_1 := \mathrm{F}^{-1}\mathrm{G}\tilde{V}^{\perp} \cap \tilde{F}.$$

Then with $\tilde{E}_\lambda := E_\lambda \cap \tilde{F}$ we have

$$\tilde{E}_\lambda \subset \tilde{F}_1, \tag{4.5}$$

for let $y \in \tilde{E}_\lambda$, then $\mathrm{F}y = \lambda \mathrm{G}y$, $y \in \tilde{F}$. Now (4.4) shows $y \in \tilde{F}_1$.

(4.6) THEOREM. $\tilde{F} = \tilde{F}_1 + (N \cap \tilde{F})$.

PROOF. Since $\tilde{F}_1 \subset \tilde{F}$ it is clear that $\tilde{F}_1 \dotplus (N \cap \tilde{F}) \subset \tilde{F}$. Conversely, let $y \in \tilde{F}$. Then $\mathrm{F}y = \mathrm{G}z$, $z \in G$. Now $\phi(G) = \mathrm{H} = \phi(\tilde{V}) \dotplus \phi(\tilde{V})^{\perp}$. Hence for $\phi(z) \in \mathrm{H}$ we find

$$\phi(z) = \phi(v) + \phi(w) \text{ with } v \in \tilde{V} \text{ and } w \in \tilde{V}^{\perp} \text{ or}$$
$$z = v + w + n,\ v \in \tilde{V},\ w \in \tilde{V}^{\perp},\ n \in N.$$

Put $w_1 = w + n$; since $N \subset \tilde{V}^{\perp}$ we find $w_1 \in \tilde{V}^{\perp}$. Therefore

$$\mathrm{F}y = \mathrm{G}z = \mathrm{G}v + \mathrm{G}w_1 = \mathrm{F}u + \mathrm{G}w_1,\ u \in N \cap \tilde{F},\ w_1 \in \tilde{V}^{\perp},$$

or

$$\mathrm{F}(y-u) = \mathrm{G}w_1,\ w_1 \in \tilde{V}^{\perp},$$

which means $y - u \in \tilde{F}_1$ and

$$y = y_1 + u,\ y_1 \in \tilde{F}_1,\ u \in N \cap \tilde{F}.$$

Therefore $\tilde{F} \subset \tilde{F}_1 + (N \cap \tilde{F})$.

(4.7) COROLLARY. *If* $\mathrm{G}N = \{0\}$, *then*

$$\text{(a)} \qquad \tilde{F} = \tilde{F}_1 \dotplus (N \cap \tilde{F}),$$

$$\text{(b)} \qquad \phi : \tilde{F}_1 \to \tilde{D} \textit{ is bijective.}$$

PROOF. To prove (a) we let $y \in \tilde{F}_1 \cap (N \cap \tilde{F}) = N \cap \tilde{F}_1$. Then $\mathrm{F}y = \mathrm{G}z$, $z \in \tilde{V}^{\perp}$. But this shows $z \in \mathrm{G}^{-1}\mathrm{F}(N \cap \tilde{F}) = \tilde{V}$. Hence $z \in \tilde{V} \cap \tilde{V}^{\perp} = N$, thus $\mathrm{G}z = 0$ and $\mathrm{F}y = 0$. Then $y \in E_\lambda$ for $\lambda = 0$. Together with $y \in N$ and (1.3) we obtain $y = 0$. Since $\tilde{D} = \phi(\tilde{F}) = \phi(\tilde{F}_1)$ the proof of (b) is complete if we show ϕ is injective. Let $y \in \tilde{F}_1$ with $\phi(y) = 0$, then $y \in N \cap \tilde{F}_1$. According to (a) it follows that $y = 0$.

(4.8) THEOREM. *Let* $GN = \{0\}$. *Then* $y \in \tilde{E}_\lambda (y \neq 0)$ *if and only if* $\tilde{A}\phi(y) = \lambda\phi(y)\,(\phi(y) \neq 0)$.

PROOF. Let $y \in \tilde{E}_\lambda$ $(y \neq 0)$, then $y \in \tilde{F}_1$ according to (4.5) and $Fy = \lambda Gy$. Then $\phi(y) \in \phi(\tilde{F}_1) = \tilde{D}$ and $\tilde{A}\phi(y) = \lambda\phi(y)$. If we had $\phi(y) = 0$, then $y \in N \cap \tilde{F}_1$ but then it follows from (4.7) (a) that $y = 0$. Hence $\phi(y) \neq 0$. Conversely, let $\tilde{A}\mathbf{y} = \lambda\mathbf{y}$, $(\mathbf{y} \neq 0)$ $\mathbf{y} \in \tilde{D}$. Corollary (4.7)(b) shows that $\mathbf{y} = \phi(y)$ for a unique $y \in \tilde{F}_1 (y \neq 0)$. Then $Fy = Gz$, $z \in \tilde{V}^\perp$, which implies $\tilde{A}\phi(y) = \phi(z)$. Hence $\lambda\phi(y) = \phi(z)$ or $\phi(z-\lambda y) = 0$; this implies $z-\lambda y \in N$ and $Gz = \lambda Gy$. Thus $Fy = \lambda Gy$, $y \in \tilde{F}_1$ and $y \in \tilde{E}_\lambda (y \neq 0)$.

(4.9) REMARK. In case $F_0 = F$ we obtain $L_0 = L$ and hence L is a self-adjoint subspace in H^2. Then $L = L_s \oplus L_\infty$ generates a densely defined self-adjoint operator A in $L(0)^\perp$ with domain $D \subset L(0)^\perp$. We define

$$V := G^{-1}F(N \cap F),$$

then $L(0) = \phi(V)$ and $L(0)^\perp = \phi(V)^\perp = \phi(V^\perp)$. Also $F \subset V^\perp$.

We define the linear manifold $F_1 := F^{-1}GV^\perp$, then $E_\lambda \subset F_1$. We have the following result.

(4.10) THEOREM.

If $F_0 = F$ *and* $\lambda \in \{i, -i\}$, *then*

(a) $$B_\lambda : G \to F \textit{ is surjective},$$

(b) $$B_\lambda : V^\perp \to F_1 \textit{ is surjective}.$$

PROOF. If $F_0 = F$ then $E_\lambda = \{0\}$ for $\lambda \in \{i, -i\}$ according to (2.9). But then theorem (2.5) shows $F = B_\lambda G$ or $B_\lambda : G \to F$ is surjective. This proves (a). In order to prove (b) we let $y \in F_1$. Then $Fy = Gz$, $z \in V^\perp$ and

$$(F-\lambda G)y = G(z-\lambda y), \; \lambda \in \{i, -i\},$$

or

$$y = B_\lambda(z-\lambda y), \text{ since } E_\lambda = \{0\}.$$

Now $z \in V^\perp$ and $y \in F_1 \subset V^\perp$, thus $z-\lambda y \in V^\perp$.

(4.11) REMARK. If G is injective, we can define an operator M on F. Let $y \in F$ then $Fy = Gz$ where $z \in G$ is unique. Define M on F by $My = z$, i.e. $M = G^{-1}F$. The operator B_λ is a right inverse for $M-\lambda$:

$$(M-\lambda)B_\lambda y = y \text{ for all } y \in G, \; \lambda \in \{i, -i\}.$$

Let M_0 be the restriction of M of F_0. Then for all $z \in G$, $y \in F_0$,

$$
\begin{aligned}
(\mathrm{B}_\lambda(M_0-\lambda)y, z) &= ((M_0-\lambda)y, \mathrm{B}_{\bar\lambda}z) \\
&= [(\mathrm{F}-\lambda\mathrm{G})y, \mathrm{S}\mathrm{B}_{\bar\lambda}z] \\
&= [\mathrm{F}y, \mathrm{S}\mathrm{B}_{\bar\lambda}z]-\lambda[\mathrm{G}y, \mathrm{S}\mathrm{B}_{\bar\lambda}z] \\
&= [\mathrm{S}y, \mathrm{F}\mathrm{B}_{\bar\lambda}z]-\langle y, \mathrm{B}_{\bar\lambda}z\rangle-\lambda[\mathrm{S}y, \mathrm{G}\mathrm{B}_{\bar\lambda}z] \\
&= [\mathrm{S}y, (\mathrm{F}-\bar\lambda\mathrm{G})B_{\bar\lambda}z] \\
&= [\mathrm{S}y, \mathrm{G}z] \\
&= (y, z).
\end{aligned} \tag{4.12}
$$

The condition $\mathrm{G}N = \{0\}$ implies that $(,)$ is an inner product for G. If $(,)$ is an inner product, i.e., if G is a Hilbert space, then (4.12) shows that B_λ is a left inverse of $M_0-\lambda$, $\lambda \in \{i, -i\}$.

(4.13) Remark. If $\dim E_i \neq \dim E_{-i}$ then L_0 does not have self-adjoint subspace extensions in H^2. We can obtain either symmetric subspace extensions in H^2 [1, theorem 4] or self-adjoint subspace extensions in a space larger than H^2 by applying [1, III section 2].

5. Boundary operators

Let $(,)_b$ be a semi-inner product on G such that G is complete with the semi-inner product $(,)_1 = (,)+(,)_b$. Let H_b be a linear space with inner product $[,]_b$. Let G_b be a linear mapping from G into H_b and let F_b, $\mathrm{S}_b : F \to H_b$ be linear mappings such that

$$[\mathrm{G}_b u, \mathrm{S}_b v]_b = (u, v)_b \qquad u \in G,\ v \in F \tag{5.1}$$

and

$$[\mathrm{F}_b u, \mathrm{S}_b v]_b-[\mathrm{S}_b u, \mathrm{F}_b v]_b = -\langle u, v\rangle, u, v \in F.^{1)} \tag{5.2}$$

By $H \times H_b$ we denote the linear space with inner product

$$[u, v]_1 = [u_1, v_1]+[u_2, v_2]_b$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ belong to } H \times H_b.$$

By $\mathbf{F}$, $\mathbf{S}$ we denote the linear mappings from F into $H \times H_b$ defined by

$$\mathbf{F}y = \begin{pmatrix} \mathrm{F}y \\ \mathrm{F}_b y \end{pmatrix},\ \mathbf{S}y = \begin{pmatrix} \mathrm{S}y \\ \mathrm{S}_b y \end{pmatrix},\ y \in F,$$

[1] It should be pointed out that we do not discuss the existense of such operators. For the case that $\dim E_i = \dim E_{-i} < \infty$, we refer to Niessen [4] for a complete treatment. We hope to treat the general case in a further paper.

and by $\mathbf{G}$ we denote the linear mapping from G into $H \times H_b$ defined by

$$\mathbf{G}y = \begin{pmatrix} \mathrm{G}y \\ \mathrm{G}_b y \end{pmatrix}, \quad y \in G.$$

The inner product $(,)_1$ has the property $(u, v)_1 = [\mathbf{G}u, \mathbf{S}v]_1$, $u \in G$, $v \in F$.

(5.3) THEOREM. $[\mathbf{F}y, \mathbf{S}z]_1 = [\mathbf{S}y, \mathbf{F}z]_1$, $y, z \in F$.

PROOF.
$$\begin{aligned} [\mathbf{F}y, \mathbf{S}z]_1 - [\mathbf{S}y, \mathbf{F}z]_1 &= [\mathrm{F}y, \mathrm{S}z] - [\mathrm{S}y, \mathrm{F}z] + [\mathrm{F}_b y, \mathrm{S}_b z]_b \\ &\qquad - [\mathrm{S}_b y, \mathrm{F}_b z]_b \\ &= \langle y, z\rangle - \langle y, z\rangle \\ &= 0, \qquad y, z \in F. \end{aligned}$$

(5.4) THEOREM. *$\mathbf{F} - \lambda\mathbf{G} : F \to H \times H_b$ is injective, $\lambda \in \{i, -i\}$.*

PROOF. Let $y \in F$ be such that $(\mathbf{F} - \lambda\mathbf{G})y = 0$. Then

$$\begin{aligned} 0 &= [\mathbf{F}y, \mathbf{S}y]_1 - [\mathbf{S}y, \mathbf{F}y]_1 \\ &= \lambda[\mathbf{G}y, \mathbf{S}y]_1 - \bar{\lambda}[\mathbf{S}y, \mathbf{G}y]_1 \\ &= (\lambda - \bar{\lambda})(y, y)_1 \\ &= (2i \operatorname{Im} \lambda)\{(y, y) + (y, y)_b\} \end{aligned}$$

and hence

$$y \in N \cap E_\lambda.$$

By (1.4) $y = 0$.

(5.5) COROLLARY. *$\mathrm{F}_b - \lambda\mathrm{G}_b$ restricted to E_λ is injective, $\lambda \in \{i, -i\}$.*
We now introduce $\boldsymbol{F} = \mathbf{F}^{-1}\mathbf{G}G$. Theorem (5.3) shows that if

$$\boldsymbol{F}_0 = \{y \in \boldsymbol{F} \mid [\mathbf{F}y, \mathbf{S}z]_1 = [\mathbf{S}y, \mathbf{F}z]_1 \text{ for all } z \in \boldsymbol{F}\}$$

then $\boldsymbol{F}_0 = \boldsymbol{F}$.

(5.6) ASSUMPTION. $\mathrm{F}_b - \lambda\mathrm{G}_b$ restricted to E_λ is bijective, $\lambda \in \{i, -i\}$.
Let $\mathbf{B}_\lambda$ be the mapping from G into $\boldsymbol{F}$ defined by

$$\mathbf{B}_\lambda y = \mathrm{B}_\lambda y + y_\lambda, \quad \lambda \in \{i, -i\},$$

where $y_\lambda \in E_\lambda$ is uniquely determined by

$$(\mathrm{F}_b - \lambda\mathrm{G}_b)y_\lambda = \mathrm{G}_b y - (\mathrm{F}_b - \lambda\mathrm{G}_b)\mathrm{B}_\lambda y.$$

(5.7) THEOREM. *If $y, z \in G$ and $\lambda \in \{i, -i\}$, then*

$$(\mathbf{F} - \lambda\mathbf{G})\mathbf{B}_\lambda y = \mathbf{G}y,$$

and

$$(\mathbf{B}_\lambda y, z)_1 = (y, \mathbf{B}_{\bar{\lambda}} z)_1.$$

We define $\boldsymbol{N} = \{y \in G \mid (y, y)_1 = 0\}$. Let Φ be the canonical mapping from

G onto G/N. We equip G/N with the obvious inner product. In $G/N \times G/N$ we define the linear manifold $\mathbf{L}$ by

$$\mathbf{L} = \{\{\Phi(y), \Phi(z)\} \in G/N \times G/N | y \in \boldsymbol{F}, z \in \mathbf{G}^{-1}\mathbf{F}\{y\}\}.$$

From the theory of the previous sections, it follows that $\mathbf{L}$ is a self-adjoint subspace. The domain $\boldsymbol{D}$ of $\boldsymbol{L}$ is given by $\boldsymbol{D} = \Phi(\boldsymbol{F})$. The subspace $\mathbf{L}_s = \mathbf{L} \ominus \mathbf{L}_\infty$ generates a densely defined self-adjoint operator $\boldsymbol{A}$ in $\mathbf{L}(0)^\perp$ with domain $\boldsymbol{D} \subset \mathrm{L}(0)^\perp$ (cf. Coddington [1]). Let the linear manifold $\boldsymbol{V}$ be defined by $\boldsymbol{V} = \mathbf{G}^{-1}\mathbf{F}(\boldsymbol{N} \cap \boldsymbol{F})$. Then $\mathbf{L}(0) = \Phi(\boldsymbol{V})$ and $\mathbf{L}(0)^\perp = (\Phi \boldsymbol{V})^\perp = \Phi(\boldsymbol{V}^\perp)$. Also $\boldsymbol{F} \subset \boldsymbol{V}^\perp$. Let $\boldsymbol{F}_1$ be defined by $\boldsymbol{F}_1 = \boldsymbol{F}^{-1} \boldsymbol{G} \boldsymbol{V}^\perp$. If $\mathbf{G}\boldsymbol{N} = \{0\}$, then the mapping $\Phi : \boldsymbol{F}_1 \to \boldsymbol{D}$ is bijective.

Let U_b be the mapping from E_i into E_{-i} defined by $\mathrm{U}_b y = z$ whenever $(\mathrm{F}_b - i\mathrm{G}_b)y = (\mathrm{F}_b + i\mathrm{G}_b)z$, $y \in E_i$, $z \in E_{-i}$. By (5.6) U_b is well-defined, surjective and the domain of U_b equals E_i.

(5.8) THEOREM $(\mathrm{U}_b y, \mathrm{U}_b z)_1 = (y, z)_1$ *for all* $y, z \in E_i$.

PROOF. Let y_λ, $z_\lambda \in E_\lambda$, $\lambda \in \{i, -i\}$ be such that $\mathrm{U}_b y_i = y_{-i}$ and $\mathrm{U}_b z_i = z_{-i}$. Then

$$\mathrm{F}_b y_{-i} = \mathrm{F}_b y_i - i\mathrm{G}_b(y_i + y_{-i}),$$
$$\mathrm{F}_b z_{-i} = \mathrm{F}_b z_i - i\mathrm{G}_b(z_i + z_{-i}).$$

By (1.2) (a) and (b), (5.1), (5.2) and the above equalities we have

$$\begin{aligned}
(y_{-i}, z_{-i}) &= -\frac{1}{2i}\langle y_{-i}, z_{-i}\rangle \\
&= \frac{1}{2i}\{[\mathrm{F}_b y_{-i}, \mathrm{S}_b z_{-i}]_b - [\mathrm{S}_b y_{-i}, \mathrm{F}_b z_{-i}]_b\} \\
&= \frac{1}{2i}\{[\mathrm{F}_b y_i, \mathrm{S}_b z_{-i}]_b - [\mathrm{S}_b y_{-i}, \mathrm{F}_b z_i]_b \\
&\quad - i(y_i + y_{-i}, z_{-i})_b - i(y_{-i}, z_i + z_{-i})_b\} \\
&= \frac{1}{2i}\{[\mathrm{S}_b y_i, \mathrm{F}_b z_{-i}]_b - [\mathrm{F}_b y_{-i}, \mathrm{S}_b z_i]_b \\
&\quad - i(y_i + y_{-i}, z_{-i})_b - i(y_{-i}, z_i + z_{-i})_b\} \\
&= \frac{1}{2i}\{[\mathrm{S}_b y_i, \mathrm{F}_b z_i]_b - [\mathrm{F}_b y_i, \mathrm{S}_b z_i] \\
&\quad + 2i(y_i, z_i)_b - 2i(y_{-i}, z_{-i})_b\} \\
&= (y_i, z_i) + (y_i, z_i)_b - (y_{-i}, z_{-i})_b.
\end{aligned}$$

Let $\mathrm{K}_\lambda : E_i + E_{-i} \to H_b \times H_b$ be defined by

$$\mathrm{K}_\lambda y = \begin{pmatrix} (\mathrm{F}_b - \lambda\mathrm{G}_b)y \\ \mathrm{S}_b y \end{pmatrix}, \quad y \in E_i + E_{-i}, \lambda \in \{i, -i\}.$$

(5.9) THEOREM. K_λ *is injective*, $\lambda \in \{i, -i\}$.

PROOF. Let $y, z \in E_i + E_{-i}$ be decomposed into $y = y_i + y_{-i}$, and $z = z_i + z_{-i}$, where $y_\lambda, z_\lambda \in E_\lambda$, $\lambda \in \{i, -i\}$. These decompositions are unique according to (2.9). Then by (5.1), (5.2) and (1.2) (a) and (b)

$$\begin{aligned}[(\mathrm{F}_b - \lambda \mathrm{G}_b)y, \mathrm{S}_b z]_b - [\mathrm{S}_b y, (\mathrm{F}_b - \bar{\lambda}\mathrm{G}_b)z]_b &= -\langle y, z\rangle \\ &= -2i(y_i, z_i) + 2i(y_{-i}, z_{-i}).\end{aligned}$$

Suppose $(\mathrm{F}_b - \lambda\mathrm{G}_b)y = 0$ and $\mathrm{S}_b y = 0$. Then

$$(y_i, z_i) = (y_{-i}, z_{-i})$$

for all $z_\lambda \in E_\lambda$, $\lambda \in \{i, -i\}$. Hence $y_i = y_{-i} = 0$.

(5.10) ASSUMPTION. K_λ is bijective, $\lambda \in \{i, -i\}$.

(5.11) REMARK. If $\dim E_i = \dim E_{-i} = r < \infty$, then we may choose $H_b = \mathbf{C}^r$; then assumption (5.6) follows from corollary (5.5) and assumption (5.10) follows from theorem (5.9).

(5.12) THEOREM. *If* $y \in F_0$ *then* $\mathrm{F}_b y = \mathrm{G}_b y = \mathrm{S}_b y = 0$.

PROOF. Let $y \in F_0$. Then

$$[(\mathrm{F}_b - \lambda\mathrm{G}_b)y, \mathrm{S}_b z]_b = [\mathrm{S}_b y, (\mathrm{F}_b - \bar{\lambda}\mathrm{G}_b)z]_b$$

for all $z \in F$, $\lambda \in \{i, -i\}$. Let $h \in H_b$ be arbitrary and let $z \in E_i + E_{-i}$ be such that

$$\mathrm{K}_\lambda(z) = \begin{pmatrix} 0 \\ h \end{pmatrix}.$$

Then

$$[(\mathrm{F}_b - \lambda\mathrm{G}_b)y, h]_b = 0, \ \lambda \in \{i, -i\}$$

and hence

$$\mathrm{F}_b y = \mathrm{G}_b y = 0.$$

Also

$$[\mathrm{S}_b y, (\mathrm{F}_b - \bar{\lambda}\mathrm{G}_b)z]_b = 0 \text{ for all } z \in E_i + E_{-i}.$$

By (5.6) $\mathrm{F}_b - \bar{\lambda}\mathrm{G}_b$ is surjective, hence $\mathrm{S}_b y = 0$.

(5.13) REMARK. Suppose $\mathrm{G}_b \equiv 0$. Then $(,)_b$ vanishes, $\boldsymbol{N} = N$ and $\boldsymbol{\Phi} = \phi$. It follows from (5.12) that $F_0 \subset \boldsymbol{F} \subset F$. Consequently $\mathbf{L}$ is a self-adjoint subspace with $\mathrm{L}_0 \subset \mathbf{L} \subset \mathrm{L}$.

By (4.2) there must exist an isometry U from E_i onto E_{-i} such that

$$\mathbf{L} = \mathrm{L}_0 \oplus (I - V)M_i$$

where $V = \phi_{-i} U \phi_i^{-1}$. It turns out that $U = U_b$.

For, if $y \in F$ has the form $y = y_0 + y_i - Uy_i$ for some $y_0 \in F_0$, $y_i \in E_i$, then $\mathrm{F}_b y = 0$ if and only if $U = U_b$.

(5.14) REMARK. Assume that $\tilde{\mathrm{L}}$ is a self-adjoint subspace extension of L_0, determined by the isometry U from E_i onto E_{-i}. Define $H_b = E_i$, and $[,]_b = (,)$. Let

$$\mathrm{G}_b \equiv 0,$$

$$\mathrm{F}_b y = \begin{cases} 0 & \text{if } y \in F_0, \\ iy & \text{if } y \in E_i, \\ iU^{-1}y & \text{if } y \in E_{-i}. \end{cases}$$

$$\mathrm{S}_b y = \begin{cases} 0 & \text{if } y \in F_0, \\ -y & \text{if } y \in E_i, \\ U^{-1}y & \text{if } y \in E_{-i}. \end{cases}$$

(Observe: $(\mathrm{F}_b y, z) = \langle y, \frac{1}{2}(I-U)z\rangle$, and $(\mathrm{S}_b y, z) = \langle y, 1/2i(I+U)z\rangle$ for all $y, z \in F$). Then (5.1) with $(,)_b \equiv 0$, (5.2) (5.6) and (5.10) are satisfied. Hence $\tilde{\mathrm{L}} = \mathbf{L}$, i.e. $y \in \tilde{F}$ if and only if $y \in F$ and $\mathrm{F}_b y = 0$.

REFERENCES

E. A. CODDINGTON

[1] Extension theory of formally normal and symmetric subspaces, Mem. Am. Math. Soc.

T. KIMURA and M. TAKAHASI

[2] Sur les opérateurs différentiels ordinaires linéaires formellement autoadjoints I. Funkcial. Ekvac. 6, 35–90 (1964)

K. KODAIRA

[3] On ordinary differential equations of any even order and the corresponding eigen. function expansions. Amer. J. Math. 72, 502–544 (1950).

H.-D. NIESSEN

[4] Singuläre S-hermitesche Rand-Eigenwertprobleme. manuscripta math. 3, 35–68 (1970).

Å. PLEIJEL

[5] Special theory for pairs of ordinary formally self-adjoint differential operators. J. Indian Math. Soc. 34, 259-268 (1970).

F. W. SCHÄFKE and A. SCHNEIDER

[6] S-hermitesche Rand-Eigenwertprobleme I. Math. Ann. 162, 9–26 (1965).

F. W. SCHÄFKE and A. SCHNEIDER:

[7] S-hermitesche Rand-Eigenwertprobleme II. Math. Ann. 165, 236–260 (1966).

F. W. SCHÄFKE and A. SCHNEIDER

[8] S-hermitesche Rand-Eigenwertprobleme III. Math. Ann. 177, 67–94 (1968).

A. SCHNEIDER

[9] Untersuchungen über singuläre reelle S-hermitesche Differentialgleichungssysteme im Normalfall. Math. Z. 107, 271–296 (1968).

A. SCHNEIDER

[10] Weitere Untersuchungen über singuläre reelle S-hermitesche Differentialgleichungssysteme im Normalfall. Math. Z. 109, 153–168 (1969).

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