AUDUN HOLME

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A GENERAL EMBEDDING THEOREM IN FORMAL GEOMETRY

by

Audun Holme

Contents

Introduction

This paper is a continuation of one aspect of an earlier paper, [1]. Classically, one has a Whitney-type *algebraic* embedding theorem for projective *varieties* over an infinite field, see E. Lluis [2]. In particular, if X is a projective smooth variety over the infinite field k, then X may be embedded as a closed subvariety of P_k^{2n+1}, where $n = \dim(X)$.

One of the main results of [1] is a *formal* embedding theorem of this type, valid for (noetherian) complete local rings with an infinite field of representatives. This contains the classical algebraic result as a special case.

In this paper we prove a formal embedding theorem, valid over any complete, noetherian local ring A, Theorem 6.1. It has the following corollaries:

1. The formal embedding theorem of [1] has an analogue over *finite fields*, Theorem 1.2 and Proposition 2.2 with A = a finite field. This result is, however, of a purely formal nature and gives nothing in the algebraic case – in fact, the algebraic embedding theorem is false in general over a finite field.

2. The algebraic embedding theorem of [2] is generalized to projective *schemes* over *artinian* local rings, Theorem 1.1 (See also the remarks in Section 11): If X is smooth and projective over the artinian local ring A, then X may be embedded as a closed subscheme of $\boldsymbol{P}^r_A$, where

$$r = 2 \dim (X) + 1 + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2)$$

and where $k = A/\mathfrak{m}_A$.

3. The formal embedding theorem in [1] holds for noetherian, complete local rings *without a field of representatives*, Theorem 10.1.

We use the notations of EGA. In particular, if $\mathcal{O}$ is a local ring, $\mathfrak{m}_{\mathcal{O}}$ denotes its maximal ideal. If x is a point of the scheme X, then $\mathcal{O}_{X,x}$ and $\mathfrak{m}_{X,x}$ denote the local ring of X at x and its maximal ideal, respectively. If $X = \operatorname{Spec}(R)$, $\mathfrak{p}(x)$ denotes the prime ideal of R which corresponds to the point x. Finally let F be an R-module, $x \in \operatorname{Spec}(R)$. Then we put

$$\tilde{F}(x) = F(x) = F \otimes_R k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$, and if $f \in F$ we let $f(x)$ denote the canonical image of f in $F(x)$.

I would like to thank Professor *Frans Oort*, Amsterdam, for suggesting the applications in Section 4. I am grateful to the referee for useful advise and criticism of an early version of this paper.

Chapter 1

EMBEDDING THEOREMS OVER ARTINIAN LOCAL RINGS

1. Main results

Throughout Chapter 1, (except in Definition 2.1) A is an artinian local ring, and $k = A/\mathfrak{m}_A$. We prove the two closely related theorems stated below:

THEOREM (1.1). *Assume that k is infinite, and let X be a projective, smooth scheme over A. Then there exists a closed A-embedding $X \hookrightarrow \boldsymbol{P}^r_A$, where*

$$r = 2 \dim (X) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2) + 1.$$

Denote by R the formal power series ring in N indeterminates over A, $R = A[[T_1, \cdots, T_N]]$, and let I be an ideal in R. Put $\mathcal{O} = R/I = A[[t_1, \cdots, t_N]]$. The scheme $PN(\mathcal{O})$ is defined as the open subscheme of Spec $(\mathcal{O})$ obtained by deleting the closed point. For any field extension

k' of k, the ring $\mathcal{O}_{k'} = \mathcal{O} \hat{\otimes}_A k'$ is local and complete for the $\mathfrak{m}_{\mathcal{O}_{k'}}$-adic topology. In fact, $\mathcal{O}_{k'} = k'[[T_1, \cdots, T_N]]/I'$, where I' is the ideal generated by the image of I under the canonical homomorphism $A[[T_1, \cdots, T_N]] \to k'[[T_1, \cdots, T_N]]$.

THEOREM (1.2). *Assume that for all field extensions k' of k, $PN(\mathcal{O}_{k'})$ is non-singular. Then there exist $M = 2 \dim (\mathcal{O}) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2)$ elements $u_1, \cdots, u_M$ in $\mathfrak{m}_{\mathcal{O}}$, which are polynomials in $t_1, \cdots, t_N$ with coefficients from A, such that the inclusion $\mathcal{Q} = A[[u_1, \cdots, u_M]] \hookrightarrow \mathcal{O}$ induces an isomorphism $f: PN(\mathcal{O}) \to PN(\mathcal{Q})$.*

If k is infinite, $u_1, \cdots, u_M$ may be chosen as linear combinations in $t_1, \cdots, t_N$ with coefficients from A.

REMARK (1.2.1). The condition of the theorem can be given various equivalent forms. In EGA IV (18.11.10) the following conditions are shown to be equivalent for a point $x \in PN(\mathcal{O}_k)$:

a) For all field extensions k' of k and all points x' over x, $\mathcal{O}_{PN(\mathcal{O}_{k'}), x'}$ is a regular local ring.

b) Let $n = \max \{\dim (X_i) | X_i$ is an irreducible component of $\operatorname{Spec} (\mathcal{O}_k)$ and $x \in X_i\}$. Then $(\hat{\Omega}^1_{\mathcal{O}_k/k})_x$ is free of rank n. [1]

If the number n in b) is equal to $\dim (\mathcal{O}_k)$, then a) and b) are equivalent to

c) There exists a local k-homomorphism $k[[X_1, \cdots, X_n]] \to \mathcal{O}_k$ such that $\mathcal{O}_k$ is finite over $k[[X_1, \cdots, X_n]]$ and such that the induced morphism $\operatorname{Spec} (\mathcal{O}_k) \to \operatorname{Spec} (k[[X_1, \cdots, X_n]])$ is étale at x.

a), b) or c) imply

d) $\mathcal{O}_{PN(\mathcal{O}_k), x}$ is geometrically regular.

Moreover, if p denotes the characteristic exponent of k (i.e., $p = 1$ if k is of characteristic zero, otherwise p is the characteristic of k), then d) is equivalent to the other conditions provided $[k : k^p] < \infty$.

Finally, still under the assumption that $[k : k^p] < \infty$, we have that if $PN(\mathcal{O})$ is *formally smooth* over $\operatorname{Spec} (A)$, i.e., if for all $x \in PN(\mathcal{O})$, $\mathcal{O}_{PN(\mathcal{O}), x}$ is formally smooth over A for the $\mathfrak{m}_{PN(\mathcal{O}), x}$ and $\mathfrak{m}_A$-adic topologies, then the conditions hold. In fact, by EGA 0_{IV} (19.3.5) iii),

$$\mathcal{O}_{PN(\mathcal{O}), x} \otimes_A k = \mathcal{O}_{PN(\mathcal{O}), x}/\mathfrak{m}_A \mathcal{O}_{PN(\mathcal{O}), x}$$

is formally smooth over k for the tensor product topology, i.e., for the

[1] For definition of $\hat{\Omega}^1_{\mathcal{O}_k/k}$, see Section 2.

$$\mathfrak{m}_{PN(\mathcal{O}),x}/\mathfrak{m}_A\,\mathcal{O}_{PN(\mathcal{O}),x}\text{-adic}$$

one. Now this ring is nothing but the local ring of $PN(\mathcal{O}_k)$ at x, and it follows that $PN(\mathcal{O}_k)$ is formally smooth over Spec (k). Hence by EGA O_{IV} (19.6.6), $\mathcal{O}_{PN(\mathcal{O}_k),x}$ is geometrically regular over k.

2. The formal case

Theorem 1.1, as well as its formal analogue Theorem 1.2, will both be deduced as corollaries of a general formal embedding theorem to be stated and proven in Chapter 2, Theorem 6.1. Proposition 2.2 below is an immediate special case of this theorem.

The remaining part of Chapter 1 – with exception of Section 4 – is devoted to showing that Proposition 2.2 implies theorems 1.1 and 1.2.

Let $\hat{\Omega}^1_{\mathcal{O}/A}$ denote the $\mathfrak{m}_{\mathcal{O}}$-adic completion of $\Omega^1_{\mathcal{O}/A}$ and let $d : \mathcal{O} \to \hat{\Omega}^1_{\mathcal{O}/A}$ denote the canonical derivation. Then $\hat{\Omega}^1_{\mathcal{O}/A}$ is generated as $\mathcal{O}$-module by $dt_1, \cdots, dt_N$. (Indeed, this follows by the more general assertion of Lemma 5.1, which is stated and proven in Chapter 2.) We now define

$$\omega(\mathcal{O}/A) = \max\,\{rk_{k(x)}\,\hat{\Omega}^1_{\mathcal{O}/A}(x)|x \in PN(\mathcal{O})\}.$$

Moreover, the statement of Theorem 1.2 suggests the following abuse of language:

DEFINITION (2.1). Let A be a complete, noetherian local ring (not necessarily artinian) and let $\mathcal{O} = A[[T_1, \cdots, T_N]]/I = A[[t_1, \cdots, t_N]]$. If $k = A/\mathfrak{m}_A$ *is infinite*, then an element $u \in \mathfrak{m}_{\mathcal{O}}$ is said to be *adequate* provided it is a *linear combination* in $t_1, \cdots, t_N$ with coefficients from A. If k is *finite and A is artinian*, u is said to be adequate if it is a *polynomial* in $t_1, \cdots, t_N$ with coefficients from A. Finally, if k *is finite but A is not artinian*, then no condition is imposed on the element u.

PROPOSITION (2.2). *There are* $s = \dim(\mathcal{O}) + \omega(\mathcal{O}/A)$ *adequate elements* $u_1, \cdots, u_s$ *in* $\mathfrak{m}_{\mathcal{O}}$ *such that the inclusion* $\mathcal{Q} = A[[u_1, \cdots, u_s]] \hookrightarrow \mathcal{O}$ *makes* $\mathcal{O}$ *to a finite* $\mathcal{Q}$*-module and induces an isomorphism* $f : PN(\mathcal{O}) \to PN(\mathcal{Q})$.

We prove first that Proposition 2.2 implies Theorem 1.2. To show is that under the assumption in Theorem 1.2, $s \leqq 2\dim(\mathcal{O}) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2)$. For this, note that there is an exact sequence:

$$\mathfrak{m}_A\,\mathcal{O}/\mathfrak{m}_A\,\mathcal{O}^2 \to \hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} \mathcal{O}_k \to \hat{\Omega}^1_{\mathcal{O}_k/A} \to 0. \tag{2.2.1}$$

Indeed, by EGA O_{IV} (20.7.20) the homomorphisms

$$\hat{\partial} = \hat{\partial}_{\mathcal{O}_k/\mathcal{O}/A} : \mathfrak{m}_A\,\mathcal{O}/\mathfrak{m}_A\,\mathcal{O}^2 \to \hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} \mathcal{O}_k$$

and

$$\hat{v} = \hat{v}_{\mathcal{O}_k/\mathcal{O}/A} : \hat{\Omega}^1_{\mathcal{O}/A}\,\hat{\otimes}_{\mathcal{O}}\,\mathcal{O}_k \to \hat{\Omega}^1_{\mathcal{O}_k/A}$$

satisfy the following: Im $(\hat{d})$ is dense in Ker $(\hat{v})$ and $\hat{v}$ is surjective. Moreover, by EGA O_I (7.7.1) and (7.7.8)

$$\hat{\Omega}^1_{\mathcal{O}/A} \mathbin{\hat{\otimes}_{\mathcal{O}}} \mathcal{O}_k = (\hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} \mathcal{O}_k)^{\wedge} = \hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} \mathcal{O}_k,$$

and the tensor product topology equals the $\mathfrak{m}_{\mathcal{O}_k}$-adic one. Therefore, since $\hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} \mathcal{O}_k$ is a finite $\mathcal{O}_k$-module, Im $(\hat{d})$ is a finite $\mathcal{O}_k$-module, thus it is complete for the $\mathfrak{m}_{\mathcal{O}_k}$-adic topology. Hence Im $(\hat{d})$ is closed in $\hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} \mathcal{O}_k$, and it follows that the sequence (2.2.1) is exact.

For all $x \in PN(\mathcal{O})$, (2.2.1) gives an exact sequence of $k(x)$-vector spaces:

(2.2.2.) $$(\mathfrak{m}_A \mathcal{O}/\mathfrak{m}_A \mathcal{O}^2)(x) \to \hat{\Omega}^1_{\mathcal{O}/A}(x) \to \hat{\Omega}^1_{\mathcal{O}_k/A}(x) \to 0.$$

Now EGA O_{IV} (20.7.17) gives an exact sequence

$$\hat{\Omega}^1_{k/A} \otimes_k \mathcal{O}_k \xrightarrow{\hat{v}_{\mathcal{O}_k/k/A}} \hat{\Omega}^1_{\mathcal{O}_k/A} \xrightarrow{\hat{u}_{\mathcal{O}_k/k/A}} \hat{\Omega}^1_{\mathcal{O}_k/k} \to 0$$

in the same way as EGA O_{IV} (20.7.20) implied (2.2.1), since the rings involved are all noetherian and the modules are of finite type. Since $\hat{\Omega}^1_{k/A} = (0)$, this shows that $\hat{\Omega}^1_{\mathcal{O}_k/A}$ and $\hat{\Omega}^1_{\mathcal{O}_k/k}$ are *canonically isomorphic.* Thus (2.2.2) gives

(2.2.3) $$rk_{k(x)}\, \hat{\Omega}^1_{\mathcal{O}/A}(x) \leqq rk_{k(x)}\, \hat{\Omega}^1_{\mathcal{O}_k/k}(x) + rk_{k(x)}(\mathfrak{m}_A \mathcal{O}/\mathfrak{m}_A \mathcal{O}^2)(x).$$

Since on the other hand

(2.2.4) $$rk_{k(x)}(\mathfrak{m}_A \mathcal{O}/\mathfrak{m}_A \mathcal{O}^2)(x) \leqq rk_k\, \mathfrak{m}_A/\mathfrak{m}_A^2,$$

we only need to prove that

(2.2.5) $$rk_{k(x)}\, \hat{\Omega}^1_{\mathcal{O}_k/k}(x) \leqq \dim(\mathcal{O}_k) = \dim(\mathcal{O}),$$

which follows by Remark 1.2.1,b).

3. Reduction to the formal case

We now prove that Proposition 2.2 implies Theorem 1.1. For this we need a lemma which provides the link between the existence of a formal embedding (Theorem 1.2 and Proposition 2.2) and the existence of an algebraic (projective) embedding as in Theorem 1.1.

Let $X = \text{Proj}(S)$ where $S = A[T_0, \cdots, T_N]/H = A[t_0, \cdots, t_N]$ is the quotient of the polynomial ring $A[T_0, \cdots, T_N]$ by a homogeneous ideal H. Denote by $\mathfrak{m}_0$ the maximal ideal of S generated by $\mathfrak{m}_A$ and $t_0, \cdots, t_N$. Finally, let $\mathcal{O}$ denote the $\mathfrak{m}_0$-adic completion of S.

Lemma (3.1). *Let $u_0, \cdots, u_r$ be linear combinations in $t_0, \cdots, t_N$ with coefficients from A. Let*

$$T = A[u_0, \cdots, u_r] \subseteq S \text{ and } \mathcal{Q} = A[[u_0, \cdots, u_r]] \subseteq \mathcal{O}.$$

Assume that $\mathcal{O}$ is finite over $\mathcal{Q}$. Then the inclusion $T \hookrightarrow S$ induces a morphism g : Proj $(S) \to$ Proj (T).

Moreover, g is an isomorphism if the canonical $f : PN(\mathcal{O}) \to PN(\mathcal{O})$ is.

PROOF. If necessary by increasing N, we may asume that $U_0 = T_0, \cdots, U_r = T_r$.

Furthermore, it suffices to prove the lemma for the case $r = N-1$. In fact, once we know this, the general case follows by repetition. So assume $r = N-1$ and $U_0 = T_0, \cdots, U_{N-1} = T_{N-1}$.

LEMMA (3.2). i) *$\mathcal{O}$ equals the completion of S at $(t_1, \cdots, t_N)S$, i.e.* $\mathcal{O} = A[[T_0, \cdots, T_N]]/HA[[t_0, \cdots, T_N]]$.

ii) *$\mathcal{Q}$ is the completion of*

$$T = A[T_0, \cdots, T_{N-1}] / H \cap A[T_0, \cdots, T_{N-1}] = A[t_0, \cdots, t_{N-1}]$$

at $(t_0, \cdots, t_{N-1})T$.

PROOF. i) follows since $\mathfrak{m}_A^s = 0$ for $s \gg 0$. If $\varphi : S \to \mathcal{O}$ denotes the canonical homomorphism, $\mathcal{Q}$ is defined as $\overline{\varphi(T)}$, the closure of $\varphi(T)$ in $\mathcal{O}$. Hence

$$\mathcal{Q} = A[[T_0, \cdots, T_{N-1}]]/(HA[[T_0, \cdots, T_N]]) \cap A[[T_0, \cdots, T_{N-1}]].$$

Thus to prove ii), it suffices to show that

$$\text{(3.2.1)} \quad \begin{aligned} &(HA[[T_0, \cdots, T_N]]) \cap A[[T_0, \cdots, T_{N-1}]] \\ &\qquad = (H \cap A[T_0, \cdots, T_{N-1}])A[[T_0, \cdots, T_N]]. \end{aligned}$$

Clearly

$$(HA[[T_0, \cdots, T_N]]) \cap A[[T_0, \cdots, T_{N-1}]] \supseteq H \cap A[T_0, \cdots, T_{N-1}],$$

so $\supseteq$ holds in (3.2.1).

Conversely, let

$$F \in (HA[[T_0, \cdots, T_N]]) \cap A[[T_0, \cdots, T_{N-1}]].$$

If $f_1, \cdots, f_m$ are homogeneous generators for H, we get

$$F = \sum_{i=1}^{m} f_i g_i = \sum_{i=1}^{m} \sum_{n=0}^{\infty} f_i g_i(n),$$

where $g(n)$ denotes the homogeneous part of g of degree n. Thus $F(n) = \Sigma f_i g_i(t)$, where Σ is taken over all i and t such that $\deg(f_i) + t = n$. This shows that $F(n) \in H$ for all n. But since T_N does not occur in F, it does not occur in $F(n)$. Hence

$$F(n) \in H \cap A[T_0, \cdots, T_{N-1}]$$

for all n, so since

$$F = \sum_{n=0}^{\infty} F(n),$$

we conclude that F is in the closure of

$$H \cap A[T_0, \cdots, T_{N-1}] \text{ in } A[[T_0, \cdots, T_{N-1}]],$$

i.e.,

$$F \in (H \cap A[T_0, \cdots, T_{N-1}])A[[T_0, \cdots, T_{N-1}]].$$

Thus Lemma 3.2 is proven.

Clearly we have a canonical morphism $g : D_+(T_+S) \to \text{Proj}\,(T)$. (If I is a homogeneous ideal, then $D_+(I) = \text{Proj}\,(S) - V_+(I)$. As always T_+ denotes the ideal in T generated by the elements of positive degree in T.) We next show that $D_+(T_+S) = \text{proj}\,(S)$. Indeed, assume the converse, and let $x \in V_+(T_+S)$. x corresponds to a point $y \in PN(S_{\mathfrak{m}_0})$ and since $\mathcal{O}$ is faithfully flat over $S_{\mathfrak{m}_0}$, there is a point $z \in PN(\mathcal{O})$ above y. If $\mathfrak{p}(z)$ is the corresponding prime in $\mathcal{O}$, then $t_0, \cdots, t_{N-1} \in \mathfrak{p}(z)$. Thus $\dim\,(\mathcal{O}/(t_0, \cdots, t_{N-1})\mathcal{O}) \geqq 1$, a contradiction since $\mathcal{O}$ is a finite $\mathcal{Q}$-module.

Now suppose that the canonical $f : PN(\mathcal{O}) \to PN(\mathcal{Q})$ is an isomorphism, and let $h \in T_+$ be a homogeneous element. The proof of Lemma 3.1 is complete once we show that

$$T_{(h)} = S_{(h)}. \tag{3.3}$$

For this, notice first that by assumption, $\mathcal{O}_h = \mathcal{Q}_h$. Hence in particular it follows that if $F/h^n \in S_{(h)}$, i.e. F is a homogeneous element of S of degree $n \deg\,(h)$, then $F/h^n \in \mathcal{Q}_h$. Thus for a suitable element $F' \in \mathcal{Q}$ and a suitable integer n',

$$F/h^n = F'/h^{n'}.$$

In order to prove (3.3), it suffices to show that $F'h^s \in T$ for some s. In fact, we then have

$$F/h^n = F'h^s/h^{n'+s} \in T_{(h)}.$$

$F'h^s$ is a polynomial in $t_0, \cdots, t_N$ for $s \gg 0$. On the other hand,

$$F'h^s = G(t_0, \cdots, t_{N-1}) \in A[[t_0, \cdots, t_{N-1}]].$$

To show is that $F'h^s$ is a polynomial in $t_0, \cdots, t_{N-1}$. We know that there exists a polynomial

$$P(T_0, \cdots, T_N) \in A[T_0, \cdots, T_N]$$

such that

$$G(t_0, \cdots, t_{N-1}) = P(t_0, \cdots, t_N),$$

i.e. such that

$$G(T_0, \cdots, T_{N-1}) - P(T_0, \cdots, T_N) = \sum_{i=1}^{m} f_i g_i$$

for suitable $g_i \in A[[T_0, \cdots, T_N]]$, where as before $f_1, \cdots, f_m$ are the homogeneous generators of the ideal H. As before, $F(r)$ denotes the homogeneous part of degree r of the power series $F \in A[[T_0, \cdots, T_N]]$. Then since $f_1, \cdots, f_m$ are homogeneous elements, we get

$$G(r) - P(r) \in HA[[T_0, \cdots, T_N]]$$

for all r. And since $P(r) = 0$ for $r \gg 0$, this shows that

$$G(r) \in HA[[T_0, \cdots, T_N]] \text{ for } r \gg 0.$$

Hence $G(t_0, \cdots, t_{N-1})$ is actually a *polynomial* in $t_0, \cdots, t_{N-1}$, and (3.3) follows. This completes the proof of Lemma 3.1.

REMARK. It is not difficult to show that in the last part of the lemma, 'if' may be replaced by 'if and only if'. However, this fact is not needed here.

By means of Lemma 3.1, we know that Proposition 2.2 implies Theorem 1.1, once we prove that

$$\omega(\mathcal{O}/A) \leqq \dim(X) + rk_k\, m_A/m_A^2 + 1. \tag{3.4}$$

To show this, we may assume that A is a field. Indeed, $X = \text{Proj}(S)$ gives $X_k = X \otimes_A k = \text{Proj}(S \otimes_A k)$. Put $S \otimes_A k = S_k$. Then with notation as before, $(S_k)^\wedge = \mathcal{O}_k$. Thus since (2.2.3), (2.2.4) and (2.2.5) give that

$$\omega(\mathcal{O}/A) \leqq \omega(\mathcal{O}_k/k) + rk_k\, \mathfrak{m}_A/\mathfrak{m}_A^2,$$

(3.4) follows once we show that

$$\omega(\mathcal{O}_k/k) \leqq \dim(X_k) + 1 = \dim(X) + 1. \tag{3.5}$$

So suppose that $X = \text{Proj}(S)$, where $S = k[T_0, \cdots, T_N]/H$ and H is generated by the homogeneous polynomials $f_1, \cdots, f_m$. Then

$$\mathcal{O} = \hat{S} = k[[T_0, \cdots, T_N]]/(f_1, \cdots, f_m) = k[[t_0, \cdots, t_N]].$$

If $x \in X$ is given by the homogeneous prime $\mathfrak{p}$, let $\mathscr{P} \subset \hat{S}$ be a prime ideal such that $\mathscr{P} \cap S = \mathfrak{p}$. Then

$$rk_{k(\mathscr{P})} \hat{\Omega}^1_{\mathcal{O}/k}(\mathscr{P}) = rk_{k(\mathfrak{p})} \Omega^1_{S/k}(\mathfrak{p}) = rk_{k(x)} \Omega^1_{X/k}(x) + 1. \tag{3.6}$$

In fact, the first equality follows since the canonical

$$\hat{v}_{\mathcal{O}/S/k} : \Omega^1_{S/k} \hat{\otimes}_S \mathcal{O} \to \hat{\Omega}^1_{\mathcal{O}/k}$$

is an isomorphism (Lemma 5.2). Moreover,

$$\Omega^1_{S/k}(\mathfrak{p}) = KD_0 \oplus \cdots \oplus KD_N/E$$

where $K = k(\mathfrak{p})$ and E is generated by the elements

$$(\partial f_j/\partial T_0)(a_0, \cdots, a_N)D_0 + \cdots + (\partial f_j/\partial T_N)(a_0, \cdots. a_N)D_N$$
$$j = 1, \cdots, m,$$

where a_i is the image of t_i in K. If necessary after a suitable change of variables, we may assume that $a_0 \neq 0$ and $D_0 \notin E$. Now

$$\Omega^1_{X/k}(x) \otimes_{k(x)} K = KD_1 \oplus \cdots \oplus KD_N/E',$$

where E' is generated by

$$\sum_{i=1}^{N} (\partial F_j/\partial T_i)(1, a_1/a_0, \cdots, a_N/a_0)D_i, \quad j = 1, \cdots, m.$$

Hence

$$\Omega^1_{X/k}(x) \otimes_{k(x)} K = \Omega^1_{S/k}(\mathfrak{p})/(D_0),$$

and the claim follows.

4. Application to the local moduli space for abelian schemes

Let k be a field and $\mathscr{C}_k$ denote the category of artinian local rings with k as residue class field.

For all R in $\mathscr{C}_k$, a scheme X over R, X/R, is said to be an *abelian scheme* over R provided it is a group scheme, smooth and proper over R with geometrically connected fiber. Let X_0 be an abelian scheme over k, and define for all R in $\mathscr{C}_k$

$$F(R) = \left\{ (X/R, \varphi_0) \,\middle|\, \begin{array}{l} X/R \text{ is an abelian scheme over } R \text{ and} \\ \varphi_0 : X x_R k \to X_0 \text{ is an isomorphism} \end{array} \right\} \Big/ \sim$$

where $\sim$ denotes R-isomorphism compatible with the φ_0's. Proofs of the following two theorems may be found in [3].

THEOREM (4.1). (*Grothendieck*). *Let W be the ring of infinite Witt-vectors if k is of characteristic $p > 0$, and $W = k$ if k is of characteristic zero. Then F is prorepresentable by*

$$W[[t]] = W[[t_{11}, \cdots, t_{gg}]],$$

where $g = \dim (X_0)$.

Now let

$$\lambda_0 = \Lambda L_0 : X_0 \to \hat{X}_0 = \mathrm{Pic}^{\tau}_{X_0/k}$$

be the quasi-polarization which corresponds to the invertible sheaf L_0, i.e. the X_0-valued point λ_0 of $\hat{X}_0$ which corresponds to

$$\mu^*(L_0) \otimes \mathrm{pr}_1^*(L_0)^{-1} \otimes \mathrm{pr}_2^*(L_0)^{-1} \in \mathrm{Pic}\,(X_0 x_k X_0)/\mathrm{Pic}\,(X_0)$$

where μ is the multiplication of X_0. Define

$$F_{\lambda_0}(R) = \left\{ (X/R, \lambda, \phi_0) \,\middle|\, \begin{matrix} (X, \lambda) \text{ is a quasi-polarized abelian scheme, and} \\ \phi_0 : (X, \lambda) \otimes_R k \to (X_0, \lambda_0) \text{ is an isomorphism} \end{matrix} \right\} \Big/ \sim$$

where $\sim$ denotes R-isomorphism compatible with λ and φ_0.

THEOREM (4.2) (*Mumford*). *The functor F_{λ_0} is a sub-functor of F, and is prorepresentable by*

$$W[[t_{11}, \cdots, t_{gg}]]/\mathfrak{a}_{\lambda_0}$$

where $\mathfrak{a}_{\lambda_0}$ is an ideal generated by $d = \frac{1}{2}g(g-1)$ elements.

Using the notations of [3], the two theorems above imply that the canonical homomorphism

$$W[[t]] \to W[[t]]/\mathfrak{a}_{\lambda_0},$$

induces the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\hat{C}_W}(W[[t]], -) & \xleftarrow[\sim]{\psi} & F(-) \\ \uparrow\cup & & \uparrow\cup \\ \mathrm{Hom}_{\hat{C}_W}(W[[t]]/\mathfrak{a}_{\lambda_0}, -) & \xleftarrow[\sim]{\psi_{\lambda_0}} & F_{\lambda_0}(-) \end{array}$$

We assume for the rest of this section that k is an *infinite field of characteristic* $p > 0$. Let X_0 be a projective abelian scheme over k. If $X/R \in F(R)$, there always exists an ample sheaf on X_0 which lifts to a relatively ample sheaf on X. We conclude that

(4.3) $$\mathrm{Hom}_{\hat{C}_W}(W[[t]], -) = \bigcup_{\lambda_0} \mathrm{Hom}_{\hat{C}_W}(W[[t]]/\mathfrak{a}_{\lambda_0}, -).$$

This implies

(4.4) $$\bigcap_{\lambda_0} \mathfrak{a}_{\lambda_0} = (0).$$

In fact, letting $\mathfrak{b} = \bigcap_{\lambda_0} \mathfrak{a}_{\lambda_0}$, (4.3) gives that for all homomorphisms $\varphi : W[[t]] \to R$, $\mathfrak{b} \subseteq \mathrm{Ker}(\varphi)$. Letting $R = W[[t]]/\mathfrak{m}^\nu$, where $\mathfrak{m} = \mathfrak{m}_{W[[t]]}$, we get $\mathfrak{b} \subseteq \mathfrak{m}^\nu$ for all ν, which implies the claim.

Using Theorem 1.1, it is now possible to refine (4.4). Indeed, let $\Lambda(N)$ denote the set of polarizations on X_0 such that the corresponding ample sheaf is very ample and induces an embedding $X_0 \hookrightarrow \boldsymbol{P}_k^m$ with $m \leqq N$. We then have the following result:

THEOREM (4.5). $\bigcap_{\lambda_0 \in \Lambda((g+1)^2)} \mathfrak{a}_{\lambda_0} = (0)$.

PROOF. Put

$$\mathfrak{c} = \bigcap_{\lambda_0 \in N((g+1)^2)} \mathfrak{a}_{\lambda_0}.$$

To show is that $\mathfrak{c} \subset \mathfrak{m}^\nu$ for all ν. Put $R = W[[t]]/m^\nu$, and let $X/R \in F(R)$

correspond to the canonical $\psi : W[[t]] \to R$. Then there is an R-embedding

$$i : X \hookrightarrow P^s_R,\ s = 2\dim(X_0)+1+rk_k\, \mathfrak{m}_R/\mathfrak{m}_R^2 = (g+1)^2.$$

Hence there is $\lambda_0 \in \Lambda((g+1)^2)$ such that $X/R \in F_{\lambda_0}(R)$: In fact, if L is the very ample sheaf of hyperplane sections which corresponds to i, and $L_0 = L \otimes_R k$, then $\lambda_0 = \Lambda(L_0)$ lifts to $\lambda = \Lambda(L)$. Thus the homomorphism which corresponds to $X/R \in F(R), \psi$, factors through the canonical $W[[t]] \to W[[t]]/\mathfrak{a}_{\lambda_0}$. Hence $\mathfrak{a}_{\lambda_0} \subseteq \text{Ker}(\psi) = \mathfrak{m}^\nu$.

Remark (4.6). From one point of view this result is somewhat surprising. If one is given an abelian scheme X over R, which lifts the projective abelian scheme X_0 over k, then the obvious way of producing a projective embedding $X \hookrightarrow P^N_R$ is to try to lift a very ample sheaf L_0 on X_0 to a very ample sheaf L on X. Of course that can't be done in general, but if $\varphi : R \to R'$ is a surjection of artinian local rings, if $\mathfrak{m}_R \text{ Ker}(\varphi) = (0)$, $X' = X \otimes_R R'$ and, finally, if L' is an invertible sheaf on X', then $(L')^{p^n}$ can be lifted to X for some n.

Proceeding in this way, one would expect to get an upper bound for the projective embedding dimension of X/R in terms of p and $l(R)$, the length of R.

Chapter 2

FORMAL EMBEDDINGS

5. Completed differentials

Let A be a noetherian, complete local ring (i.e., A is not assumed to be artinian as in Chapter 1.) R denotes the formal power series ring $A[[T_1, \cdots, T_N]]$ in N indeterminates over A. Let I be an ideal in R, and put $\mathcal{O} = R/I = A[[t_1, \cdots, t_N]]$.

As in Chapter 1, $PN(\mathcal{O})$ denotes the open subscheme of Spec $(\mathcal{O})$ obtained by deleting the closed point, and $\hat{\Omega}^1_{\mathcal{O}/A}$ denotes the $\mathfrak{m}_{\mathcal{O}}$-adic conpletion of $\Omega^1_{\mathcal{O}/A}$. As before, $k = A/\mathfrak{m}_A$.

Under the assumptions above, $\hat{\Omega}^1_{\mathcal{O}/A}$ is an $\mathcal{O}$-module of finite type. More precisely, we have the following:

Proposition (5.1). *Let $d : \mathcal{O} \to \hat{\Omega}^1_{\mathcal{O}/A}$ denote the canonical derivation. Then $\hat{\Omega}^1_{\mathcal{O}/A}$ is generated over $\mathcal{O}$ by $dt_1, \cdots, dt_N$.*

Proof. By EGA 0_{IV} (20.7.17), the canonical homomorphism $\hat{v} : \hat{v}_{\mathcal{O}/R/A} : \hat{\Omega}^1_{R/A} \hat{\otimes}_R \mathcal{O} \to \hat{\Omega}^1_{\mathcal{O}/A}$ has dense image in $\hat{\Omega}^1_{\mathcal{O}/A}$. Assume the claim for R, i.e. for $I = (0)$. Then $\hat{\Omega}^1_{R/A}$ is a finite R-module, so $\hat{\Omega}^1_{R/S} \hat{\otimes}_R \mathcal{O} = \hat{\Omega}^1_{R/A} \otimes_R \mathcal{O}$ is a finite $\mathcal{O}$-module. Hence Im $(\hat{v})$ is a finite $\mathcal{O}$-module, i.e., Im $(\hat{v}) = \hat{\Omega}^1_{\mathcal{O}/A}$. Thus the claim follows for $\mathcal{O}$. So we may assume that $I = (0)$. It suffices to show the following.

Lemma (5.2). *If A is a topological ring, and B is a topological A-algebra, then*

$$v_{\hat{B}/B/A} : \Omega^1_{B/A} \otimes_B \hat{B} \to \Omega^1_{\hat{B}/A}$$

is a formal bimorphism (i.e. $\hat{v}_{\hat{B}/B/A}$ is bijective).

Indeed, the lemma applied to $B = A[T_1, \cdots, T_N]$ with the $(\mathfrak{m}_A, T_1, \cdots, T_N)$-adic topology gives $\hat{\Omega}^1_{\hat{B}/A} = \Omega^1_{B/A} \otimes_B \hat{B}$, and since A is complete, $R = \hat{B}$.

Proof of the lemma. By EGA 0_{IV} (20.7.6) it suffices to note that the canonical $B \to \hat{B}$ makes $\hat{B}$ to a formally étale B-algebra. This is immediate. (EGA 0_{IV} (19.3.6) and (19.10.2).)

6. Main theorem. Outline of proof

As before, we put $\mathcal{O}_k = \mathcal{O} \hat{\otimes}_A k$ and $\omega(\mathcal{O}/A) = \max \{rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}/A}(x) \mid x \in PN(\mathcal{O})\}$. Define

$$\boldsymbol{d}(\mathcal{O}/A) = \max \{\omega(\mathcal{O}/A), \dim (\mathcal{O}_k)\}.$$

Let $f : X \to Y$ be a morphism of schemes, and let $x \in X$. We say that f *is an isomorphism* at x provided there exists *an open subset* V of Y containing $f(x)$, such that the restriction of f to $f^{-1}(V)$, $f' : f^{-1}(V) \to V$ is an isomorphism. The set of points in X at which f is not an isomorphism, is denoted by $C(f)$. By definition, this is a closed subset of X.

The aim of Chapter 2 is to prove the following:

Theorem (6.1). *There exists* $\boldsymbol{d}(\mathcal{O}/A) + \dim(\mathcal{O}) = M$ *adequate elements* $u_1, \cdots, u_M$ in $\mathfrak{m}_{\mathcal{O}}$, *such that $\mathcal{O}$ is finite over the subring* $\mathcal{Q} = A[[u_1, \cdots, u_M]]$, *and such that the canonical morphism* $f : PN(\mathcal{O}) \to PN(\mathcal{Q})$ *is an isomorphism.*

The proof of this theorem is rather technical, but in outline it runs as follows:

Step 1 consists in finding $\boldsymbol{d} = \boldsymbol{d}(\mathcal{O}/A)$ adequate elements

$$u_1, \cdots, u_{\boldsymbol{d}} \tag{6.2}$$

such that $\mathcal{O}$ is finite over the subring $\mathcal{Q}_1 = A[[u_1, \cdots, u_{\boldsymbol{d}}]]$, and such that

(6.3) *$du_1, \cdots, du_d$ generate a sufficiently large part of $\hat{\Omega}^1_{\mathcal{O}/A}$ on sufficiently large pieces of $PN(\mathcal{O})$.*

Condition (6.3) is made precise in the statement of Lemma 7.3, and implies in particular that the elements in (6.3) generate $\hat{\Omega}^1_{\mathcal{O}/A}$ *at all generic points* of $PN(\mathcal{O})$, and hence (Lemma 8.4.3) that the canonical $f_1 : PN(\mathcal{O}) \to PN(\mathcal{Q}_1)$ is *unramified* there. The elements $u_1, \cdots, u_d$ are picked inductively as follows:

First u_1 is chosen outside all minimal primes of $\mathfrak{m}_A \mathcal{O}$ and such that du_1 does not vanish at any generic point of $PN(\mathcal{O})$. A difficulty occurs here if k is *finite*: Namely, since a vector space over k may in this case be the union of a finite set of proper subspaces, one has to seek a replacement for arguments using 'generic conditions' over k. This is achieved by Lemmas 7.4 and 7.4.2, but of course at the expense of having 'linear combination' replaced by 'adequate element' in Theorem 6.1.

Now u_2 is found outside all minimal primes of $(\mathfrak{m}_A, u_1)\mathcal{O}$, and such that du_1, du_2 satisfy a condition like (6.3.) For this we use a technique similar to one developed by *J. P. Serre* ([4], Théorème 2), here isolated as Lemma 7.5. Repetition of the process yields the elements (6.2).

STEP 2. To find the remaining elements $u_{d+1}, \cdots, u_M$, we again proceed inductively. First choose u_{d+1} such that

(6.4) u_{d+1} separates the generic points of $PN(\mathcal{O})$,

(6.5) for all generic points x of $PN(\mathcal{O})$, $u_{d+1}(x)$ generates $k(x)$ over $k(f_1(x))$,

and finally such that $du_1, \cdots, du_{d+1}$ satisfy a condition of the same type as (6.3), namely (8.4.2). (6.4) and (6.5) are possible because f_1 is unramified at the generic points of $PN(\mathcal{O})$, Lemma 8.2. This implies that the canonical $f_2 : PN(\mathcal{O}) \to PN(\mathcal{Q}_1[[u_{d+1}]])$ is an isomorphism at all generic points of $PN(\mathcal{O})$, Lemma 8.1.

Now the (6.3)-condition on $du_1, \cdots, du_{d+1}$ implies that f_2 is unramified at all generic points to those irreducible components of $C(f_2)$ which are of dimension equal to $\dim(PN(\mathcal{O}))-1$ (cf. (8.4.2). This is what happens when $PN(\mathcal{O})$ is *equidimensional*. The general case is slightly more complicated.) Thus the process may be repeated: We get u_{d+2} which separates these points, and also satisfy (6.5) there. It follows that the canonical morphism $f_3 : PN(\mathcal{O}) \to PN(A[[u_1, \cdots, u_{d+2}]]$ has $C(f_3)$ of dimension $\leqq \dim(PN(\mathcal{O})-2 = \dim(\mathcal{O})-3$. Repeating this $n = \dim(\mathcal{O})$ times, we finally get

$$f = f_{n+1} : PN(\mathcal{O}) \to PN(A[[u_1, \cdots, u_M]])$$

with $C(f) = \emptyset$, i.e. f is *an isomorphism.*

7. Proof of the theorem. Step 1

We now turn to the details. In order to make the loosely phrased condition (6.3) precise, we need the following notation:

DEFINITION (7.1). Let $X_1, \cdots, X_r$ be the irreducible components of $X = PN(\mathcal{O})$. Put

$$X(j, d) = \{x \in X_j | rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}/A}(x) \geqq d\}$$

for all $j = 1, \cdots, r$ and all integers d. We denote the irreducible components of $X(j, d)$ by $Y_s = \overline{\{y_s\}}$, where s runs through the index set $I(j, d)$.

REMARK. Of course this definition does not make sense until we prove that $X(j, d)$ is a *closed* subset of X. But this is easily seen: In fact, we have the following:

LEMMA (7.2). *Let F be an $\mathcal{O}$-module of finite type. Then for all integers d, the subset of $X = PN(\mathcal{O})$*

$$U_d = \{x \in X | rk_{k(x)} F(x) < d\}$$

is open in X.

PROOF. Suppose that F is generated as $\mathcal{O}$-module by $f_1, \cdots, f_m$. Let $x \in U_d$, and let $\varphi_1, \cdots, \varphi_l$ $(l < d)$ be elements of F such that $\varphi_1(x), \cdots, \varphi_l(x)$ generate $F(x)$. By Nakayamas' Lemma this implies that the images of $\varphi_1, \cdots, \varphi_l$ in $F_{\mathfrak{p}(x)}$, $\varphi_1/1, \cdots, \varphi_l/1$ generate $F_{\mathfrak{p}(x)}$ as $\mathcal{O}_{\mathfrak{p}(x)}$-module. Thus there are elements $a_{ij} \in \mathcal{O}_{\mathfrak{p}(x)}$ such that

$$f_i/1 = \sum_{j=1}^{l} a_{ij}(\varphi_i/1).$$

Now $a_{ij} = b_{ij}/c$, where $b_{ij} \in \mathcal{O}$ and $c \in \mathcal{O} - \mathfrak{p}(x)$. Clearly there exists $d_i \in \mathcal{O} - \mathfrak{p}(x)$, $i = 1, \cdots, m$, such that $d_i(f_i c - \Sigma b_{ij}\varphi_i) = 0$. Let $a = d_1 \cdots d_m c$. Then $x \in D(a) \cap PN(\mathcal{O}) \subseteq U_d$.

We also use the following notation:

$$rk_{k(x)}(\hat{\Omega}^1_{\mathcal{O}/A}(x)) = r(x)$$
$$rk_{k(x)}(\hat{\Omega}^1_{\mathcal{O}/A}/(du_1, \cdots, du_l))(x) = r(u_1, \cdots, u_l; x)$$

for all $x \in PN(\mathcal{O})$ and all elements $u_1, \cdots, u_l \in \mathcal{O}$. Moreover, define

$$E(X_j; u_1, \cdots, u_l; i) = \{x \in X_j | r(u_1, \cdots, u_l; x) \geqq i\}.$$

As before, $X_1, \cdots, X_r$ are the irreducible components of $X = PN(\mathcal{O})$. Lemma 7.2 applied to the module $F = \hat{\Omega}^1_{\mathcal{O}/A}/(du_1, \cdots, du_l)$ with $d = i$, gives that $E(X_j; u_1, \cdots, u_l; i)$ *is a closed subset of* X.

Step 1 of the outline in Section 6 amounts to proving the lemma below:

LEMMA (7.3). *There exists* $\mathbf{d}(\mathcal{O}/A) = \mathbf{d}$ *adequate elements in* $\mathfrak{m}_{\mathcal{O}}$ *such that* $\mathcal{O}$ *is finite over the subring* $\mathcal{Q}_1 = A[[u_1, \cdots, u_{\mathbf{d}}]]$ *and such that*

(7.3.1) $$\dim (E(X_j; u_1, \cdots, u_{\mathbf{d}}; i)) \leqq \max \{\dim (X_j)-i, -1\}$$

for all $j = 0, \cdots, r$ *and all* $i = 1, \cdots, \mathbf{d}$.

PROOF. Let $\mathbf{l} \leqq \mathbf{d}$ and $u_1, \cdots, u_{\mathbf{l}}$ be adequate elements in $\mathfrak{m}_{\mathcal{O}}$. Denote the following statement by $P(u_1, \cdots, u_{\mathbf{l}})$:

i) $\dim (\mathcal{O}/(m_A, u_1, \cdots, u_{\mathbf{l}})\mathcal{O}) \leqq \max \{\dim (\mathcal{O}_k)-\mathbf{l}, 0\}$

ii) Let $\mathbf{l} \leqslant d \leqslant \mathbf{d}$ and define $F(s; u_1, \cdots, u_{\mathbf{l}}; i) = \{x \in Y_s | r(u_1, \cdots, u_{\mathbf{l}}; x) \geqq d-\mathbf{l}+i\}$.

Then $\dim (F(s; u_1, \cdots, u_{\mathbf{l}}; i)) \leqq \max \{\dim (Y_s)-i, -1\}$ for all $i = 1, \cdots, \mathbf{l}$, all d such that $\mathbf{l} \leqq d \leqq \mathbf{d}$ and for all $s \in I(j, d)$ for which $r(y_s) = d$.

By Lemma 7.2, $F(s; u_1, \cdots, u_{\mathbf{l}}; i)$ is a closed subset of Y_s. We first show that it suffices to find adequate elements $u_1, \cdots, u_{\mathbf{d}}$ in $\mathfrak{m}_{\mathcal{O}}$ such that $P(u_1, \cdots, u_{\mathbf{l}})$ holds for all $\mathbf{l} \leqq \mathbf{d}$: these elements satisfy the claim of the lemma. In fact, assume that we have $u_1, \cdots, u_{\mathbf{d}}$ such that $P(u_1, \cdots, u_{\mathbf{l}})$ holds for all $\mathbf{l} \leqq \mathbf{d}$. First, it is clear that in order to prove (7.3.1), it suffices to show

(7.3.2) $$E(X_j; u_1, \cdots, u_{\mathbf{d}}; i) \subseteq \cup F(s; u_1, \cdots, u_{\mathbf{d}}; i)$$

where the union is taken over all $d \leqq \mathbf{d}$ *and* $s \in I(j, d)$ *for which* $r(y_s) = d$.

To show (7.3.2) let $x \in E(X_j; u_1, \cdots, u_{\mathbf{d}}; i)$. Then there exists d and $s \in I(j, d)$ such that $x \in Y_s$. Now we may first of all assume that $r(y_s) = d$: Let $d' = r(y_s)$. Since $d \leqq d'$, $X(j, d') \subseteq X(j, d)$. Thus since $\overline{\{y_s\}} \subseteq X(j, d')$ (Lemma 7.2), we conclude that Y_s is an irreducible component of $X(j, d')$. Replacing d by d', we get what we want. Now, $r(u_1, \cdots, u_{\mathbf{d}}; x) \leqq r(u_1, \cdots, u_d; x)$, so $r(u_1, \cdots, u_d; x) \geqq i = i+d-d$, i.e. $x \in F(s; u_1, \cdots, u_d; i)$.

For $\mathbf{l} \geqq \dim (\mathcal{O}_k)$, $P(u_1, \cdots, u_{\mathbf{l}})$ implies that $\mathcal{O}/(\mathfrak{m}_A, u_1, \cdots, u_{\mathbf{l}})\mathcal{O}$ is artinian. Hence so is $\mathcal{O}/\mathfrak{m}_{\mathcal{Q}_1}\mathcal{O}$, i.e. the $\mathcal{Q}_1$-module $\mathcal{O}$ is quasi-finite. Since $\mathcal{Q}_1$ is noetherian and complete, we conclude that $\mathcal{O}$ is a finite $\mathcal{Q}_1$-module (EGA O_I (7.4.3)).

To find $u_1, \cdots, u_{\mathbf{d}}$ as above, we proceed by induction on $\mathbf{l}$. For $\mathbf{l} = 1$, we want an adequate element $u_1 \in \mathfrak{m}_{\mathcal{O}}$ such that u_1 is not contained in any minimal prime of $\mathfrak{m}_A \mathcal{O}$ (so $P(u_1)$ i) holds), and such that $du_1(y_s) \neq 0$ for all j, d and s such that $r(y_s) = d$. This gives $r(u_1; y_s) = d-1$ and hence $y_s \notin F(s; u_1; 1)$. Thus ii) holds in $P(u_1)$.

The existence of such an adequate element u_1, while easy if k is infinite, is somewhat more complicated to prove if k is finite. We need the following lemma:

LEMMA (7.4). *Let* $S = \{x_1, \cdots, x_h\} \subseteq PN(\mathcal{O})$, *and* $V(x)$ *be a proper* $k(x)$*-subspace of* $\hat{\Omega}^1_{\mathcal{O}/A}(x)$ *for all* $x \in S$. *Then there exists an adequate element* $u \in \mathfrak{m}_{\mathcal{O}}$ *such that*

$$(7.4.1) \qquad u \notin \mathfrak{p}(x) \text{ and } du(x) \notin V(x) \text{ for all } x \in S.$$

REMARK. Moreover, if k is infinite, and if $P(X_1, \cdots, X_N)$ is a non-zero polynomial with coefficients from some field extension L of k, then $u = a_1 t_1 + \cdots + a_N t_N$ may be so chosen that $P(\bar{a}_1, \cdots, \bar{a}_N) \neq 0$, ($\bar{a}$ denotes the image of a in k).

PROOF. Assume first that k is infinite. Clearly, for all $i = 1, \cdots, h$ there exists a non-zero polynomial $P_i(X_1, \cdots, X_N) \in k(x_i)[X_1, \cdots, X_N]$ such that if $P_i(\bar{a}_1, \cdots, \bar{a}_N) \neq 0$, then $u = a_1 t_1 + \cdots + a_N t_N$ satisfies (7.4.1) at x_i. $\bar{a}_j$ denotes the image of a_j in $k(x_i)$. Hence it suffices to show that there exist $a_1, \cdots, a_N \in A$ such that

$$P_i(\bar{a}_1, \cdots, \bar{a}_N) \neq 0 \text{ for } i = 0, \cdots, h,$$

where $P_0 = P$. By induction it suffices to find $a_1 \in A$ such that

$$P_i(\bar{a}_1, X_2, \cdots, X_N), \; i = 0, \cdots, h$$

are non zero polynomials. The set of elements α in k (respectively, $k(x_i)$) such that $P_0(\alpha, X_2, \cdots, X_N)$ (respectively, $P_i(\alpha, X_2, \cdots, X_N)$) is the zero polynomial, is a finite set. Thus we need only to show that A is not contained in a finite union of subsets of the form $b + \mathfrak{p}$, where $\mathfrak{p}$ is a prime in $\mathcal{O}$ and $b \in A$. If this were so, then A would be contained in a finite union of subsets $b + \mathfrak{m}_{\mathcal{O}}$ where $b \in A$, and hence A would equal a finite union of subsets of the form $b + \mathfrak{m}_A$, a contradiction since $k = A/\mathfrak{m}_A$ is infinite.

Assume next that $A/\mathfrak{m}_A = k$ is finite of characteristic p. We proceed by induction on h. For $h = 1$, pick u such that $du(x_1) \notin V(x_1)$. If $u \notin \mathfrak{p}(x_1)$, we are done. If $u \in \mathfrak{p}(x_1)$, pick $t \in \mathfrak{m}_A \cup \{t_1, \cdots, t_N\}$ outside $\mathfrak{p}(x_1)$, and if $\mathfrak{m}_A \nsubseteq \mathfrak{p}(x_1)$, pick $t \in \mathfrak{m}_A$. Then $u + t^p$ satisfies (7.4.1): Indeed, $u + t^p \notin \mathfrak{p}(x_1)$ and if $\mathfrak{m}_A \subset \mathfrak{p}(x_1)$, then $k(x_1)$ is of characteristic p, so $du(x_1) = d(u+t^p)(x_1)$. If $\mathfrak{m}_A \nsubseteq \mathfrak{p}(x_1)$, then $t \in A$, so $du = d(u+t^p)$.

It now suffices to show the following lemma:

LEMMA (7.4.2). *Let* k *be finite of characteristic* p, *and assume that* $\mathfrak{p}(x_h) \nsupseteq \mathfrak{p}(x_i)$ *for all* $i < h$.

i) *Suppose that* u'' *satisfies* (7.4.1) *at* $x_1, \cdots, x_{h-1}$ *and that* $u'' \in \mathfrak{p}(x_h)$.

Then there exists an adequate element $t \in \mathfrak{m}_{\mathcal{O}}$ *and a finite set* J_1 *of integers, such that if* $m \notin J_1$, *then*

$$u' = u'' + t^m \notin \mathfrak{p}(x_h)$$

and satisfies (7.4.1) *at* $x_1, \cdots, x_{h-1}$.

ii) *Let* u' *be as in* i), *and assume that* u' *does not satisfy* (7.4.1) *at* x_h. *Then there exists an adequate element* $t \in \mathfrak{m}_{\mathcal{O}}$ *and a finite set* J_2 *of integers, such that if* $m \notin J_2$ *and* $p \nmid m$, *then*

$$u = (u')^m + t$$

satisfies (7.4.1) *at* $x_1, \cdots, x_h$.

PROOF. By assumption $\mathfrak{p}(x_h) \not\supseteq \bigcap_{i=1}^{h-1} \mathfrak{p}(x_i)$. Pick $t \notin \mathfrak{p}(x_h)$, $t \in \bigcap_{i=1}^{h-1} \mathfrak{p}(x_i)$. We first show i) for non-artiman A. It suffices to show that for all i there exists m_i such that

(7.4.3) $$u'' + t^m \notin \mathfrak{p}(x_i) \text{ for all } m \neq m_i,$$

since for all

$m > 1$ and $i < h$, $d(u'' + t^m)(x_i) = du''(x_i) + mt^{m-1}(x_i)dt(x_i) = du''(x_i)$.

To show (7.4.3), assume the converse for some i. Then

$$t^m(1 - t^r) \in \mathfrak{p}(x_i)$$

for some m and $r \geqq 1$. Hence $t \in \mathfrak{p}(x_i)$, i.e. $i \neq h$.

But since $u'' \notin \mathfrak{p}(x_i)$ for all $i \neq h$, we conclude that $u'' + t^m \notin \mathfrak{p}(x_i)$ for all m, a contradiction. Thus i) follows in the case that A is not artinian, cf. Definition 2.1.

Now assume that A is artinian. Pick $\varphi \in \{t_1, \cdots, t_N\}$ outside $\mathfrak{p}(x_h)$. Since $\mathfrak{m}_A$ is contained in all prime ideals of $\mathcal{O}$, $k(x_i)$ is of characteristic p for all i. Thus, since (7.4.3) depends only on $t \notin \mathfrak{p}(x_h)$, $t = \varphi^p$ gives what we want: $d(u'' + \varphi^{mp})(x_i) = du''(x_i)$ for all $i = 1, \cdots, h$.

To show ii), note first that

(7.4.4) $$m_1(u')^{m_1-1} - m_2(u')^{m_2-1} \notin \mathfrak{p}(x_i)$$

for all $i = 1, \cdots, h$, and all positive integers $m_1 > m_2$ where $p \nmid m_2$: The converse implies, since $u' \notin \mathfrak{p}(x_i)$, that

$$m_1(u')^{m_1-m_2} - m_2 \in p(x_i) \subset \mathfrak{m}_{\mathcal{O}}, \text{ a contradiction.}$$

Now pick $t = t_j$ such that $dt(x_h) \notin V(x_h)$. Since $du'(x_h) \in V(x_h)$ by assumption, we get $d((u')^m + t)(x_h) \notin V(x_h)$ for all m. Furthermore, if $i < h$, then there is at most one m_i not divisible by p such that $d((u')^{m_i} + t)(x_i) \in V_i$: Indeed, the converse together with (7.4.4) implies that

$du'(x_i) \in V_i$, a contradiction. Finally, since $(u')^m + t \in \mathfrak{p}(x_i)$ for at most one integer $m = m_i'$, ii) follows. Thus the proof of Lemma 7.4 is complete.

We return to the proof of Lemma 7.3. Since an element u_1 has now been produced such that $P(u_1)$ holds, we may assume that there exist $u_1, \cdots, u_l$ such that $P(u_1, \cdots, u_l)$ holds; and it remains to show that this implies the existence of an element u_{l+1} such that $P(u_1, \cdots, u_{l+1})$ holds. For this we need a modification of Théorème 2 in [4].

LEMMA (7.5). *Let F_s be a closed subset of Y_s for all $s \in I(j, d)$ with $j = 1, \cdots, r$ and $d = 1, \cdots, \omega(\mathcal{O}/A)$, and let $u_1, \cdots, u_l \in \mathfrak{m}_{\mathcal{O}}$ be such that $r(u_1, \cdots, u_l; x) = d-l$ for all $x \in Y_s - F_s$* with $d \geqq l$.

Then there exists an adequate element $u_{l+1} \in \mathfrak{m}_{\mathcal{O}}$ and for all $s \in I(j, d)$ with $j = 1, \cdots, r$ and $d \geqq l+1$ there exists a proper closed subset F_s' of Y_s such that

$$r(u_1, \cdots, u_{l+1}; x) = d-(l+1)$$

for all $x \in Y_s - (F_s \cup F_s')$.

REMARK (7.5.1). If, in the addition to the above, we are in the situation of Lemma 7.4, then u_{l+1} may be so chosen that the conclusion of Lemma 7.4 holds as well.

PROOF OF THE LEMMA AND THE REMARK. Let S be the subset in Lemma 7.4. Put

$$B = \{y_s | s \in I(j, d) \text{ with } d \geqq l+1 \text{ and } j = 1, \cdots, r\}.$$

For all $x \in B$, let $V(x)$ be the subspace of $\hat{\Omega}^1_{\mathcal{O}/A}(x)$ generated by $du_1(x), \cdots, du_l(x)$. This is a proper subspace since $d \geqq l+1$.

There is an adequate element $v \in \mathfrak{m}_{\mathcal{O}}$ which satisfies the conclusion of Lemma 7.4 with $S' = S \cup B$. Let

$$K_s = \{x \in Y_s | r(u_1, \cdots, u_l, v; x) \geqq d-l\},$$

which is closed by Lemma 7.2. Now let F_s' be the union of all those irreducible components of K_s which are not contained in F_s. We show that F_s' and $v = u_{l+1}$ satisfy the claims of the lemma and the remark.

First, clearly the conclusion of the remark holds. To show the claim of the lemma, we may assume that $F_s \neq Y_s$, since otherwise $F_s' = \emptyset$.

Let $x \in Y_s - (F_s \cup F_s') = Y_s - (F_s \cup K_s)$. Then $r(u_1, \cdots, u_l, v; x) \leqq d-(l+1)$, and equality holds because $r(u_1, \cdots, u_l; x) = d-l$ by the assumption. Thus it remains to show that $F_s' \neq Y_s$, i.e. that $y_s \notin K_s$.

Assume the converse. Then $r(u_1, \cdots, u_l, v; y_s) \geqq d-l$. Since $r(u_1, \cdots, u_l; y_s) = d-l$, this gives $r(u_1, \cdots, u_l, v; y_s) = d-l$, which contradicts $dv(y_s) \notin V(y_s)$ since

$$\hat{\Omega}^1_{\mathcal{O}/A}(y_s)/V(y_s) = (\hat{\Omega}^1_{\mathcal{O}/A}/(du_1, \cdots, du_l, dv))(y_s).$$

This completes the proof of Lemma 7.5 and Remark 7.5.1.

We return to the proof of Lemma 7.3: Let T_1 be the set of generic points of all irreducible components of $F(s; u_1, \cdots, u_l; i)$ which are of dimension equal to $\dim(Y_s)-i$, for all $j, d \geqq l+1, s \in I(j, d)$ with $r(y_s) = d$ and for all $i = 1, \cdots, l$.

Let $V(x)$ denote the subspace of $\hat{\Omega}^1_{\mathcal{O}/A}(x)$ generated by $du_1(x), \cdots, du_l(x)$ for all $x \in T_1$. Since $l < d$, this is a proper subspace.

If $l < \dim(\mathcal{O}_k)$, let T_2 be the set of points in $PN(\mathcal{O})$ which correspond to minimal primes of the ideal

$$(\mathfrak{m}_A, u_1, \cdots, u_l)\mathcal{O}.$$

If $l \geqq \dim(\mathcal{O}_k)$, let $T_2 = \emptyset$.

Further, for all $x \in T_2 - T_1$ we pick an arbitrary, proper subspace $V(x)$ of $\hat{\Omega}^1_{\mathcal{O}/A}(x)$, for example $V(x) = (0)$.

Now apply Lemma 7.5 and the remark to $u_1, \cdots, u_l$, $F_s = F(s; u_1, \cdots, u_l; 1)$ and $S = T_1 \cup T_2$. We get an adequate element $u_{l+1} \notin \mathfrak{p}(x)$ for all $x \in S$ such that

$$du_{l+1}(x) \notin V(x) \text{ for all } x \in S$$

and

$$r(u_1, \cdots, u_{l+1}; x) = d-(l+1)$$

for all $x \in Y_s - (F(s; u_1, \cdots, u_l; 1) \cup F'_s)$, where $\dim(F'_s) \leqq \dim(Y_s)-1$.

Then $P(u_1, \cdots, u_{l+1})$ holds: In fact, i) is immediate, and ii) follows for $i = 1$ since by the above

$$F(s; u_1, \cdots, u_{l+1}; 1) \subseteq F(s; u_1, \cdots, u_l; 1) \cup F'_s$$

and thus $\dim(F(s; u_1, \cdots, u_{l+1}; 1)) < \dim(Y_s)$ by the induction assumption.

For $l+1 \geqq i > 1$, we have

$$F(s; u_1, \cdots, u_{l+1}; i) \subseteq F(s; u_1, \cdots, u_l; i-1) \tag{7.6}$$

and the induction assumption gives

$$\dim(F(s; u_1, \cdots, u_{l+1}; i)) \leqq \max\{\dim(Y_s)-i+1, -1\}. \tag{7.7}$$

Thus the claim is trivial if $i \geqq \dim(Y_s)+2$. For $i < \dim(Y_s)+2$, assume that equality holds in (7.7). Then there is an irreducible component G of $F(s; u_1, \cdots, u_l; i-1)$ which is contained in $F(s; u_1, \cdots, u_{l+1}; i)$ and which is of dimension equal to $\dim(Y_s)-(i-1)$. Hence the generic point x of G is in T_1, which implies that

$$r(u_1, \cdots, u_{l+1}; x) = d-l+i-2. \tag{7.8}$$

Indeed, this follows by the choice of u_{l+1} once we notice that

$$(7.9) \qquad r(u_1, \cdots, u_l; x) = d-l+i-1.$$

To show this, assume the converse, i.e. ($x \in F(s; u_1, \cdots, u_l; i-1)$) that $r(u_1, \cdots, u_l; x) \geqq d-l+i$. Then $x \in F(s; u_1, \cdots, u_l; i)$, hence if $i \leqq l$, $\dim(G) \leqq \dim(Y_s)-i$ by the induction assumption, a contradiction. If, on the other hand, $i = l+1$, we argue as follows: The converse of (7.9) gives $r(u_1, \cdots, u_l; x) \geqq d-l+l+1 = d+1$. In particular this implies the existence of $d' \geqq d+1$ and $s' \in I(j, d')$ which satisfies

$$x \in Y_{s'} \subset Y_s$$

where the inclusion is proper since we assume $r(y_s) = d$. Now

$$x \in F(s'; u_1, \cdots, u_l; l),$$

hence the induction assumption gives $\dim(G) \leqq \dim(Y_{s'})-l \leqq \dim(Y_s)-(l+1) = \dim(Y_s)-i$, a contradiction.

Thus (7.9) – and hence (7.8) – follow. But (7.8) gives $x \notin F(s; u_1, \cdots, u_{l+1}; i)$ a contradiction.

This completes the proof of Lemma 7.3, and the first step in the proof of the theorem is completed.

8. Step 2. The critical subsets of a formal projection

We first list three lemmas, which will be proven in section 9. The following situation will remain fixed in lemmas 8.1, 8.2 and 8.3: $\mathcal{O}$ is finite over the subring $\mathcal{Q} = A[[u_1, \cdots, u_m]]$, where $u_1, \cdots, u_m$ are adequate elements in $\mathfrak{m}_{\mathcal{O}}$. As is easily seen, we then have

$$\mathcal{O} = \mathcal{Q}[t_1, \cdots, t_N].$$

Let $f: PN(\mathcal{O}) \to PN(\mathcal{Q})$ denote the morphism induced by the inclusion.

LEMMA (8.1). *The following are equivalent for a point $x \in PN(\mathcal{O})$ at which f is unramified*:

i) *f is an isomorphism at x.*
ii) *$k(f(x)) = k(x)$ and $f^{-1}(f(x)) = \{x\}$.*

LEMMA (8.2). *Let S be a finite set of points in $PN(\mathcal{O})$ such that*

(8.2.1) *$k(x)$ is a (finite) separable extension of $k(f(x))$ for all $x \in S$.*

Then there exist field extensions K_x of $k(x)$, and non-zero polynomials $F_x \in K_x[X_1, \cdots, X_N]$ for each $x \in S$ with the property that if $a_i \in \mathcal{Q}$ are such that

$$F_x(a_1(x), \cdots, a_N(x)) \neq 0 \quad \textit{for all } x \in S$$

then the element $u = a_1 t_1 + \cdots + a_N t_N$ *satisfies the following two conditions for all* $x \in S$:

(8.2.2) $u(x)$ *generate* $k(x)$ *over* $k(f(x))$.

(8.2.3) *If* $f' : PN(\mathcal{O}) \to PN(Q[u]) = PN(A[[u_1, \cdots, u_m, u]]$ *is the canonical morphism, then* $f'^{-1}(f'(x)) = \{x\}$.

LEMMA (8.3). *Let* S *be a finite subset of* $PN(\mathcal{O})$, *and let* K_x *be a field extensions of* $k(x)$ *and* $F_x \in K_x[X_1, \cdots, X_N]$ *be a non-zero polynomial for all* $x \in S$. *Put*

$$\mathcal{Q}_0 = A[u_1, \cdots, u_m].$$

Then there exist $a_1, \cdots, a_N \in \mathcal{Q}_0$ *such that for all* $x \in S$ *we have* $F_x(a_1(x), \cdots, a_N(x) \neq 0$. *Moreover, if* k *is infinite, then we may assume that* $a_1, \cdots, a_N \in A$.

To complete the proof of Theorem 6.1, we show the following, more general.

THEOREM (8.4). *For all* $0 \leqq h \leqq n = \dim(\mathcal{O})$, *there exist* $d+h$ *adequate elements* $u_1, \cdots, u_{d+h} \in \mathfrak{m}_{\mathcal{O}}$, *such that* $\mathcal{O}$ *is finite over the subring* $\mathcal{Q} = A[[u_1, \cdots, u_{d+h}]]$ *and such that the canonical morphism*

$$f : PN(\mathcal{Q}) \to PN(\mathcal{Q})$$

satisfies

$$\text{(8.4.1)} \qquad \dim(C(f) \cap X_j) \leqq \dim(X_j) - h$$

and

$$\text{(8.4.2)} \qquad \dim(E(X_j; u_1, \cdots, u_{d+h}; i)) \leqq \max\{-1, \dim(X_j) - i - h\}$$

for all j *and all* $i = 1, \cdots, d$.

REMARK. For $h = n$, Theorem 8.4 yields Theorem 6.1. But this result contains more information: The E-sets above are sometimes referred to as the *critical subsets* of the morphism f, and Theorem 8.4 for $h < n$ shows the existence of *formal projections* with *critical subsets of low dimension.*

Moreover, if $PN(\mathcal{O})$ is non singular, then the singular locus Sing $(PN(\mathcal{Q}))$ is contained in $f(C(f))$. In particular, then, Theorem 8.4 implies that dim (Sing$(PN(\mathcal{Q}))) \leqq n-h$.

PROOF OF THE THEOREM. We proceed by induction on h. For $h = 0$, the claim is just Lemma 7.3.

Now assume the theorem for $h-1$, and let $u_1, \cdots, u_{d+h-1}$ be adequate elements such that the conclusion holds.

Let $f_h : PN(\mathcal{O}) \to PN(A[[u_1, \cdots, u_{d+h-1}]])$ be the corresponding morphism.

Let S_1 be the set of all generic points of those components of $C(f_h) \cap X_j$ which are of dimension equal to $\dim (X_j)-h+1$, for $j = 1, \cdots, r$. Further, let S_2 be the generic points of $E(X_j; u_1, \cdots, u_{d+h-1}; i)$ for all j and $i = 1, \cdots, d$. For all $x \in S_2$, let $V(x)$ be the (proper) subspace of $\hat{\Omega}^1_{\mathcal{O}/A}(x)$ generated by $du_1(x), \cdots, du_{d+h-1}(x)$. By (8.4.1) and (8.4.2) of the induction assumption, we have

$$S_1 \cap E(X_j; u_1, \cdots, u_{d+h-1}; 1) = \emptyset \text{ for all } j = 1, \cdots, r.$$

It follows that f_h is *unramified* at all points of S_1: Indeed, we have the following lemma:

LEMMA (8.4.3). *With situation as in lemmas* 8.1, 8.2 *and* 8.3, *assume that* $du_1(x), \cdots, du_m(x)$ *generate* $\hat{\Omega}^1_{\mathcal{O}/A}(x)$ *for some* $x \in PN(\mathcal{O})$. *Then* f *is unramified at* x.

PROOF. $\hat{\Omega}^1_{\mathcal{O}/A}$ is a finite $\mathcal{O}$-module, so EGA O_{IV} (20.7.17) gives an exact sequence

$$\hat{\Omega}^1_{\mathcal{Q}/A} \otimes_{\mathcal{Q}} \mathcal{O} \xrightarrow{u} \hat{\Omega}^1_{\mathcal{O}/A} \to \hat{\Omega}^1_{\mathcal{O}/\mathcal{Q}} \to 0.$$

Since $\mathcal{O}$ is a finite $\mathcal{Q}$-module, $\hat{\Omega}^1_{\mathcal{O}/\mathcal{Q}} = \Omega^1_{\mathcal{O}}/_{\mathcal{Q}}$. Hence

$$(\hat{\Omega}_{\mathcal{Q}/A} \otimes_{\mathcal{Q}} \mathcal{O})(x) \xrightarrow{u(x)} \hat{\Omega}^1_{\mathcal{O}/A}(x) \to \Omega^1_{\mathcal{O}/\mathcal{Q}}(x) \to 0$$

is exact. But $u(x)$ is surjective by the assumption, so $\Omega^1_{\mathcal{O}/\mathcal{Q}}(x) = (0)$, and f is unramified at x.

It follows that the conclusion of Lemma 8.2 holds for $S = S_1$. Let K_x, F_x be the field extensions and polynomials, respectively.

Moreover, for all $x \in S_2$ there is a non-zero polynomial $G_x \in k(x)[X_1, \cdots, X_N]$, such that if $G_x(\alpha_1, \cdots, \alpha_N) \neq 0$, then

$$\alpha_1 dt(x) + \cdots + \alpha_N dt_N(x) \notin V(x).$$

By Lemma 8.3 there exist $a_1, \cdots, a_N \in \mathcal{Q}_0$, which we may even assume to be elements of A if k is infinite, such that

(8.4.4) $$F_x(a_1(x), \cdots, a_N(x)) \neq 0 \text{ for all } x \in S_1.$$

(8.4.5) $$G_x(a_1(x), \cdots, a_N(x)) \neq 0 \text{ for all } x \in S_2.$$

Put $u_{d+h} = a_1 t_1 + \cdots + a_N t_N$, and let

$$f_{h+1} : PN(\mathcal{O}) \to PN(A[[u_1, \cdots, u_{d+h}]])$$

be the canonical morphism.

Then $S_1 \cap C(f_{h+1}) = \emptyset$: Indeed, (8.4.4) implies (8.2.2) and (8.2.3) for $u = u_{d+h}$ and $f' = f_{h+1}$, respectively. Hence the claim follows by Lemma 8.1.

Moreover, Lemma 8.1 gives that

$$C(f_{h+1}) \subseteq C(f_h),$$

and (8.4.1) follows for f_{h+1}.

It remains to show (8.4.2). Note first that

$$E(X_j; u_1, \cdots, u_{d+h}; i) \subseteq E(X_j; u_1, \cdots, u_{d+h-1}; i).$$

Thus it suffices to show that if G is an irreducible component of $E(X_j; u_1, \cdots, u_{d+h-1}; i)$ of dimension equal to $\dim(X_j)-i$, then $G \nsubseteq E(X_j; u_1, \cdots, u_{d+h}; i)$. Assume the converse. Then the generic point y of G is in $E(X_j; u_1, \cdots, u_{d+h}; i)$. Moreover, $r(u_1, \cdots, u_{d+h-1}) \leqq i$: If otherwise $y \in E(u_1, \cdots, u_{d+h-1}; i+1)$ so by the induction assumption, $\dim(G) \leqq \dim(X_j)-i-1$, a contradiction. But by the choice of u_{d+h}, this gives $r(u_1, \cdots, u_{d+h}; y) \leqq i-1$, which contradicts

$$y \in E(X_j, u_1, \cdots, u_{d+h}; i).$$

Hence lemmas 8.1, 8.2 and 8.3 imply Theorem 8.4.

9. Proof of the lemmas

PROOF OF LEMMA 8.1. Clearly i) implies ii). For the converse, we show that ii) implies

i′) There exists $g \in \mathcal{Q}$, $g \notin \mathfrak{p}(x)$, such that $\mathcal{Q}_g = \mathcal{O}_g$.

This is enough, since obviously i) and i′) are equivalent.

For this, we first show that ii) implies

i″) $f_x : \mathcal{O}_{PN(\mathcal{Q}), f(x)} \to \mathcal{O}_{PN(\mathcal{O}), x}$ is bijective.

Let $\mathfrak{p} = \mathfrak{p}(x)$ and $\mathfrak{q} = \mathfrak{p} \cap \mathcal{Q}$. Since $f^{-1}(f(x)) = \{x\}$, it follows that $\mathcal{O}_\mathfrak{q} = \mathcal{O}_\mathfrak{p}$. Indeed, it suffices to show that $\{D(g) \subseteq \operatorname{Spec}(\mathcal{O}) | g \in Q, g \notin \mathfrak{q}\}$ is cofinal in the neighborhood system of x. So let U be an open subset of Spec $(\mathcal{O})$ containing x. Then $f(\operatorname{Spec}(\mathcal{O})-U) = F$ is a closed subset of Spec $(\mathcal{Q})$, and $f(x) \notin F$ since $f^{-1}(f(x)) = \{x\}$. Let $g \in \mathcal{Q}$, $g \notin \mathfrak{q}$ be such that $F \cap D(g) = \emptyset$. Then $f^{-1}(D(g)) \ni x$, and $f^{-1}(D(g)) \subseteq U$. Thus the claim follows. In particular, it follows that the canonical

$$f_x : \mathcal{O}_{PN(\mathcal{Q}), f(x)} = \mathcal{Q}_\mathfrak{q} \to \mathcal{O}_{PN(\mathcal{O}), x} = \mathcal{O}_\mathfrak{p} = \mathcal{O}_\mathfrak{q}$$

is injective and makes $\mathcal{O}_{PN(\mathcal{O}), x}$ to a finite $\mathcal{O}_{PN(\mathcal{Q}). f(x)}$ module. Since f is unramified at x and $k(f(x)) = k(x)$, this gives that $f_x \otimes_{\mathcal{O}_{f(x)}} k(f(x))$ is an isomorphism, and hence f_x is onto, by Nakayama's Lemma.

Note that we have not only shown i''), we know also that ii) implies

$$\mathcal{O}_{PN(\mathcal{O}),x} = \mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{q}}.$$

Thus i'') amounts to

i''') $$\mathcal{O}_{\mathfrak{q}} = \mathcal{Q}_{\mathfrak{q}}.$$

But clearly i''') implies i'): There are elements $s, n_j \in \mathcal{Q}$, $s \notin \mathfrak{q}$ such that $t_j/1 = n_j/s$ for all $j = 1, \cdots, N$ as elements of $\mathcal{O}_{\mathfrak{q}}$. This means that there exists $t \in \mathcal{Q}$, $t \notin \mathfrak{q}$, such that $t(st_j - n_j) = 0$ for all $j = 1, \cdots, N$ i.e., $t_j/1 = tn_j/st$, as elements of $\mathcal{O}_{st}$. Thus $g = st$ satisfies i').

This completes the proof of Lemma 8.1.

Proof of Lemma 8.2. Clearly we may assume $f(S) = \{y\}$. There is a finite, normal extension K of $L = k(y)$ such that for all $x \in T = f^{-1}(y)$ there exists an L-injection $k(x) \to K$. Denote the finite number of such L-injections by

$$i(x,j) : k(x) \to K, \quad j = 1, \cdots, j(x).$$

Let $h(x,j) : \mathcal{O} \to K$ be the composition of $i(x,j)$ with the canonical homomorphism $\mathcal{O} \to k(x)$. For all $x \neq z$ in T, define

$$W(x, z, j_1, j_2) = \{\lambda \in \mathfrak{m}_{\mathcal{O}} | h(x, j_1)(\lambda) = h(z, j_2)(\lambda)\}.$$

Then there exists $t_i \notin W(x, z, j_1, j_2)$: In fact, choose $\lambda \in \mathfrak{p}(x)$, $\lambda \notin \mathfrak{p}(z)$. Then $\lambda = H(t_1, \cdots, t_N)$, where $H \in \mathcal{Q}[X_1, \cdots, X_N]$. So if $\bar{H}$ denotes the polynomial over $k(y)$ obtained by reducing H modulo $p(y)$ we get

$$h(x, j_1)(\lambda) = \bar{H}(h(x, j_1)(t_1), \cdots, h(x, j_1)(t_N))$$
$$h(z, j_2)(\lambda) = \bar{H}(h(z, j_2)(t_1), \cdots, h(z, j_2)(t_N)),$$

and the claim follows since $h(x, j_1)(\lambda) \neq h(z, j_2)(\lambda)$.

Now

(9.1) $$\sigma \notin \bigcup_{\substack{x \neq z \\ \text{all } j_1 \text{ and } j_2}} W(x, z, j_1, j_2) \Rightarrow \sigma \text{ satisfies (8.2.3)}$$

Indeed, assume that σ is in no $W(x, z, j_1, j_2)$ for $x \neq z$. Then

(9.2) $h(x, j_1)(\sigma) = u$ is not conjugate to $h(z, j_2)(\sigma)$ over L for any $z \neq x$, and any $j_2 \leqq j(z)$.

Namely, assume that u and $u' = h(z, j_2)(\sigma)$ are conjugate, and let $v : K \to K$ be an L-automorphism of K such that $v(u) = u'$. Then $v \cdot i(x, j_1) : k(x) \to K$ maps the canonical image of σ in $k(x)$ to u'. Since $v \cdot i(x, j_1) = i(x, j_1')$ for some $j_1' \leqq j(x)$, we get $\sigma \in W(x, z, j_1', j_2')$, a contradiction. Thus (9.2) follows. Now let $g(Z)$ be the minimal polynomial of u over L. We may assume that the coefficients of $g(Z)$ are in

$\mathcal{Q}/\mathfrak{p}(y)$. By (9.2) $g(h(z, j_2)(\lambda)) \neq 0$ for all $z \neq x$, so if $G \in \mathcal{Q}[Z]$ corresponds to g by reduction modulo $\mathfrak{p}(y)$, then $G(\sigma) \in \mathfrak{p}(x)$, $G(\sigma) \notin \mathfrak{p}(z)$ for all $z \neq x$, and (9.1) follows.

Next, let $x \in S \subseteq f^{-1}(y)$. Let $\mathscr{W}$ denote the set of all $W(x, z, j_1, j_2)$ where $z \neq x$. The two homomorphisms defining $W \in \mathscr{W}$ are denoted by φ_W and θ_W. Put

$$\bar{F}_x(X_1, \cdots, X_N) = \prod_{W \in \mathscr{W}} \Big(\sum_{s=1}^{N} X_s(\varphi_W(t_s) - \theta_W(t_s))\Big)$$

By the above, this is a *non-zero* polynomial, and by (9.1), the polynomial $\bar{F}_x$ has the property of the lemma with respect to condition (8.2.3).

Finally, $\bar{t}_1, \cdots, \bar{t}_N$ generate $k(x)$ over $k(y) = L$, so we get a non-zero polynomial $G_x \in k(x)[X_1, \cdots, X_N]$, such that if $G_x(\alpha_1, \cdots, \alpha_N) \neq 0$, where $\alpha_1, \cdots, \alpha_N \in k(x)$, then $\alpha_1 \bar{t}_1 + \cdots + \alpha_N \bar{t}_N$ generates $k(x)$ over $k(y)$, cf. [5] page 85.

Now $F_x = \bar{F}_x \cdot G_x$ gives what we want.

Proof of Lemma 8.3. By induction it suffices to show that there exists an element $a_1 \in \mathcal{Q}_0$ (respectively, $a_1 \in A$ if k is infinite) such that all polynomials $F_x(\bar{a}_1, X_2, \cdots, X_N)$ are non-zero.

Since the set of all $\alpha \in K_x$ such that $F_x(\alpha, X_2, \cdots, X_N)$ is the zero polynomial is finite, it suffices to show that $\mathcal{Q}_0$ (respectively, A) is *not covered* by a finite number of sets $g + \mathfrak{p}$, where $g \in \mathcal{Q}_0$ (respectively, $g \in A$) and $\mathfrak{p}$ is a prime in $\mathcal{O}$, different from $\mathfrak{m}_{\mathcal{O}}$.

Assume first that A is covered by a finite number of such cosets. Then

$$A = g_1 + \mathfrak{m}_A \cup \cdots \cup g_h + \mathfrak{m}_A,$$

thus $k = A/\mathfrak{m}_A$ is finite, and the claim follows for k infinite.

Next, assume

$$\mathcal{Q}_0 \subseteq g_1 + \mathfrak{p}_1 \cup \cdots \cup g_h + \mathfrak{p}_h,$$

where $g_i \in \mathcal{Q}_0$ and $\mathfrak{p}_i$ are primes in $\mathcal{O}$ different from $\mathfrak{m}_{\mathcal{O}}$. Since $g_i + \mathfrak{p}_i$ is closed in $\mathcal{O}$ and $\mathcal{Q}_0$ is dense in $\mathcal{Q}$ and the $\mathfrak{m}_{\mathcal{Q}}$-adic topology on $\mathcal{Q}$ equals the topology induced from the $\mathfrak{m}_{\mathcal{O}}$-adic topology on $\mathcal{O}$, we get

$$\mathcal{Q} \subseteq g_1 + \mathfrak{p}_1 \cup \cdots \cup g_h + \mathfrak{p}_h$$

Let $\mathfrak{q}_i = \mathfrak{p}_i \cap \mathcal{Q}$. Since $g_i \in \mathcal{Q}$, we have

$$(g_i + \mathfrak{p}_i) \cap \mathcal{Q} = g_i + \mathfrak{q}_i.$$

In particular, it follows that

$$\mathfrak{m}_{\mathcal{O}} \subseteq g_1 + \mathfrak{q}_1 \cup \cdots \cup g_h + \mathfrak{q}_h.$$

Of course we may assume that the $g_i + \mathfrak{q}_i$ occurring here have at least one element in common with $\mathfrak{m}_{\mathscr{Q}}$, which implies that $g_i + \mathfrak{q}_i \subseteq \mathfrak{m}_{\mathscr{Q}}$, i.e. $g_i \in \mathfrak{m}_{\mathscr{Q}}$. Thus

$$\mathfrak{m}_{\mathscr{Q}} = g_1 + \mathfrak{q}_1 \cup \cdots \cup g_h + \mathfrak{q}_h.$$

Moreover, since $\mathscr{Q}$ is finite over $\mathscr{O}$, $\mathfrak{q}_i \neq \mathfrak{m}_{\mathscr{Q}}$ for all $i = 1, \cdots, h$. Hence

$$\mathfrak{m}_{\mathscr{Q}} \neq \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_h.$$

Pick $g \in \mathfrak{m}_{\mathscr{Q}}$ outside all $\mathfrak{q}_i$. Then $g^m - g_i \in \mathfrak{q}_i$ *for at most one integer* $m = m_i$: If not, then

$$g^m(1 - g^{m'-m}) \in \mathfrak{q}_i$$

so either $g \in \mathfrak{q}_i$ or $1 - g^{m'-m} \in \mathfrak{q}_i$, both of which are impossible. Thus for $m \gg 0$,

$$g^m \notin g_1 + \mathfrak{q}_1 \cup \cdots \cup g_h + \mathfrak{q}_h,$$

a contradiction.

This completes the proof of Lemma 10.

10. The non-equicharacteristic case

We consider the following situation: $\mathscr{O}$ is a noetherian, complete local ring which is an integral domain, and A is a Cohen subring. We may assume that A is not a field, otherwise we are in the situation of Proposition 2.2. Let $k = A/\mathfrak{m}_A$, and let K be the quotient field of A. Let

$$r = \dim(\mathscr{O}) + \max\{\omega(\mathscr{O}_k/k)+1, \omega(\hat{\mathscr{O}} \otimes_A K(K/K)\}.$$

THEOREM (10.1). *There exist r elements $u_1, \cdots, u_r$ in $\mathfrak{m}_{\mathscr{O}}$, such that if $\mathscr{Q} = A[[u_1, \cdots, u_r]]$, then the inclusion $\mathscr{Q} \hookrightarrow \mathscr{O}$ induces an isomorphism $PN(\mathscr{O}) \to PN(\mathscr{Q})$.*

If $PN(\mathscr{O} \otimes_A k)$ and $PN(\mathscr{O} \hat{\otimes}_A K)$ satisfy the conditions in Remark 1.2.1., then we may take $r = 2 \dim(\mathscr{O})$.

PROOF. Spec (A) has the generic point g and the special point s. $PN(\mathscr{O}) = PN(\mathscr{O})_s \cup PN(\mathscr{O})_g$. By Theorem 8.4 it suffices to show that for all $x \in PN(\mathscr{O}_s)$, $rk_{k(x)}(\hat{\Omega}^1_{\mathscr{O}/A}(x)) \leqq \omega(\mathscr{O}_k/k)+1$ and for all $x \in PN(\mathscr{O}_g)$, $rk_{k(x)}(\hat{\Omega}^1_{\mathscr{O}/A}(x)) \leqq \omega(\mathscr{O} \hat{\otimes}_A K/K)$.

The first of these inequalities is shown as follows: We get, in the same way as (2.2.1), the exact sequence

$$\mathfrak{m}_A/\mathfrak{m}_A\mathscr{O}^2 \to \hat{\Omega}^1_{\mathscr{O}/A} \otimes_{\mathscr{O}} \mathscr{O}_k \to \hat{\Omega}^1_{\mathscr{O}_k/k} \to 0 \tag{10.1.1}$$

which gives the first inequality since

$$rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = 1,$$

A being a discrete valuation ring of rank 1.

To show the second inequality, note first of all that

(10.1.2) $$\Omega^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} (\mathcal{O} \otimes_A K) = \Omega^1_{(\mathcal{O} \otimes_A K)/K}$$

cf. EGA IV (16.4.5), and

(10.1.3) $$\Omega^1_{B/K} \hat{\otimes}_B \hat{B} = \hat{\Omega}^1_{\hat{B}/K}$$

for any topological K-algebra B, see Lemma 5.2. Now (10.1.2) and (10.1.3) give

(10.1.4) $$\hat{\Omega}^1_{\mathcal{O}/A} \otimes_{\mathcal{O}} (\mathcal{O} \hat{\otimes}_A K) = \hat{\Omega}^1_{(\mathcal{O} \hat{\otimes}_A K)/K}$$

which immediately implies the claim. Thus the first part of the theorem follows.

To show the last part, we get as above for all $x \in PN(\mathcal{O})_s$

$$rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}/A}(x) \leqq rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}_k/k}(x) + 1.$$

By the assumption on $PN(\mathcal{O}_k)$, we thus have

$$rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}/A}(x) \leqq \dim(\mathcal{O}_k) + 1 = \dim(\mathcal{O}).$$

Now assume that $x \in PN(\mathcal{O})_g$. Then there is a point y in $PN(\mathcal{O} \otimes_A K)$ above x, and a point $z \in PN(\mathcal{O} \hat{\otimes}_A K)$ above y. By (10.1.4)

(10.1.5) $$\hat{\Omega}^1_{\mathcal{O} \hat{\otimes}_A K/K}(z) = \hat{\Omega}^1_{\mathcal{O}/A}(x) \otimes_{k(x)} k(z),$$

and

$$rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}/A}(x) = rk_{k(z)} \hat{\Omega}^1_{\mathcal{O} \hat{\otimes}_A K/K}(z) = \dim(\mathcal{O} \hat{\otimes}_A K) \leqq \dim(\mathcal{O}).$$

This completes the proof of the theorem.

11. The non-smooth case

If the projective A-scheme in Theorem 1.1 is non-smooth over A, then it may be embedded in P^s_A, where

$$s = \dim(X) + \max\{rk_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2) | x \in X_k\} + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2) + 1$$

and $X_k = X \otimes_A k$. The proof is almost the same as that of Theorem 1.1; we give a rapid outline below.

Instead of (3.5) one shows

(3.5)* $$rk_{k(x)} \hat{\Omega}^1_{\mathcal{O}_k/k}(x) \leqq \max\{rk_{k(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2) | y \in X_k\} + 1 = d(X)$$

for all $x \in PN(\mathcal{O})$. Here all notations are as before, in particular $X = \mathrm{Proj}(S)$ and $\mathcal{O} = \hat{S}$.

Furthermore, if there exists an open dense subset U of X, such that for all $x \in U$

$$rk_{k(x)}\,\Omega^1_{X/A}(x) = \dim(X) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2),$$

then we may take

$$s = \max\{2\dim(X)+1, \dim(X)+d(X)-1\} + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2).$$

In fact, note first that there exists a point y_j in each irreducible component X_j of $PN(\mathcal{O})$ such that

$$rk_{k(y)}\,\hat{\Omega}^1_{\mathcal{O}/A}(y_j) = \dim(\mathcal{O}) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2).$$

This is proven in the same way as (3.6).

Hence there is a non-empty open subset V of $PN(\mathcal{O})$, such that for all $y \in V$,

$$rk_{k(y)}\,\hat{\Omega}^1_{\mathcal{O}/A}(y) \leqq \dim(\mathcal{O}) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2)$$

(cf. Lemma 7.2). Using this, one can show that Lemma 7.3 holds with $\boldsymbol{d}$ replaced by $\boldsymbol{d}-1$ if $\boldsymbol{d} > \dim(\mathcal{O}) + rk_k(\mathfrak{m}_A/\mathfrak{m}_A^2)$.

The rest of the proof is the same as that of Theorem 1.1.

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University of Bergen
Department of Mathematics
Bergen, Norway