COMPOSITIO MATHEMATICA

C. M. PAREEK

The inverse image of a metric space under a biquotient compact mapping

Compositio Mathematica, tome 25, nº 3 (1972), p. 323-328
<http://www.numdam.org/item?id=CM_1972__25_3_323_0>

COMPOSITIO MATHEMATICA, Vol. 25, Fasc. 3, 1972, pag. 323–328
Wolters-Noordhoff Publishing
Printed in the Netherlands

THE INVERSE IMAGE OF A METRIC SPACE UNDER A BIQUOTIENT COMPACT MAPPING

by

C. M. Pareek

1. Introduction

In this note we define the notion of a P_1-space which is a generalization of the notion of paracompact p-space by Arhangel'skii [1], and prove that a regular space admits a biquotient compact mapping onto a metric space if and only if it is a P_1-space.

All maps are assumed continuous and onto. A regular space is also a T_1-space. For the definition of p-space, see [1]. The notation and terminology which is not defined here will follow that of [4].

2. Preliminaries

We shall need the following definitions:

A topological space X is called a P_1-space if there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the following conditions:

(a) $\mathscr{V}_{i+1}$ star refines $\mathscr{V}_i$ for each i;
(b) for each $x \in X$, $L(x) = \bigcap_{i=1}^{\infty} st(x, \mathscr{V}_i)$ is compact;
(c) for each $A \subset X$ and any $x \in X$, if $st(x, \mathscr{V}_i) \cap L(A) \neq \phi$ for each i, then $L(x) \cap cl(L(A)) \neq \phi$ where $L(A) = \cup \{L(x) | x \in A\}$.

The mapping $f: X \to Y$ is called:

quotient if the set $M \subset Y$ is closed if and only if $f^{-1}M$ is closed (this is equivalent to the condition: the set $M \subset Y$ is open if and only if $f^{-1}M$ is open);

pseudo-open (pre-closed or hereditarily quotient) if for an arbitrary neighborhood U of the inverse $f^{-1}y$ of an arbitrary point y from Y, the interior of the set fU contains the point.

open if images of open sets are open;

closed if images of closed sets are closed;

compact if the inverse image of any point is compact;

perfect if, it is simultaneously closed and compact;

biquotient if, for any point y in Y and any open cover $\mathscr{U}$ of $f^{-1}y$ there exists a finite number of members of $\mathscr{U}$ such that the point y is interior to the image of their union.

PROPOSITION 2.1. A topological space X is a P_1-space if and only if there exist a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the following conditions:

(a) $\mathscr{V}_{i+1}$ star refines $\mathscr{V}_i$ for each i;
(b) for each $x \in X$, $L(x) = \bigcap_{i=1}^{\infty} st(x, \mathscr{V}_i)$ is compact;
(c) for each $x \in X$ and any neighborhood U of $L(x)$, there is an i such that $st(x, \mathscr{V}_i) \subset L(U)$ where $L(U) = \cup\{L(x)|x \in U\}$.

PROOF. Let X be a P_1-space. Then there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the required conditions. Let x be a fixed but arbitrary point of X and let U be a neighborhood of $L(x)$. Suppose $st(x, \mathscr{V}_i) \cap (X-L(U)) \neq \phi$ for each i. Then by the hypothesis $L(x) \cap cl(L(X-L(U))) \neq \phi$. Since it is easy to see that $L(X-L(U)) = X-L(U)$, therefore $L(x) \cap cl(X-L(U)) \neq \phi$. But $L(x) \cap cl(X-L(U)) \neq \phi$ implies every neighborhood of $L(x)$ has a nonempty intersection with $X-L(U)$, which is not true as U is a neighborhood of $L(x)$ and $U \subset L(U)$. Hence for some i, $st(x, \mathscr{V}_i) \cap X-L(U) = \phi$, *i.e.*, $st(x, \mathscr{V}_i) \subset L(U)$.

Conversely, suppose there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying conditions (a), (b) and (c) of the hypothesis. Let $x \in X$ and let A be any subset of X such that $st(x, \mathscr{V}_i) \cap L(A) \neq \phi$ for each i. Suppose $L(x) \cap cl(L(A)) = \phi$. Then $X-cl(L(A))$ is an open neighborhood of $L(x)$. Yence by the hypothesis there exists an i such that $st(x, \mathscr{V}_i) \subset L(X-cl(L(A))) \subset L(X-L(A)) = X-L(A)$, a contradiction to the fact that $st(x, \mathscr{V}_i) \cap L(A) \neq \phi$ for each i. Hence if $st(x, \mathscr{V}_i) \cap L(A) \neq \phi$ for each i, then $L(x) \cap cl(L(A)) \neq \phi$. This proves the proposition.

The above proposition suggests the definition:

A topological space X is called a *p_1-space* if there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ open covers of X satisfying the following conditions

(a) $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;
(b) for each $x \in X$, $L(x) = \bigcap_{i=1}^{\infty} st(x, \mathscr{V}_i)$ is compact;
(c) for each $x \in X$ and every neighborhood U of $L(x)$, there is an i such that $st(x, \mathscr{V}_i) \subset L(U)$.

The following is an immediate consequence of Theorem 2.2 [3, p. 605].

PROPOSITION 2.2. *Every strict p-space is a p_1-space.*

3. Biquotient mappings

THEOREM 3.1. *Let f be a pseudo-open compact mapping of a regular space X onto a paracompact Hausdorff space Y. Then X is a paracompact space.*

PROOF. Let $\mathscr{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an open cover of a space X. For each $y \in Y$ choose a finite cover $U_1, \cdots, U_{n_y}$ of $f^{-1}y$ from $\mathscr{U}$ and set $O_y = \bigcup_{i=1}^{n_y} U_{y,i}$. Then $\mathscr{P} = \{P_y = \operatorname{int} fO_y | y \in Y\}$ is an open cover of Y. Since Y is paracompact, there exists a locally finite refinement $\mathscr{W} = \{W_y | y \in Y'\}$ of $\mathscr{P}$ where $Y' \subset Y$, $y \in W_y$ and $W_y \neq W'_y$ for distinct points y, y' of Y'. Let $\mathscr{R} = \{f^{-1}W_y \cap U_{y,i} | y \in Y$ and $i = 1, 2, \cdots, n_y\}$. Then it is easy to see that $\mathscr{R}$ is a locally finite open refinement of $\mathscr{U}$. Consequently, X is a paracompact space. Hence the theorem is proved.

PROPOSITION 3.2. *If $f : X \to Y$ is a compact mapping, then the following statements are equivalent*:

(i) *f is biquotient.*
(ii) *f is pseudo-open.*
(iii) *for each $M \subset Y$ and $y \in Y$, $y \in clM$ if and only if $f^{-1}y \cap clf^{-1}M \neq \phi$.*

PROOF. (i) $\Leftrightarrow$ (ii) This is trivial.
(ii) $\Leftrightarrow$(iii) See lemma 1.3 of [5]. One may note that the compactness of f is not needed to show that (ii) $\Leftrightarrow$ (iii).

If $f_\alpha : X_\alpha \to Y_\alpha$ is a mapping of X_α onto Y_α for all $\alpha \in \Lambda$, then the product map $f = \prod_{\alpha \in \Lambda} f_\alpha$ from $\prod_{\alpha \in \Lambda} X_\alpha$ to $\prod_{\alpha \in \Lambda} Y_\alpha$ is defined by $fx = (f_\alpha x_\alpha)_\alpha$.

PROPOSITION 3.3. (*E. Michael* [6]) *Any product* (*finite of infinite*) *of biquotient maps is a biquotient map.*

The following is the main theorem of this note.

THEOREM 3.4. *A topological space X is a P_1-space if and only if there exists a biquotient compact mapping f of X onto some metric space Y.*

We shall divide the proof of the theorem in two lemmas.

LEMMA 1. *If there exists a biquotient compact mapping f of a space X onto a metric space Y, then X is a P_1-space.*

PROOF. Let $f : X \to Y$ be a biquotient compact mapping of a space X onto a metric space Y. Since Y is a metric space, Y is a T_1-space and there exists a sequence $\{\mathscr{W}_i\}_{i=1}^\infty$ of open covers of Y such that (1) $\mathscr{W}_{i+1}$ star refines $\mathscr{W}_i$ for each i, and (2) for each $y \in Y$, $\{st(y, \mathscr{W}_i)\}_{i=1}^\infty$ is a base for the neighborhood system at y. Consider the sequence $\{\mathscr{V}_i\}_{i=1}^\infty$ of open covers of X, where $\mathscr{V}_i = f^{-1}\mathscr{W}_i$ for each i. Now it is easy to see that $\mathscr{V}_{i+1}$ star refines $\mathscr{V}_i$ for each i and $f^{-1}fx = \bigcap_{i=1}^\infty st(x, \mathscr{V}_i)$ for each x in X. Because $f^{-1}y$ is compact we have $\bigcap_{i=1}^\infty st(x, \mathscr{V}_i)$ is compact for each x in X. Let M be a subset of X and let x be a point in X such that $st(x, \mathscr{V}_i) \cap L(M) \neq \phi$ for each i. By the choice of the sequence

$\{\mathscr{V}_i\}_{i=1}^{\infty}$ of covers of X it is easy to see that $fx = z$ is a limit point of $fM = fL(M)$.

Now by proposition 3.2 we have $f^{-1}z \cap clf^{-1}fM \neq \phi$. But $f^{-1}z = \bigcap_{i=1}^{\infty} st(x, \mathscr{V}_i)$ and $clf^{-1}fM = clL(M)$; therefore $L(x) \cap clL(M) \neq \phi$. Consequently, $\{\mathscr{V}_i\}_{i=1}^{\infty}$ is the required sequence of open covers of X. Hence X is a P_1-space.

LEMMA 2. *If X is a P_1-space, then there exists a biquotient compact mapping of X onto some metric space Y.*

PROOF. Let X be a P_1-space and let $\{\mathscr{V}_i\}_{i=1}^{\infty}$ be a sequence of open covers of X satisfying the required conditions.

We shall denote by (X, τ) the topological space obtained from X by taking $\{st(x, \mathscr{V}_i)\}_{i=1}^{\infty}$ as a base for the neighborhood system at $x \in X$. Let $\mathbf{X}$ be the quotient space obtained from (X, τ) by defining two points x and y to be equivalent if and only if $y \in st(x, \mathscr{V}_i)$ for each $i = 1, 2, \cdots$. Let ϕ be the quotient map of (X, τ) onto $\mathbf{X}$. Let ψ be the identity map of X onto (X, τ). Then ψ is obviously continuous. Let us define $f = \phi \circ \psi$ and $Y = \mathbf{X}$. Then it is obvious that Y is a metric space and f is a continuous compact map. Now we need to show that f is a biquotient map. In view of proposition 3.2, we need only show that $y \in Y$ belongs to clM if and only if $f^{-1}y \cap clf^{-1}M \neq \phi$, where M is a subset of Y. Observe that for any subset M of $Y, f^{-1}M = L(f^{-1}M)$. Then $y \in cl M$ if and only if for some $z \in f^{-1}y$ we have $st(z, \mathscr{V}_i) \cap f^{-1}M \neq \phi$ for each i, i.e., $st(z, \mathscr{V}_i) \cap L(f^{-1}M) \neq \phi$ for each i. Since X is a P_1-space we have

$$L(z) \cap cl\, L(f^{-1}M) \neq \phi.$$

But $L(z) = f^{-1}y$ implies $f^{-1}y \cap clf^{-1}M \neq \phi$. Hence the lemma is proved.

The proof of the theorem 3.4 follows immediately from lemmas 1 and 2.

THEOREM 3.5. *Let f be a pseudo-open mapping of a topological space X onto a paracompact space Y satisfying the following condition*:

(i) *for each $y \in Y$ and C a closed subset of $f^{-1}y$, if $C \subset U$ where U is open in X, then there is a V open in X such that $C \subset V \subset cl\, V \subset U$. Then X is a normal space.*

PROOF. To prove that X is normal it is enough to show that every finite open cover of X has a locally finite closed refinement. Let $\mathscr{U} = \{U_i\}_{i=1}^{n}$ be a finite open cover of X. It follows easily from condition (i) that $f^{-1}y$ is normal for each y in Y. Hence $\mathscr{U}|_{f^{-1}y} = \{U_i \cap f^{-1}y\}_{i=1}^{n}$ has a closed refinement $\mathscr{F}_y = \{F_{y,i}\}_{i=1}^{n}$. Then by condition (i) we obtain an open cover $\mathscr{V}_y = \{V_{y,i}\}_{i=1}^n$ of $f^{-1}y$ such that $F_{y,i} \subset V_{y,i} \subset cl\, V_{y,i} \subset U_i$ for all i and $y \in Y$. Let $O_y = \cup\, (V_{y,i}|i = 1, 2, \cdots, n)$ and let $P_y = \operatorname{int} fO_y$ for $y \in Y$. Then $\mathscr{P} = \{P_y|y \in Y\}$ is an open cover of the paracompact space Y. Therefore there exists a locally finite open refinement $\mathscr{W}$ of $\mathscr{P}$. For each W in $\mathscr{W}$ we choose one point y in W and write $W = W_y$. The set of these y is denoted by $Y' \subset Y$, thus $\mathscr{W} = \{W_y|y \in Y'\}$. Let $\mathscr{R} = \{f^{-1}W_y \cap V_{y,i}|y \in Y'$ and $i = 1, \cdots, n\}$. It is easy to see that $\mathscr{R}$ is an open locally finite cover of X. Also, we have $f^{-1}W_y \cap V_{y,i} \subset cl\, V_{y,i} \subset U_i$ for each y and i. Therefore $S = \{cl(f^{-1}W_y \cap V_{y,i})|y \in Y'$ and $i = 1, \cdots, n\}$ is a locally finite closed refinement of $\mathscr{U}$. Hence the theorem is proved.

THEOREM 3.6. *If a completely regular p-space X is an inverse image of a metric space Y under an open finite-to-one mapping, then X is a metric space.*

PROOF. It follows immediately from [2], [3] and theorem 3.1.

REMARK 2. In [2] Arhangel'skii showed that the inverse image of a metric space under an open finite-to-one mapping need not be metrizable. In view of Arhangel'skii's results and the results of this note it is obvious that a regular P_1-space need not be a p-space in the sense of Arhangel'skii.

PROPOSITION 3.7. *Let X_i $(i = 1, 2, \cdots)$ be a P_1-space. Then the topological product of the spaces X_i $(i = 1, 2, \cdots)$ is a P_1-space.*

PROOF. From proposition 3.3 it is easy to conclude that any product (finite or infinite) of biquotient compact maps is a biquotient compact map. Now the proof follows immediately form theorem 3.4.

PROPOSITION 3.8. *Every regular P_1-space is a paracompact p_1-space.*

PROOF. That every P_1-space is a p_1-space follows immediately from the definition of P_1-space and proposition 2.1. That every P_1-space is paracompact follows from theorem 3.1 and theorem 3.4.

QUESTION. Is the converse of proposition 3.8 true?

REFERENCES

A. V. ARHANGEL'SKII

[1] Mappings and space, Russian Math. Surveys, 21 (1966), 115–162.

A. V. ARHANGEL'SKII

[2] A theorem on the metrizability of the inverse metric space under an open closed finite-to-one mapping. Example and unsolved problems, Soviet Math. Dokl., 7 (1966), 1258–1261.

D. K. BURKE and R. A. STOLTENBERG

[3] A note on p-space and Moore space, Pacific J. Math. 30 (1969), 601–608.

R. ENGELKING
[4] Outline of general topology, North-Holland, 1968.

V. V. FILIPPOV
[5] Quotient spaces and multiplicity of a base, Math. USSR Sbornik, 9 (1969), 487–496.

E. MICHAEL
[6] Biquotient maps and cartesian product of quotient maps, Extrait Des Ann. de l'Inst. Fourier de l'Uni. De Grenoble, 18 (1969), 287–302.

(Oblatum 11–III–1971 & 6–VI–1972)

Department of Mathematics
Kuwait University
Kuwait

This work was supported by a research associateship at the University of Saskatchewan, Regina, Canada.