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# Algebraic equivalence modulo rational equivalence on a cubic threefold

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# ALGEBRAIC EQUIVALENCE MODULO RATIONAL EQUIVALENCE ON A CUBIC THREEFOLD

by

J. P. Murre [1]

## Contents

Introduction . . . . . . . . . . 161
1. Preliminary results . . . . . . . . . . 163
2. The tangent bundle restricted to a line . . . . . . . . . . 173
3. The blow-up $X'$ of $V$ . . . . . . . . . . 179
4. The rational transformation $\phi : X' \to X$ . . . . . . . . . . 179
5. A fibration by conics and a double covering of $X$ . . . . . . . . . . 183
6. The birational transformation $\tilde{\phi} : X' \to \tilde{X}$ . . . . . . . . . . 187
7. The involutions on $X'$ and $\tilde{X}$ . . . . . . . . . . 191
8. Behaviour of some cycles on $X'$ under the involution . . . . . . . . . . 192
9. Relations between the curves $H$, $\mathscr{H}$, $\hat{H}$ and $\tilde{H}$ . . . . . . . . . . 194
10. Algebraic equivalence modulo rational equivalence . . . . . . . . . . 195
11. Algebraic families of cycles . . . . . . . . . . 202
References . . . . . . . . . . 205

## Introduction

Consider a non-singular variety $X$ of *dimension three and degree three* in projective four space, defined over a field $k$ with characteristic not two. The purpose of this paper is to study those 1-dimensional (algebraic) cycles on $X$ which are algebraically equivalent to zero.

Our main result (theorem 10.8 and corollary 10.10) is the following. Consider the group of *rational* equivalence classes of 1-dimensional cycles on $X$ in the sense of Chow [2]. Consider in this group the subgroup of those classes which are algebraically equivalent to zero. Then this subgroup is isomorphic to the direct product of an abelian variety and a group consisting of elements of order two. Moreover the abelian variety is a so-called *Prym variety*. The construction of this Prym variety and the mapping of the cycle classes is not rational over the ground field; it is rational over an extension field of this as soon as this is a field of definition of a sufficient-

[1] This research was partially done during the spring of 1971 at the University of Warwick. The author thanks the Mathematics Institute of the University of Warwick for the support and for the stimulating atmosphere.

ly general line on $X$ (see 2.1).[2]

The theory of Prym varieties has been developed by Mumford [5]. Such a Prym variety is always obtained from a system of two curves, one of which is an étale covering of the other. In our case we obtain these curves as follows. Fix a sufficiently general line $l$ on $X$ and consider on the Fano surface of lines on $X$ the set consisting of those lines which meet $l$. This is a curve and this curve has a natural involution without fixed elements, obtained by considering three coplanar lines $l$, $l'$ and $l''$. This curve together with its involution determines the above mentioned Prym variety.

As to the torsion group, entering in our main result, we do not have any further information about it, except that its elements are of order two.

The main motivation for our investigation was Mumford's recent result that the intermediate Jacobian of the cubic threefold is a Prym variety ([6], see also appendix C of [3]). Our proposition 10.6, which is a key result in this paper, together with the arguments used in the proof of 10.8, give – in our opinion – a reasonable geometric insight in this fact. In order to avoid misunderstanding we explicitly mention that in our algebraic approach the intermediate Jacobian does not actually enter into the discussion.

Some remarks on the various sections. For convenience of the reader we have collected in section 1 those properties of the lines on $X$ which we need in this paper, stated in the form in which they are used. Next we fix a sufficiently general (see 1.25) line $l$ on $X$; we enlarge the groundfield $k$ such that $l$ is defined over $k$. After this we construct a 'covering' $X'$ of $X$, 2 to 1 over $X$ (but with exceptions). Following a suggestion of Clemens this covering is constructed in section 3 via the tangent bundle to $X$ along $l$ (see also footnote 2 on page 14). To be precise, $X'$ is a blow up of the projective bundle $V$ associated with the above tangent bundle. We blow up along a curve and in section 2 we show that this curve, and hence also $X'$, is non-singular. Section 4 studies the morphism $X' \to X$. Again using the line $l$ and a 'fibration' of $X$ by means of conics we construct in section 5 another 2 to 1 covering (with exceptions) $\tilde{X}$ of $X$. This $\tilde{X}$ is intuitively simpler than $X'$ but technically more complicated because $\tilde{X}$ has singularities. It turns out (section 6) that $X'$ is a desingularization of $\tilde{X}$. The varieties $X'$ and $\tilde{X}$ are rational varieties. On these 2 to 1 coverings $X'$, or $\tilde{X}$, of $X$ we have in a natural way an involution over $X$. This involution plays a crucial part in the paper (section 7–10), es-

[2] After completion of the paper I learnt from Manin about a recent paper of Šermenev [12] which is closely related with this paper. Using the theory of motifs (Grotendieck, Manin) Šermenev determined the motif of a cubic threefold. The motif essentially determines the Chow ring tensored with $Q$, i.e., determines the Chow ring up to torsion.

pecially the behaviour of the exceptional locus of $X' \to V$ with respect to this involution is of importance. Finally section 11, entirely technical, is the proof of proposition 10.5 which says that the mapping of the 1-cycles of $X$ to the Prym variety is 'algebraic'. As mentioned above, the main results of the paper are 10.8 and 10.10.

We often use specializations of points or cycles in the sense of Weil (see [10]); if there is no field mentioned for such a specialization then we always mean specialization over the groundfield $k$. In the Chow ring $A(X)$, etc., we mean to take the cycles rational over a fixed sufficiently large and algebraically closed overfield of $k$ (a 'universal domain').

The problem of investigating the Chow ring of the cubic threefold was suggested by Mumford. I like to thank Mumford and Clemens for valuable help and Griffiths for stimulating conversations on the topic.

## 1. Preliminary results

1.1. Let $X$ be a *non-singular cubic threefold* in projective 4-space $\boldsymbol{P}^4$, defined over a field $k$ of characteristic not 2. Note that $X$ is absolutely irreducible because it is a non-singular 3-dimensional variety in $\boldsymbol{P}^4$. Let

$$\mathscr{F} = \{l;\, l \text{ a line on } X\}.$$

The following result is classical ([1], lemma 3; [3], thm. 7.8): *$\mathscr{F}$ is a non-singular, absolutely irreducible projective surface defined over $k$, the so-called Fano surface.*

*Section 1A. Linear 2-spaces going through a line on X*

1.2. Let $(x, y, z, u, v)$ be projective coordinates in $\boldsymbol{P}^4$ and let

$$F(x, y, z, u, v) = 0$$

be the equation for $X$. Let $l$ be a line in $\boldsymbol{P}^4$ not meeting the linear space $\{u = 0,\ v = 0\}$. Let $(x', y', z', 1, 0)$ be the point $l \cap \{v = 0\}$, resp. $(x'', y'', z'', 0, 1)$ the point $l \cap \{u = 0\}$. Then $(x', y', z', x'', y'', z'')$ can be used as local coordinates on the Grassmannian of lines in $\boldsymbol{P}^4$ (cf. [1], p. 4). An arbitrary point $P \in l$ has coordinates:

$$(1) \qquad \begin{aligned} x &= ux' + vx'' \\ y &= uy' + vy'' \\ z &= uz' + vz'' \\ u &= u \\ v &= v. \end{aligned}$$

The line $l$ is on $X$ if and only if identically in $u$ and $v$

$$F(ux'+vx'', uy'+vy'', uz'+vz'', u, v) = 0.$$

From the coefficients of $u^3$, $u^2v$, $uv^2$ and $v^3$ we get 4 equations:

$$\phi_1(x', y', z', x'', y'', z'') = 0, \phi_2(\cdots) = 0, \phi_3(\cdots) = 0, \phi_4(\cdots) = 0,$$

which are necessary and sufficient conditions for $l$ in order to be on $X$.

1.3. Let $l_0$, given by $(x_0', y_0', z_0', x_0'', y_0'', z_0'')$, be a line on $X$. When does there exist a linear 2-dimensional space (shortly: 2-plane) $L_0^2$ such that

$$X \cdot L_0 = 2l_0 + l' \,?$$

If $T_0 = (\xi, \eta, \zeta, 0, 0)$ is the point of $L_0 \cap \{u = 0, v = 0\}$, then $L_0$ is spanned by $l_0$ and $T_0$ (we write $L_0 = \operatorname{span}(l_0, T_0)$). Therefore a point $Q \in L_0$ is given symbolically by:

$$Q = P + t(\xi, \eta, \zeta, 0, 0),$$

where $P \in l_0$ and where $u$, $v$ (hidden in $P$) and $t$ are projective coordinates in $L_0$. Substituting in $F = 0$ we get $Q \in X$ if

$$F(Q) = F(P) + t\left\{\xi \frac{\partial F}{\partial x}(P) + \eta \frac{\partial F}{\partial y}(P) + \zeta \frac{\partial F}{\partial z}(P)\right\}$$
$$+ t^2\{\cdots\} + t^3\{\cdots\} = 0.$$

Since $F(P) = 0$, the intersection $X \cap L_0$ consists of $l_0$, given by $t = 0$, and a conic given by

$$\xi \frac{\partial F}{\partial x}(P) + \eta \frac{\partial F}{\partial y}(P) + \zeta \frac{\partial F}{\partial z}(P) + t\{\cdots\} + t^2\{\cdots\} = 0. \tag{2}$$

Therefore $L_0 \cdot X = 2l_0 + l_0'$ if and only if

$$\xi \frac{\partial F}{\partial x}(P) + \eta \frac{\partial F}{\partial y}(P) + \zeta \frac{\partial F}{\partial z}(P) \equiv 0 \text{ identically in } u \text{ and } v. \tag{3}$$

Writing

$$\frac{\partial F}{\partial x}(P) = u^2\psi_{11}(x_0', \cdots z_0'') + uv\psi_{12}(x_0' \cdots z_0'') + v^2\psi_{13}(\cdots)$$

etc., we get that there exists a 2-plane $L_0$ such that $L_0 \cdot X = 2l_0 + l_0'$ if and only if

$$\det \begin{pmatrix} \psi_{11} & \psi_{21} & \psi_{31} \\ \psi_{12} & \psi_{22} & \psi_{32} \\ \psi_{13} & \psi_{23} & \psi_{33} \end{pmatrix} = \psi(x_0', \cdots, z_0'') = 0. \tag{4}$$

So far we have neglected the lines meeting the 2-plane $\{u = 0, v = 0\}$. These lines are in a Zariski-closed subset of $\mathscr{F}$; i.e., we have worked Zariski-locally on $\mathscr{F}$. Summarizing we have:

LEMMA (1.4). *Put* $\mathscr{F}_0 = \{l; l \in F \text{ s.t. } \exists L^2 \textit{ with } L \cdot X = 2l + l'\}$. *Then* $\mathscr{F}_0$ *is a Zariski closed set on* $\mathscr{F}$. *Moreover locally on* $\mathscr{F}$ *the set* $\mathscr{F}_0$ *is given by one equation (namely* $\psi = 0$).

1.5. The following lemma, which was communicated to me by Mumford, gives more precise information on $\mathscr{F}_0$ (cf. also [3], 7.6):

LEMMA (1.5). $\dim \mathscr{F}_0 \leqq 1$ (*and hence* $= 1$ *by* 1.4).

PROOF. Let $l_0 \in \mathscr{F}$. Enlarging, if necessary, the groundfield we can assume that $l_0$ is given by

$$x = 0,\ y = 0,\ z = 0. \tag{5}$$

Using the notations of 1.2 we have that $x', y', \cdots, z''$ are the (local) coordinates on the Grassmannian $G(2, 5)$; $l_0$ is given on $G(2.5)$ by

$$x' = 0, y' = 0, \cdots, z'' = 0.$$

For $X$ we have now an equation

$$F(x, y, z, u, v) = xf(x, \cdots, v) + yg(\cdots) + zh(\cdots) \tag{6}$$

with $f$, $g$ and $h$ quadratic. Put

$$\begin{aligned} f(x, \cdots, v) &= \lambda u^2 + \lambda' uv + \lambda'' v^2 + \cdots \\ g(x, \cdots, v) &= \mu u^2 + \mu' uv + \mu'' v^2 + \cdots \\ h(x, \cdots, v) &= \nu u^2 + \nu' uv + \nu'' v^2 + \cdots . \end{aligned} \tag{7}$$

REMARK 1.6. The rank of the matrix

$$\begin{pmatrix} \lambda & \lambda' & \lambda'' \\ \mu & \mu' & \mu'' \\ \nu & \nu' & \nu'' \end{pmatrix} \tag{8}$$

is always at least 2, because

$$f(0, 0, 0, u, v) = 0,\ g(0, 0, 0, u, v) = 0,\ h(0, 0, 0, u, v) = 0$$

do not have a common zero. For, in a point $P = (0, 0, 0, u, v)$, we have

$$\frac{\partial F}{\partial x}(P) = f(0, 0, 0, u, v),\ \frac{\partial F}{\partial y}(P) = g(0, 0, 0, u, v),$$

$$\frac{\partial F}{\partial z}(P) = h(0, 0, 0, u, v),\ \frac{\partial F}{\partial u}(P) = \frac{\partial F}{\partial v}(P) = 0,$$

and $X$ is non-singular. Clearly if the rank of (8) is at most one then we can find $u_0$, $v_0$ such that $f(0, 0, 0, u_0, v_0) = g(0, 0, 0, u_0, v_0) = h(0, 0, 0, u_0, v_0) = 0$.

1.7. Returning to the proof of 1.5, let $l_0 \in \mathscr{F}$ be fixed as above and $l$ a variable line in $\boldsymbol{P}^4$. Substituting (1) in (6) and evaluating the coefficients of $u^3$, $u^2v$, $uv^2$ and $v^3$ we get the equations $\phi_1 = 0, \cdots, \phi_4 = 0$ of 1.2, i.e. the equations of $\mathscr{F}$:

$$
(9) \quad
\begin{aligned}
\phi_1(x', \cdots, z'') &= \lambda x' + \mu y' + \nu z' + \cdots \text{higher terms} \cdots = 0 \\
\phi_2(x', \cdots, z'') &= \lambda' x' + \lambda x'' + \mu' y' + \mu y'' + \nu' z' + \nu z'' + \cdots \text{higher terms} \cdots = 0 \\
\phi_3(x', \cdots, z'') &= \lambda'' x' + \lambda' x'' + \mu'' y' + \mu' y'' + \nu'' z' + \nu' z'' + \cdots \text{higher terms} \cdots = 0 \\
\phi_4(x', \cdots, z'') &= \lambda'' x'' + \mu'' y'' + \nu'' z'' + \cdots \text{higher terms} \cdots = 0
\end{aligned}
$$

The tangent space to $\mathscr{F}$ at $l_0$ is given by the linear terms in $x', \cdots, z''$ of (9). In order to determine its dimension we have to consider the rank of the corresponding 4 by 6 matrix. Now consider, for instance

$$
(10) \det \begin{pmatrix} \mu & 0 & \nu & 0 \\ \mu' & \mu & \nu' & \nu \\ \mu'' & \mu' & \nu'' & \nu' \\ 0 & \mu'' & 0 & \nu'' \end{pmatrix} = (\mu\nu'' - \mu''\nu)^2 - (\mu\nu' - \mu'\nu)(\mu'\nu'' - \mu''\nu').
$$

This is the resultant of $g(0, 0, 0, u, v) = 0$ and $h(0, 0, 0, u, v) = 0$. Therefore if $f$, $g$ and $h$ do not have 2 by 2 a common zero on $l_0$ then the rank of the matrix in (9) is 4. Otherwise we can assume that $(u, v) = (1, 0)$ is the common zero of $f$ and $g$ on $l_0$, similarly $(0, 1)$ for $g$ and $h$. Then $\lambda = \mu = \mu'' = \nu'' = 0$ and it is easily checked (as in 1.6) that $\nu \neq 0$, $\lambda'' \neq 0$ and $\mu' \neq 0$. Then the 1st, 2nd, 3rd and 5th column are independent and the rank is again 4. Hence $\mathscr{F}$ is a non-singular surface.

1.8. Let $T_0 = (\xi, \eta, \zeta, 0, 0)$ and $L_0 = \text{span}\,(l_0, T_0)$. We get for the equation (2) of 1.3:

$$
\xi f(0, 0, 0, u, v) + \eta g(0, 0, 0, u, v) + \zeta h(0, 0, 0, u, v) + \cdots \text{terms with } t \cdots = 0.
$$

Therefore by (3) of 1.3 we have $l_0 \in \mathscr{F}_0$ if and only if we have identically in $u$ and $v$:

$$(11) \qquad \xi f(0,0,0,u,v)+\eta g(0,0,0,u,v)+\zeta h(0,0,0,u,v) = 0,$$

i.e. $l_0 \in \mathscr{F}_0$ if and only if the rank of (8) is 2.

Let now $l_0 \in \mathscr{F}_0$; then we can assume after a change of variables $x, y$ and $z$ that $f(0,0,0,u,v) = 0$. Since now $\lambda = \lambda' = \lambda'' = 0$ we have that the tangent space to $\mathscr{F}$ at $l_0$ is given by

$$(12) \qquad y' = 0,\ y'' = 0,\ z' = 0,\ z'' = 0.$$

Moreover *we have now that the determinant in* (10) *is different from zero*. Also the equation (6) simplifies to

$$(13) \qquad F(x,y,z,u,v) = x^2 l(u,v)+yg(x,\cdots,v)+zh(x,\cdots,v) = 0,$$

with $l(u,v)$ linear, $g$ and $h$ quadratic.

Finally we want to make explicite the equation $\psi = 0$ of (4), defining the set $\mathscr{F}_0$ in $\mathscr{F}$. We use the notation of 1.3, except that we take a 'variable' line $l \in \mathscr{F}$. Substituting the coordinates (1) of $P \in l$ in (13) we get that $l \in \mathscr{F}_0$ if and only if (3) is satisfied.

In our case we get:

$$\frac{\partial F}{\partial x}(P) = 2(ux'+vx'')l(u,v)+(uy'+vy'')\frac{\partial g}{\partial x}(ux'+vx'',\cdots) + (uz'+vz'')\frac{\partial h}{\partial x}(ux'+vx'',\cdots)$$

$$\frac{\partial F}{\partial y}(P) = g(ux'+vx'',\cdots,v)+(uy'+vy'')\frac{\partial g}{\partial y}(\cdots)+(uz'+vz'')\frac{\partial h}{\partial y}(\cdots)$$

$$\frac{\partial F}{\partial z}(P) = (uy'+vy'')\frac{\partial g}{\partial z}(ux'+vx'',\cdots)+h(\cdots)+(uz'+vz'')\frac{\partial h}{\partial z}(\cdots).$$

By (3) we have to compute the coefficients of $u^2$, $uv$ and $v^2$. This is rather involved, *however we are only interested in $\psi$ in a neighborhood* of $l_0$. Therefore, using (12) it suffices to consider

$$\bar{\psi} = \psi \bmod (y', y'', z', z'', x'^2, x''^2, x'x'').$$

This gives

$$\bar{\psi} = \det \begin{pmatrix} 2x'l(1,0) & \mu & \nu \\ 2x'l(0,1)+2x''l(1,0) & \mu' & \nu' \\ 2x''l(0,1) & \mu'' & \nu'' \end{pmatrix}.$$

Therefore

$$\bar{\psi} = 2x'\{l(1,0)(\mu'\nu''-\mu''\nu')-l(0,1)(\mu\nu''-\mu''\nu)\}+2x''\{l(0,1)(\mu\nu'-\mu'\nu) - l(1,0)(\mu\nu''-\mu''\nu)\}.$$

It follows from (10) and the first line following equation (12) that *not* both coefficients in $\bar{\psi}$ are zero unless $l \equiv 0$, i.e. unless $X$ contains the linear space $\{y = 0, z = 0\}$; but this is impossible (see lemma 1.17 below). Since the tangent space to $\mathscr{F}_0$ at $l_0$ is given by (cf. (12)):

$$y' = y'' = z' = z'' = 0 \text{ and } \bar{\psi} = 0,$$

we have that the dimension of this tangent space is one. Hence the dimension of $\mathscr{F}_0$ is at most (but by 1.4 also at least) one. *This completes the proof of* 1.5.

In fact we have proved:

COROLLARY (1.9). $\mathscr{F}_0$ is a non-singular curve (not necessarily connected).

1.10. Let $l_0 \in \mathscr{F}_0$. If $T_0 = (\xi, \eta, \zeta, 0, 0)$ and $L_0 = \text{span}\,(l_0, T_0)$ then

$$L_0 \cdot X = 2l_0 + l_0'$$

if $T_0$ satisfies (11), i.e. the linear equations determined by the matrix in (7). By 1.6 the point $T_0$, and hence $L_0$ and $l_0'$, is unique. From this we have

LEMMA (1.11). *Let* $\mathscr{F}_0' = \{m; m \in \mathscr{F}$ *such that* $\exists L^2$ *with* $X \cdot L = m + 2m'$, $m' \in \mathscr{F}\}$.

*Then* $\mathscr{F}_0'$ *is Zariski closed on* $\mathscr{F}$ *and of dimension at most one.*

1.12. Let $l$ be a line on $X$. Assume for simplicity that $l$ is defined over $k$ (otherwise we have to work over an extension of $k$); we can arrange then that $l$ has equations

$$x = 0,\ y = 0,\ z = 0. \tag{14}$$

We can write then for the equation of $X$:

$$\begin{aligned} F(x, y, z, u, v) = {} & u^2 l_1(x, y, z) + 2uvl_2(\cdots) + v^2 l_3(\cdots) \\ & + 2uQ_1(x, y, z) + 2vQ_2(\cdots) + C(x, y, z) = 0 \end{aligned} \tag{15}$$

with:

$l_i(x, y, z)$ homogeneous linear, $i = 1, 2, 3$,

$Q_i(x, y, z)$ homogeneous quadratic, $i = 1, 2$,

$C(x, y, z)$ homogeneous cubic.

DEFINITION (1.13). (cf. [3], 6.6): $l$ is of the *first type* (resp. *second type*) if $l_i(x, y, z)$, $i = 1, 2, 3$, are linearly independent (resp. linearly dependent) over $k$.

LEMMA (1.14). *$l$ is of the second type* $\Leftrightarrow l \in \mathscr{F}_0$.

PROOF. Comparing the equations (6), (7) and (15) we see that

$$\begin{aligned} l_1(x, y, z) &= \lambda x + \mu y + \nu z \\ \tfrac{1}{2} l_2(x, y, z) &= \lambda' x + \mu' y + \nu' z \\ l_3(x, y, z) &= \lambda'' x + \mu'' y + \nu'' z \end{aligned}$$

Therefore $l$ is of the second type $\Leftrightarrow$ determinant of (8) is zero $\Leftrightarrow$ $l \in \mathscr{F}_0$ by (11).

1.15. In case $l$ is of type 1 we can make a change of variables, within the $x$, $y$ and $z$, in order to simplify equation (15) to

$$F(x, y, z, u, v) = u^2 x + 2uvy + v^2 z + 2uQ_1(x, y, z) + 2vQ_2(x, y, z) + C(x, y, z) = 0 \tag{16}$$

with $Q_i(x, y, z)(i = 1, 2)$ quadratic and $C(x, y, z)$ cubic (all homogeneous).

The tangent space to $X$ in $P = (0, 0, 0, u_0, v_0)$ is given by

$$u_0^2 x + 2u_0 v_0 y + v_0^2 z = 0. \tag{17}$$

It is clear that there is no point common to all hyperplanes of type 17) except the points of $l$. Therefore:

LEMMA (1.16). *If $l$ is a line of the first type then there is no 2-plane tangent to $X$ in all points of $l$.*

*Section 1B: The lines going through a fixed point.*

LEMMA (1.17). *$X$ does not contain a linear 2-space.*

PROOF. If $L^2$ defined by $x = 0$, $y = 0$ is on $X$ then every term in the equation $F = 0$ of $X$ contains $x$ or (and) $y$. Therefore in $P = (0, 0, z, u, v)$ we have

$$\frac{\partial F}{\partial z}(P) = \frac{\partial F}{\partial u}(P) = \frac{\partial F}{\partial v}(P) = 0.$$

Therefore if $P$ is in the intersection of $\partial F/\partial x = 0$, $\partial F/\partial y = 0$ and $L$ then $P$ is a singular point.

LEMMA (1.18). *If $l$ is a line of the first type on $X$ and $P \in l$ then there are only finitely many (and in fact at most 6) lines on $X$ through $P$.*

PROOF. Let $P = (0, 0, 0, 0, 1)$. Without loss of generality we can assume

$$F(x, y, z, u, v) = v^2 z + vG_2(x, y, z, u) + G_3(x, y, z, u) = 0 \tag{18}$$

with $G_2$ (resp. $G_3$) quadratic (resp. cubic) homogeneous. The lines through $P$ are given by

$$z = 0, G_2 = 0, G_3 = 0. \tag{19}$$

Therefore there are at most 6 lines or infinitely many going through $P$ (in the latter case $P$ is a so-called *Eckardt* point, see [3], no. 8); in the latter case $G_2(x, y, 0, u)$ and $G_3(x, y, 0, u)$ have a common factor. If this factor is linear then $X$ contains a 2-plane, if the factor is quadratic then $X \cap \{z = 0\}$ contains a quadratic cone and hence again a 2-plane. This is impossible by 1.17. Therefore if $P$ is a Eckardt point then $G_2(x, y, 0, u) \equiv 0$. Therefore $X \cap \{z = 0\}$ is a cubic cone with $P$ as vertex and 'base curve' $\{G_3(x, y, 0, u) = 0, v = 0\}$. But along every line of the cone, in particular along $l$, there is a 2-plane tangent to this cone. By 1.16 this completes the proof of 1.18.

LEMMA (1.19). *Let $Y = \{P$; through $P$ goes a line of the second type$\}$. $Y$ is Zariski closed and of dimension at most* 2. *Let $P \notin Y$, then there are* 6 *different lines on $X$ going through $P$. Also if $l$ is a line of the first type then $l$ counts with 'multiplicity one' in each of its points.*

PROOF. The assertion about $Y$ follows from 1.9 and 1.14. Let $P \notin Y$ and $l$ a line through $P$. We can assume that $l$ is given by $x = 0, y = 0$, and $z = 0$ and $P = (0, 0, 0, 0, 1)$. Since $l$ is of the first type we can assume that the equation of $X$ is given by (16); writing this as in (18) we have

$$
\text{(20)} \qquad \begin{aligned} G_2(x, y, 0, u) &= 2uy + 2Q_2(x, y, 0) \\ G_3(x, y, 0, u) &= u^2x + 2uQ_1(x, y, 0) + C(x, y, 0). \end{aligned}
$$

The lines through $P$ are given by (19). It suffices to see that the point $S = (0, 0, 0, 1, 0)$, which is the point in (19) corresponding with $l$, is a point of multiplicity 1 of $G_2 = 0$, $G_3 = 0$ in $z = 0$, $v = 0$. From (20) we see that this point is non-singular on $G_2 = 0$ and on $G_3 = 0$; if the intersection multiplicity is larger than 1 there exists $v_0 \neq 0, \infty$ such that

$$
\text{(21)} \qquad \begin{aligned} v_0 \frac{\partial G_2}{\partial x}(S) + \frac{\partial G_3}{\partial v}(S) &= 0 \\ v_0 \frac{\partial G_2}{\partial y}(S) + \frac{\partial G_3}{\partial y}(S) &= 0 \\ v_0 \frac{\partial G_2}{\partial z}(S) + \frac{\partial G_3}{\partial z}(S) &= 0. \end{aligned}
$$

However from (20) follows immediately that $\partial G_2/\partial x\ (S) = 0$ and $\partial G_3/\partial x(S) = 1$. Therefore there is not such a $v_0$. This completes the proof of 1.19.

*Section* 1*C*: *The curve* $\mathscr{H}(l)$.

1.20. *From now on we assume that $l$ is a line of the first type* on $X$ (i.e. $l \notin \mathscr{F}_0$, see 1.4) *and also that* $l \notin \mathscr{F}'_0$ (sec. 1.11). Furthermore for simplicity we assume that $l$ is defined over $k$ (otherwise enlarge $k$). Let

(22) $$N = \{L; L \text{ a linear 2 dim. space through } l\}.$$

$N$ is a projective 2-space. If $l$ has equations $x = 0$, $y = 0$, $z = 0$ then we can identify $N$ with the 2-plane $N'$ defined by $u = 0$, $v = 0$ and we introduce coordinates in $N$, using the coordinates in $\boldsymbol{P}^4$, as follows:

$$T = (\xi, \eta, \zeta) \in N \rightleftarrows T = (\xi, \eta, \zeta, 0, 0) \in N'.$$

Let $L_T = \text{span}\,(l, T)$. We have

(23) $$L_T \cdot X = l + K_T,$$

where $K_T$ is a conic. A point in $L_T$ has projective coordinates (cf. 1.3) $(\xi t, \eta t, \zeta t, u, v)$ and we can use $(t, u, v)$ as (projective) coordinates in $L_T$. If the equation of $X$ is given by (16) then the equation of $K_T$ is given by

(24) $$u^2\xi + 2uv\eta + v^2\zeta + 2utQ_1(\xi, \eta, \zeta) + 2vtQ_2(\xi, \eta, \zeta) + t^2C(\xi, \eta, \zeta) = 0.$$

1.21. The conic $K_T$ *degenerates if* $T$ *is on the curve* $H$ in $N$, where $H$ has equation:

(25) $$\det \begin{pmatrix} \xi & \eta & Q_1(\cdots) \\ \eta & \zeta & Q_2(\cdots) \\ Q_1(\cdots) & Q_2(\cdots) & C(\cdots) \end{pmatrix} = 0.$$

For $T \in H$ we have

(26) $$L_T \cdot X = l + l'_T + l''_T.$$

Since $l \notin \mathscr{F}_0$, $l \notin \mathscr{F}'_0$ we have (1.4 and 1.11) that $l \neq l'_T$, $l \neq l''_T$ and $l'_T \neq l''_T$. We apply now the results of [1], p. 6 below (our curve $H$ corresponds with $\Gamma$ there). This gives:

PROPOSITION (1.22). *$H$ is a non-singular (and hence), absolutely irreducible curve defined over $k$. The degree is* 5 *and (hence) the genus* 6.

1.23. Consider on the Fano surface $\mathscr{F}$ the following curve

(27) $$\mathscr{H} = \mathscr{H}(l) = \{l'; l' \cap l \neq \emptyset\}.$$

On $\mathscr{H}$ we have *an involution* $\sigma$ namely if $L_T$ is the 2-plane spanned by $l$ and $l'$ then $L_T \cdot X = l + l' + l''$ and put $\sigma(l') = l''$. (cf. [1]; p. 5, in the notation there $l = L_u$, $\mathscr{H}(l) = C_u$ and $j = \sigma$). The quotient of $\mathscr{H}$ under $\sigma$ is the curve $H$ of (25) (cf. again [1], p. 5). Since $H$ is absolutely irreducible we have that either $\mathscr{H}$ is absolutely irreducible or $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$ with $\mathscr{H}_i$ $(i = 1, 2)$ absolutely irreducible and $\sigma$ interchanges $\mathscr{H}_1$ and $\mathscr{H}_2$.

1.24. Finally consider on $\mathscr{F}$ the set $U'$ of lines $l$ on $X$ such that through $l$ passes a *non-singular* hyperplane section $\Sigma$. It is 'well-known' that $U'$ is an open set on $\mathscr{F}$ (non-empty!). Put

$$U = U' - U' \cap \mathscr{F}_0 - U' \cap \mathscr{F}'_0,$$

where $\mathscr{F}_0$ (resp. $\mathscr{F}'_0$) are as above (see 1.4, resp. 1.11). If $l \in U$ then we can apply the results of 1.20–1.23. Moreover since there is a non-singular hyperplane section $\Sigma$ through $l$ we can apply the argument of [1], p. 10 [3] in order to rule out the possibility $\mathscr{H} = \mathscr{H}_1 \cap \mathscr{H}_2$. Hence $\mathscr{H} = \mathscr{H}(l)$ is an absolutely irreducible curve. Moreover $\mathscr{H}$ is a covering of degree 2 of $H$, namely above $T \in H$ we have by (26) the two points $l'_T$ and $l''_T$ of $\mathscr{H}$. Since $l'_T \neq l''_T$ we have that $\mathscr{H}$ is an *étale covering* of $H$ of degree 2. Summarizing we have:

PROPOSITION (1.25). *There exists a non empty open set $U$ on the Fano surface $\mathscr{F}$ with the following properties. For $l \in U$, let $N$ be the projective 2-space of 2-planes through $l$ and for $T \in N$ let $L_T$ denote the corresponding 2-plane. Then*:

(i) $L_T \cdot X = l + K_T$ *with* $K_T$ *a conic.*

(ii) *Let* $H = \{T;\ K_T \text{ degenerates}\}$, *then $H$ is a non-singular, absolutely irreducible curve in $N$ of degree 5 and genus 6.*

(iii) *For $T \in H$ we have $K_T = l'_T + l''_T$ with $l \neq l'_T$, $l \neq l''_T$ and $l'_T \neq l''_T$.*

(iv) *If $\mathscr{H} = \{l';\ l' \cap l \neq \emptyset\}$ then $\mathscr{H}$ is an absolutely irreducible curve in $\mathscr{F}$.*
*Moreover $\mathscr{H}$ is an étale covering of $H$ of degree 2 and hence non-singular. The fibre over $T \in H$ consists of $l'_T$ and $l''_T$. The genus of $\mathscr{H}$ is 11 (by the Hurwitz formula).*

(v) *$l$ is of first type.*

(vi) *Through all points of $l$ go at most 6 lines on $X$ (by 1.18) and through almost all points of $l$ go exactly 6 lines (the possible exceptions are the points of $Y \cap l$; $Y$ defined in 1.19).*

[3] There is a little slip in the argument in the middle of page 10 in [1], namely there is not a line on $\Sigma$ meeting $M'_1, \ldots, M'_5$. However there are several ways of correcting the argument; the following one was communicated to me by Bombieri. By [1] one must have $C_u = C_u^1 + C_u^2$ (with the notations from [1]), where $j_u$ interchanges $C_u^1$ and $C_u^2$. Now note that if $u$ and $t$ are two points on the Fano surface such that $L_u$ and $L_t$ do not go through an Eckardt point, then $C_u^1 \cdot C_t^2$ is defined for $u \neq t$ (cf. lemma 1.17 and 1.18). Since $C_u^i$ ($i = 1, 2$) is a 2-dimensional family of curves on a surface one has $C_u^1 \cdot C_u^2 \geqq 2$ and $(C_u^i)^2 \geqq 1$ for $i = 1$ or 2. This gives

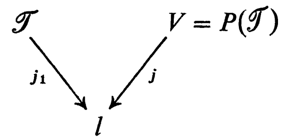
$$5 = (C_u)^2 = (C_u^1)^2 + 2C_u^1 \cdot C_u^2 + (C_u^2)^2 \geqq 6.$$

Therefore we get a contradiction by assuming that $C_u$ has 2 components.

## 2. The tangent bundle restricted to a line [4]

2.1. *Notations.* From now on $l$ denotes a fixed line on $X$ *contained in the open set* $U$ of proposition 1.25. We assume that $l$ is defined over the ground-field (otherwise we enlarge the groundfield), i.e. the Plücker-coordinates of $l$ are rational over $k$. We can assume now that $l$ is given by $x = 0, y = 0, z = 0$; the equation of $X$ is (15), or even (16) if we want.

Let $\mathscr{T}$ be the restriction of the tangent bundle of $X$ to $l$ and $V = P(\mathscr{T})$ the bundle of associated projective spaces of 1-dimensional linear subspaces:




For $S \in l$, $\mathscr{T}_S$ (resp. $V_S$) denotes the fibre of $\mathscr{T}$ (resp. $V$) over $S$. By definition $V_S$ is the projective space of the 1-dimensional linear subspaces, associated with $\mathscr{T}_S$. Furthermore $\mathscr{T}_S^*$ denotes the tangent hyperplane to $X$ at $S$ and put

$$X_S^* = X \cap \mathscr{T}_S^*.$$

2.2. *A canonical identification.* The *lines through S in the tangent hyperplane* $\mathscr{T}_S^*$ correspond canonically with the points of $V_S$. Namely such a line determines a tangent vector to $X$ at $S$ up to a scalar multiple and hence determines uniquely a point in $V_S$, and conversely.

2.3. *Extra structure in* $V_S$. Using 2.2 we have in $V_S$:

(a) a point $I_S$ corresponding with the line $l$ in $\mathscr{T}_S^*$,

(b) five points $M_{S,i}$ $(i = 1, \cdots 5)$ corresponding with the other 5 lines $m_{S,i}$ through $S$ on $X$ (see 1.25 (vi)).

Moreover these 6 points are on a conic $W_S$ in $V_S$ (possibly degenerated) corresponding with the tangent cone of $X_S^*$ at $S$ (compare with (18) and (19); the tangent cone is given by $z = 0$, $G_2 = 0$). For a special point $S \in l$ some of the points $M_{S,i}$ may coincide but in a generic point there are 6 different points in $V_S$ (1.25 vi). Also by 1.19 $I_S$ itself never coincides with one of the points $M_{S,i}$.

Introduce in $V$ the curves

$$I = \bigcup_{S \in l} I_S \text{ and } \hat{H} = \bigcup_{S \in l} \{ \bigcup_i M_{S,i} \}. \tag{28}$$

[4] Clemens suggested to me to study this tangent bundle. My original approach was along a desingularization of $\tilde{X}$ introduced in § 5. From 6.8 (ii) it is clear that the suggestion of Clemens is very important.

LEMMA (2.4). *Both $I$ and $\hat{H}$ are absolutely irreducible over $k$; $I$ is a rational curve and $\hat{H}$ is birational with the curve $\mathscr{H}$ of* (27).

PROOF. Let $k(S)$ denote the field obtained by adjoining the non-homogeneous coordinates of $S$ to $k$. The point $I_S$ is rational in $k(S)$; $I$ itself is the locus (in the sense of Weil) of $I_S$ over $k$. Hence $I$ is absolutely irreducible, and in fact birational with $l$ itself.

Next $\hat{H} : \hat{H}$ is birational equivalent with the curve $\mathscr{H}$ $(= \mathscr{H}(l))$ of (27), namely

$$M_{S,i} \in \hat{H} \leftrightarrow \text{line } m_{S,i} \text{ in } \mathscr{T}_S^* \text{ on } X \leftrightarrow m_{S,i} = l' \in \mathscr{H}.$$

Since $\mathscr{H}$ is absolutely irreducible (1.25 (iv)) the same is true for $\hat{H}$.

PROPOSITION (2.5). *$\hat{H}$ and $I$ are non-singular curves in $V$ and $\hat{H} \cap I = \emptyset$.*

PROOF. $\hat{H} \cap I = \emptyset$: this we have seen already in 2.3.

Next: the non-singularity of $\hat{H} \cup I$. This requires some computation and occupies the rest of section 2. Several of the steps are also useful further on.

2.6. Let $M_{S,i} \in \hat{H}$; let $m_{S,i}$ be the corresponding line on $X$. For simplicity write $m_{S,i} = l'$. We have seen already in 2.3 that $l' = m_{S,i} \neq l$; let $L$ be the 2-space spanned by $l$ and $l'$. There are two different cases now for the intersection $X \cap L$ (cf. 1.25 (iii)):

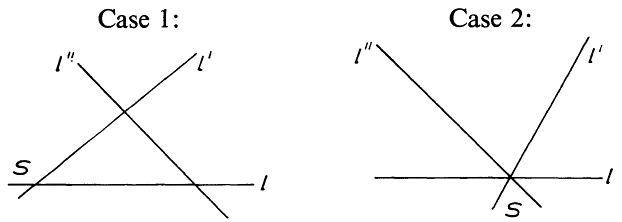



Figure 1

2.7. *Suitable coordinates in case* 1:

Take coordinates such that $L$ is given by $x = 0$, $z = 0$ and

$$l : \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \quad l' : \begin{cases} x = 0 \\ z = 0 \\ u = 0 \end{cases} \quad l'' : \begin{cases} x = 0 \\ z = 0 \\ v = 0 \end{cases}$$

Our point $S$ is now $S = (0, 0, 0, 0, 1)$.

Also the equation $F$ is (15). Substituting $x = 0$, $z = 0$ should give $yuv = 0$. Therefore:

$$l_1(0,1,0) = l_3(0,1,0) = Q_1(0,1,0) = Q_2(0,1,0) = C(0,1,0) = 0$$
$$l_2(0,1,0) \neq 0.$$

Making a change of variables in $x$, $y$ and $z$ we can assume

$$l_1(x,y,z) = x,\ l_2(x,y,z) = y,\ l_3(x,y,z) = z.$$

Without disturbing the conditions above we still have freedom for a transformation

$$u \mapsto u+\alpha x+\beta z$$
$$v \mapsto v+\gamma x+\partial z.$$

Using this we can achieve that

$$Q_1(x,y,z) = Q_1(x,z)$$
$$Q_2(x,y,z) = Q_2(x,z).$$

Therefore we arrive at the equation (cf. 1.15):

$$(16') \qquad F(x,y,z,u,v) = u^2x+2uvy+v^2z+2uQ_1(x,z)+2vQ_2(x,z) + C(x,y,z) = 0$$

with

$$(29) \qquad C(x,y,z) = y^2(ax+bz)+yQ^*(x,z)+C^*(x,z).$$

LEMMA 2.8. *X non-singular* $\Rightarrow$ *not* $a = 0$ *and* $b = 0$ *at the same time.*

PROOF. Take $P = (0, 1, 0, 0, 0)$. Then $P \in X$ and

$$\frac{\partial F}{\partial y}(P) = \frac{\partial F}{\partial u}(P) = \frac{\partial F}{\partial v}(P) = 0,\ \frac{\partial F}{\partial x}(P) = a,\ \frac{\partial F}{\partial z}(P) = b.$$

2.9. *Suitable coordinates in case* 2.
Take coordinates such that $L$ is given by $y = 0$, $z = 0$ and

$$l:\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \quad l':\begin{cases} u = 0 \\ y = 0 \\ z = 0 \end{cases} \quad l'':\begin{cases} 2x+u = 0 \\ y = 0 \\ z = 0. \end{cases}$$

Again $S = (0, 0, 0, 0, 1)$. Substituting $y = 0$, $z = 0$ in the equation (15) for $F$ we should get $xu(u+2x) = 0$. This gives

$$l_2(1,0,0) = l_3(1,0,0) = Q_2(1,0,0) = C(1,0,0) = 0,$$
$$l_1(1,0,0) = Q_1(1,0,0) \neq 0.$$

Making a change of variables in $x$, $y$ and $z$ we can assume

$$l_1(x,y,z) = x,\ l_2(x,y,z) = y,\ l_3(x,y,z) = z.$$

Further simplification is also possible here, but not necessary. We have again

$$(16'')\qquad F(x, y, z, u, v) = u^2x+2uvy+v^2z+2uQ_1(x, y, z) \\ +2vQ_2(x, y, z)+C(x, y, z) = 0$$

with the usual meaning (see (16)) of $Q_i$ and $C$. Moreover we are going to use

$$(30)\qquad Q_1(1, 0, 0) = 1.$$

2.10. *Suitable coordinates in the bundles $\mathscr{T}$ and $V$* (in both cases). We have the point $S = (0, 0, 0, 0, 1)$ on $l$; take an affine neighborhood $U$ of $S$ on $l$ and $S_\alpha \in U$, then we can write $S_\alpha = (0, 0, 0, \alpha, 1)$. We use then notations $\mathscr{T}_\alpha$ (resp. $V_\alpha$, $\mathscr{T}^*_\alpha$, etc. ...) instead of $\mathscr{T}_{S_\alpha}$ (resp. $V_{S_\alpha}$, $\mathscr{T}^*_{S_\alpha}$ etc....) introduced in 2.1. In a neighborhood of $S_0$ on $X$ we use affine coordinates $(x, y, z, u, 1)$.

Furthermore the tangent space to $X$ at $S_\alpha$ is given by

$$(17')\qquad \alpha^2x+2\alpha y+z = 0.$$

Therefore we can use $(x, y, u)$ as uniformizing parameters for $X$ at $S$. Finally put

$$u^* = u-\alpha v.$$

*Coordinates in $\mathscr{T}$* (in $j_1^{-1}(U)$) : $(\alpha; \bar{x}, \bar{y}, \bar{u})$.

*Coordinates in $V$* (in $j^{-1}(U)$) : $(\alpha; \bar{x} : \bar{y} : \bar{u})$; i.e. the latter are homogeneous coordinates.

2.11. The canonical identification of 2.2 between points in $V_\alpha$ and lines through $S_\alpha$ in $\mathscr{T}^*_\alpha$ is obtained via the following *non-canonical* (i.e., depending on the choice of affine coordinates) identification

$$\mathscr{T}_\alpha \rightleftarrows \mathscr{T}^*_\alpha - \{v = 0\},$$

with

$$(\alpha; \bar{x}, \bar{y}, \bar{u}) \rightleftarrows (x, y, z, u, v)$$

with

$$(31)\qquad \begin{aligned} v &= 1 \\ x &= \bar{x} \\ y &= \bar{y} \\ u-\alpha = u^* &= \bar{u} \\ z &= -\alpha^2\bar{x}-2\alpha\bar{y} \end{aligned}$$

2.12. *Equation for $X^*_\alpha = X \cap \mathscr{T}^*_\alpha$ in $\mathscr{T}^*_\alpha$*: Substituting (17′) of 2.10 in (16′) of 2.7 or (16″) of 2.9 and replacing $u$ by $u = u^*+\alpha v$ gives the equation for $X_\alpha^*$. Ordering according to decreasing powers of $v$ (compare 1.18) gives

$$G_2(x, y, u^*; \alpha)v + G_3(x, y, u^*; \alpha) = 0 \tag{32}$$

with

$$\begin{aligned} G_2(x, y, u^*; \alpha) &= 2\alpha x u^* + 2yu^* + 2\alpha Q_1(x, y, -\alpha^2 x - 2\alpha y) \\ &\quad + 2Q_2(x, y, -\alpha^2 x - 2\alpha y) \\ G_3(x, y, u^*; \alpha) &= x(u^*)^2 + 2u^* Q_1(x, y, -\alpha^2 x - 2\alpha y) \\ &\quad + C(x, y, -\alpha^2 x - 2\alpha y). \end{aligned} \tag{33}$$

2.13. Using the above equation for $X_\alpha^*$ and the identification (31) we get: equation for $W_\alpha$ in $V_\alpha$ (the cone of the tangents!):

$$G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) = 0.$$

Equations for $\hat{H} \cup I$ in $V$:

$$\begin{aligned} G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) &= 0 \\ G_3(\bar{x}, \bar{y}, \bar{u}; \alpha) &= 0. \end{aligned} \tag{34}$$

Note that $G_2$ is homogeneous of degree 2 and $G_3$ of degree 3 in $\bar{x}$, $\bar{y}$ and $\bar{u}$, and that by 1.25 (vi) $G_2$ and $G_3$ have no common factor.

2.14. *The non-singularity of $\hat{H} \cup I$ in case* 1: It suffices, since we started with an arbitrary point $S = S_0 \in l$, to prove the non-singularity in the following two points: $M_{S_0, i} \in \hat{H}$ and $I_{S_0} \in I$. Note also that the tangent hyperplane at $X$ in $S_0$ is $z = 0$. *The point* $M_{S_0, i} \in \hat{H}$: this point corresponds with the line $m_{S_0, i} = l'$ and the equations of $l'$ are (see 2,7): $x = 0$, $z = 0$, $u = 0$. That means that we have for $M_{S_0, i}$ the following coordinates:

$$M_{S_0, i} = (\alpha = 0;\ \bar{x} : \bar{y} : \bar{u} = 0 : 1 : 0).$$

We have to consider the matrix

$$\begin{pmatrix} \dfrac{\partial G_2}{\partial \bar{x}} & \dfrac{\partial G_2}{\partial \bar{u}} & \dfrac{\partial G_2}{\partial \alpha} \\ \dfrac{\partial G_3}{\partial \bar{x}} & \dfrac{\partial G_3}{\partial \bar{u}} & \dfrac{\partial G_3}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \cdots & 2 + \cdots & \cdots \\ a + \cdots & \cdots & -2b + \cdots \end{pmatrix}$$

where $a$ and $b$ are from (29) and where $\cdots$ stand for terms containing at least one $\bar{x}$, $\bar{u}$ or $\alpha$ (we have used for this the special fact that $Q_1 = Q_1(x, z)$ and $Q_2 = Q_2(x, z)$ in (16′)). Evaluating at $M_{S_0, i}$, only the indicated terms contribute and the matrix has rank 2 by lemma 2.8. Hence the point $M_{S_0, i}$ is non-singular on $\hat{H}$.

*The point* $I_{S_0} \in I$: this corresponds with the line $l$ and therefore, by 2.7, we have

$$I_{S_0} = (\alpha = 0;\ \bar{x} : \bar{y} : \bar{u} = 0 : 0 : 1).$$

We have to consider

$$\begin{pmatrix} \frac{\partial G_2}{\partial \bar{x}} & \frac{\partial G_2}{\partial \bar{y}} & \frac{\partial G_2}{\partial \alpha} \\ \frac{\partial G_3}{\partial \bar{x}} & \frac{\partial G_3}{\partial \bar{y}} & \frac{\partial G_3}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \cdots & 2+\cdots & ? \\ 1+\cdots & \cdots & ? \end{pmatrix}$$

Now $\cdots$ means terms with at least one $\bar{x}$, $\bar{y}$ or $\alpha$ and ? means: of no importance! Evaluating at $I_{S_0}$ we see that the rank is 2. Hence $I_{S_0}$ is non-singular on $I$.

2.15. *The non-singularity of* $\hat{H} \cup I$ *in case* 2. Similarly to 2.14 we have to consider two points.

$M_{S_{0,i}} \in \hat{H}$: this point corresponds with $l'$ and by 2.9 $l'$ has equations $u = 0, y = 0, z = 0$. Therefore

$$M_{S_{0,i}} = (\alpha = 0;\ \bar{x} : \bar{y} : \bar{u} = 1 : 0 : 0).$$

Consider the matrix

$$\begin{pmatrix} \frac{\partial G_2}{\partial \bar{y}} & \frac{\partial G_2}{\partial \bar{u}} & \frac{\partial G_2}{\partial \alpha} \\ \frac{\partial G_3}{\partial \bar{y}} & \frac{\partial G_3}{\partial \bar{u}} & \frac{\partial G_3}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} ? & \cdots & Q_1(1, \cdots)+\cdots \\ ? & 2Q_1(1, \cdots)+\cdots & \cdots \end{pmatrix}$$

where $\cdots$ means: at least one $\bar{y}$, $\bar{u}$ or $\alpha$ and ? is of no importance. Evaluating at $M_{S_{0,i}}$ and using $Q_1(1, 0, 0) \neq 0$ by (30), we get rank 2. Hence the point is non-singular on $\hat{H}$.

Finally $I_{S_0} \in I$: we have by 2.9

$$I_{S_0} = (\alpha = 0;\ \bar{x} : \bar{y} : \bar{u} = 0 : 0 : 1),$$

therefore consider

$$\begin{pmatrix} \frac{\partial G_2}{\partial \bar{x}} & \frac{\partial G_2}{\partial \bar{y}} & \frac{\partial G_2}{\partial \alpha} \\ \frac{\partial G_3}{\partial \bar{x}} & \frac{\partial G_3}{\partial \bar{y}} & \frac{\partial G_3}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \cdots & 2+\cdots & ? \\ 1+\cdots & \cdots & ? \end{pmatrix}.$$

Evaluating gives rank 2, hence a non-singular point. This completes the proof of 2.5.

## 3. The blow-up $X'$ of $V$

3.1. Blow up the projective bundle $V$ along the curve $\hat{H} \cup I$. We get the following diagram:

$$\begin{array}{ccc} D \cup D_* & \hookrightarrow & X' = B_{\hat{H} \cup I}(V) \\ \downarrow & & \downarrow p \\ \hat{H} \cup I & \hookrightarrow & V \\ & & \downarrow j \\ & & l \end{array}$$

From proposition 2.5 we have at once:

PROPOSITION 3.2. (i) *$X'$ is a non-singular threefold.*
(ii) $D = p^{-1}(\hat{H})$ and $D_* = p^{-1}(I)$ are non-singular divisors; $D \cap D_* = \emptyset$.

3.3. *Coordinates in $X'$*: Let $(\alpha; \bar{x} : \bar{y} : \bar{u})$ be coordinates in the open set $j^{-1}(U)$ of $V$ (see 2.10; $U$ an open set on $l$). We use bihomogeneous coordinates in $p^{-1}(j^{-1}(U))$ on $X'$ as follows (cf. [11], p. 35):

$$(\alpha; \bar{x} : \bar{y} : \bar{u}; \lambda_1 : \lambda_2 : \lambda_3 : \mu)$$

with

$$(35) \qquad \lambda_1 : \lambda_2 : \lambda_3 : \mu = \bar{x}G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) : \bar{y}G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) : \bar{u}G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) : G_3(\bar{x}, \bar{y}, \bar{u}; \alpha)$$

with $G_2$ and $G_3$ as in 2.12–2.13.

## 4. The rational transformation $\phi : X' \to X$

4.1. *Introduction of $\phi$.* Take a point $P' \in X'$ such that $p(P') \notin \hat{H} \cup I$ (see the diagram of 3.1). Let $S = j \cdot p(P')$. Consider the point $m = p(P')$ as a line in the tangent hyperplane $\mathscr{T}_S^*$ to $X$ at $S$ (i.e., use the identification 2.2). Since by assumption $m \notin X$, the line $m$ intersects $X_S^* = X \cap \mathscr{T}_S^*$ in a 'third' point $P$. (Note also that in case $P \neq S$ we have $m = p(P') = PS$.) Consider on $X' \times X$ the ($k-$) Zariski closure $\Gamma_\phi$ of the couples $(P', P)$ obtained in this way.

PROPOSITION (4.2). (i) *$\Gamma_\phi$ is the graph of a rational transformation $\phi$ : $X' \to X$.* (ii) *$\phi$ is everywhere regular (i.e. everywhere 'defined') and if $P_0 = \phi(P_0')$ then $P_0 \in X_{S_0}^*$ with $S_0 = j \cdot p(P_0')$.*

PROOF. (i) Is almost clear by construction. For further purpose we compute the actual transformation formulas. Choose coordinates as in 3.3 on $X'$. Take a generic point $P' \in X'$:

$$P' = (\alpha; \bar{x} : \bar{y} : \bar{u}; \lambda_1 : \lambda_2 : \lambda_3 : \mu).$$

In order to compute $P = \phi(P')$ we have to consider $p(P')$ as a line in the tangent hyperplane $\mathscr{T}^*_\alpha$ to $X$ at $S = j \cdot p(P)$; i.e., we have to use the identification (see 2.11)

$$\mathscr{T}_\alpha \rightleftarrows \mathscr{T}^*_\alpha - \{v = 0\}.$$

Working non-homogeneously we get

$$(36) \quad P = (x, y, z, u, 1) = (0, 0, 0, \alpha, 1) + t(\bar{x}, \bar{y}, -(\alpha^2\bar{x} + 2\alpha\bar{y}), \bar{u}, 0)$$

Substituting this into equation (32) of $X^*_\alpha$ we get

$$t^2 G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) + t^3 G_3(\bar{x}, \bar{y}, \bar{u}; \alpha) = 0$$

i.e.,

$$t = -\frac{G_2(\bar{x}, \bar{y}, \bar{u}; \alpha)}{G_3(\bar{x}, \bar{y}, \bar{u}; \alpha)}.$$

Substituting this into (36), and making the coordinates homogeneous, we get:

$$(37) \quad \begin{aligned} xG_3(\bar{x}, \bar{y}, \bar{u}; \alpha) &= -\bar{x}G_2(\bar{x}, \bar{y}, \bar{u}; \alpha)v \\ yG_3(\bar{x}, \bar{y}, \bar{u}; \alpha) &= -\bar{y}G_2(\bar{x}, \bar{y}, \bar{u}; \alpha)v \\ uG_3(\bar{x}, \bar{y}, \bar{u}; \alpha) &= \{-\bar{u}G_2(\bar{x}, \bar{y}, \bar{u}; \alpha) + \alpha G_3(\bar{x}, \bar{y}, \bar{u}; \alpha)\}v \\ zG_3(\bar{x}, \bar{y}, \bar{u}; \alpha) &= (\alpha^2\bar{x} + 2\alpha\bar{y})G_2(\bar{x}, \bar{y}, \bar{u}; \alpha)v. \end{aligned}$$

Using (35) this may be rewritten as

$$(37') \quad \begin{aligned} x\mu &= -\lambda_1 v \\ y\mu &= -\lambda_2 v \\ u\mu &= (-\lambda_3 + \alpha\mu)v \\ z\mu &= (\alpha^2\lambda_1 + 2\alpha\lambda_2)v. \end{aligned}$$

(ii) To prove: $\phi$ is everywhere regular. Take

$$P'_0 = (\alpha_0; \bar{x}_0 : \bar{y}_0 : \bar{u}_0; \lambda_{10} : \lambda_{20} : \lambda_{30} : \mu_0) \in X'.$$

CASE a: $\mu_0 \neq 0$. Then it follows from (37') immediately that $x/v$, $y/v$, $z/v$ and $u/v$ are all in $0_{X', P'_0}$.

CASE b: $\mu_0 = 0$. Then some $\lambda_{i0} \neq 0$, for instance $\lambda_{10} \neq 0$. We rewrite (37') as follows:

$$(37'')\quad \begin{aligned} \frac{y}{x} &= \frac{\lambda_2}{\lambda_1} \\ \frac{z}{x} &= -\frac{\alpha^2\lambda_1+2\alpha\lambda_2}{\lambda_1} \\ \frac{u}{x} &= \frac{\lambda_3-\alpha\mu}{\lambda_1} \\ \frac{v}{x} &= -\frac{\mu}{\lambda_1}. \end{aligned}$$

Hence all are in $O_{X',P'_0}$. The cases $\lambda_{20} \neq 0$ or $\lambda_{30} \neq 0$ are treated similarly. The last assertion of (ii) is obvious.

4.3. Let $S_\alpha \in l$, with $S_\alpha = (0, 0, 0, \alpha, 1)$. Write, as in 2.10, $X^*_\alpha$ instead of $X^*_{S_\alpha}$. Put (set theoretically) $X'_{S_\alpha} = X'_\alpha = (j \cdot p)^{-1}(S_\alpha)$ and let $\phi_\alpha$ be the set theoretical restriction of $\phi$ to $X'_\alpha$. Consider also in $V_\alpha = j^{-1}(S_\alpha)$ the cone $W_\alpha$ of tangents (see 2.13) and let $p^{-1}[W_\alpha]$ denote the *proper transform* of $W_\alpha$ in $X'_\alpha$ (see for instance [11], p. 4). Finally note that the restriction of $p$ to $p^{-1}[W_\alpha] \to W_\alpha$ is one to one except in a possible double point of $W_\alpha$. We need the following

LEMMA (4.4). *Let* $P'_0 = (\alpha_0;\ \bar{x} : \bar{y}_0 : \bar{u}_0; \lambda_{10} : \lambda_{20} : \lambda_{30} : \mu_0) \in X'_{\alpha_0}$. Then

$$P'_0 \in p^{-1}[W_{\alpha_0}] \Leftrightarrow \lambda_{10} = \lambda_{20} = \lambda_{30} = 0.$$

PROOF. Recall that the equation of $W_{\alpha_0}$ is $G_2(\bar{x}, \bar{y}, \bar{u}; \alpha_0) = 0$ (see 2.13).

$\Rightarrow$: Follows directly from (35) and from the fact that in a generic point of $W_{\alpha_0}$ the $G_3(\bar{x}, \bar{y}, \bar{u}; \alpha_0) \neq 0$.

$\Leftarrow$: Looking to a generic point of $p^{-1}[W_{\alpha_0}]$ we see that the equations are $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $G_2 = 0$. Since not at *the same time* $\bar{x}_0 = 0$, $\bar{y}_0 = 0$, $\bar{u}_0 = 0$ we have by (35) that $G_2(\bar{x}_0, \bar{y}_0, \bar{u}_0; \alpha_0) = 0$.

LEMMA (4.5). *The set theoretical map* $\phi_\alpha : X'_\alpha \to X^*_\alpha$ *has the properties*:

(i) *the restriction of* $\phi_\alpha$ *to* $X'_\alpha - p^{-1}[W_\alpha]$ *is one to one*:

$$X'_\alpha - p^{-1}[W_\alpha] \rightleftarrows X^*_\alpha - S_\alpha.$$

(ii) $\phi_\alpha^{-1}(S_\alpha) = p^{-1}[W_\alpha]$.

PROOF. First note that by 4.2 (ii) the image of $\phi_\alpha$ is in $X^*_\alpha$.

(i) Let $P'_0 = (\alpha_0;\ \bar{x}_0 : \bar{y}_0 : \bar{u}_0; \lambda_{10} : \lambda_{20} : \lambda_{30} : \mu_0) \notin p^{-1}[W_{\alpha_0}]$; put $P_0 = \phi(P'_0)$ and let $P_0 = (x_0 : y_0 : z_0 : u_0 : v_0)$. By 4.4 at least one $\lambda_{i0} \neq 0$. If $\mu_0 \neq 0$ then by (37') we don't have $x_0 = y_0 = 0$, $u_0 = \alpha_0$; i.e. $P_0 \neq S_{\alpha_0}$. If $\mu_0 = 0$ then some $\lambda_{i0} \neq 0$. It then follows that $v_0 = 0$

(for instance if $\lambda_{10} \neq 0$ then use (37″), otherwise we have similar equations). Hence $P_0 \neq S_{\alpha_0}$.

SURJECTIVITY. Let $P_0 \in X^*_{\alpha_0} - S_{\alpha_0}$ be given. Take a generic point $P' \in X'$, then $P = \phi(P')$ is generic on $X$; put $S_\alpha = j \cdot p(P')$. Counting dimensions one sees that $S_\alpha$ is generic on $l$ and $P$ generic in $X^*_\alpha$. It follows that $(P, S_\alpha) \to (P_0, S_{\alpha_0})$ is a specialization. Extend this to a specialization $(P, S_\alpha, P') \to (P_0, S_{\alpha_0}, P'_0)$, then $P_0 = \phi(P'_0)$ and $j \cdot p(P'_0) = S_{\alpha_0}$. Now $P'_0 \notin p^{-1}[W_{\alpha_0}]$ because if this was the case then, with the usual coordinates, $\lambda_{i0} = 0$ for $i = 1, 2, 3$ by 4.4 and hence $\mu_0 \neq 0$. Then using (37′) we see $P_0 = S_{\alpha_0}$ contrary to the assumption. Hence the (restriction of the) map $\phi_{\alpha_0}$ is onto.

INJECTIVITY. Let $P_0 = (x_0 : y_0 : z_0 : u_0 : v_0)$ be given in $X^*_{\alpha_0} - S_0$ and $P'_0 = (\alpha_0; \bar{x} : \bar{y}_0 : \bar{u}_0; \lambda_{10} : \lambda_{20} : \lambda_{30} : \mu_0)$ such that $\phi(P'_0) = P_0$. If $v_0 \neq 0$ then by (37′) necessarily $\mu_0 \neq 0$; put $\mu_0 = 1$, then by (37′) the $\lambda_{i0}$ are uniquely determined and not all zero, hence by (35) also $\bar{x}_0 : \bar{y}_0 : \bar{u}_0$ is unique. Next if $v_0 = 0$ then again by (37′) we have $\mu_0 = 0$. Some $\lambda_{i0} \neq 0$, for instance $\lambda_{10} \neq 0$. Then since the right hand side of (37″) is finite we must have $x_0 \neq 0$. It follows then by (37″) that $\lambda_{10} : \lambda_{20} : \lambda_{30}$ is unique in terms of $P_0$, hence by (35) also $\bar{x}_0 : \bar{y}_0 : \bar{u}_0$; hence $P_0$ uniquely determines $P'_0$ (always with a fixed $\alpha_0$).

(ii) From (i) follows that $\phi_\alpha^{-1}(S_\alpha) \subset p^{-1}[W_\alpha]$. Conversely if $P'_0 \in p^{-1}[W_\alpha]$ then by 4.4 we have $P'_0 = (\alpha; \bar{x}_0 : \bar{y}_0 : \bar{u}_0; 0 : 0 : 0 : 1)$. Using (37′) we see $P_0 = \phi(P'_0) = S_\alpha$.

PROPOSITION (4.6). *The rational transformation $\phi : X' \to X$ is generically two-to-one* (*cf. also* [3], *App. B*).

PROOF. Take $P \in X$ generic and let $P' \in X'$ be such that $\phi(P') = P$. Consider the 2-plane $L$ spanned by $l$ and $P$. We have

$$L \cdot X = l + K$$

with $K$ a conic and $P \in K$.

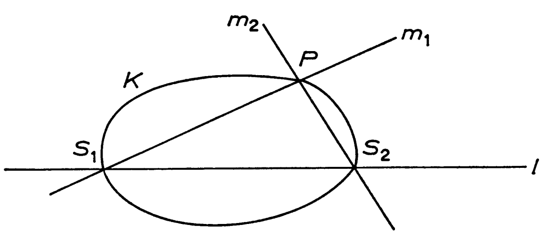



Figure 2

There are two lines, $m_1$ and $m_2$, through $P$ tangent to $X$ in a point of $l$. If $\phi(P') = P$ then, by the construction of $\phi$ in 4.1, we have that $p(P')$ either is $m_1$ or $m_2$. Therefore there are two possibilities for $p(P')$, hence two possibilities for $S_i = j \cdot p(P')$. The proposition follows then from 4.5(i).

Remark (4.7). In fact the proof works for all $P \notin l$, i.e., $\phi$ is *two-to-one outside* $\phi^{-1}(l)$.

## 5. A fibration by conics and a double covering of $X$

5.1. In 1.20 we introduced the projective 2-space $N$ of linear 2-planes through the fixed line $l$ (see (22)). Blow $X$ up along the line $l$:

$$\begin{array}{ccc} & B_l(X) & \\ {}^{i}\swarrow & & \searrow^{\pi} \\ X & & N \end{array}$$

Recall that for $T \in N$, if $L_T$ denotes the corresponding 2-plane through $N$, we have $X \cdot L_T = l + K_T$, where $K_T$ is a conic and $K_T$ degenerates into $l'_T + l''_T$ for $T \in H$, where $H$ is the curve (25) of 1.21. From the construction of $B_l(X)$ follows at once for the *cycle* $\pi^{-1}(T)$:

(i) $\pi^{-1}(T) = i^{-1}[K_T]$ is an irreducible variety (counted once) if $T \notin H$.

(ii) $\pi^{-1}(T) = i^{-1}[l'_T] + i^{-1}[l''_T]$ is a sum of two irreducible varieties (each counted once) if $T \in H$.

(Recall that $i^{-1}[\cdots]$ means *proper* transform. Note also that the components of $\pi^{-1}(T)$, for $T \in H$, are – in general – not defined over the field $k(T)$ itself).

5.2. *Base change*. The morphism $\pi$ gives, up to the birational transformation $i$, a 'fibration' of $X$ by conics $K_T$ over the projective 2-space $N$. However the conics $K_T$ don't have – in general – a rational point over $k(T)$. In order to 'ameliorate this situation' we enlarge the field $k(T)$ in such a way that the points $K_T \cap l$ become rational, or – geometrically speaking – we replace $N$ by a covering as follows. Let

$$\tilde{N} = \{(T, S);\ T \in N,\ S \in K_T \cap l\} \subset N \times l. \tag{38}$$

Consider the diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\ i\ } & B_l(X) & \xleftarrow{\ \tilde{\rho}\ } & B_l(X) \times_N \tilde{N} = \tilde{X} \\ & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ & & N & \xleftarrow{\ \rho\ } & \tilde{N} \\ & & \cup\!\uparrow & & \cup\!\uparrow \\ & & H & & \tilde{H} \end{array}$$

where $\tilde{H} = \rho^{-1}(H)$. From the statements (i) and (ii) of 5.1 we get at once for the *cycle* $\tilde{\pi}^{-1}(\tilde{T})$, with $\tilde{T} \in \tilde{N}$:

(i) this cycle is an irreducible 1-dimensional variety counted once if $\tilde{T} \notin \tilde{H}$.

(ii) this cycle is the sum of two irreducible 1-dimensional varieties, each counted once, if $\tilde{T} \in \tilde{H}$ (note that now each component is in fact defined over $k(\tilde{T})$ because $\tilde{T} = (T, S)$ and $S$ is one of the points $l \cap l'_T$ or $l \cap l''_T$).

5.3. *The ramification divisor and the involutions*:

$$\Delta = \{T; T \in N \text{ such that } K_T \cap l \text{ consists of one point}\}$$

is clearly the *ramification divisor* for $\rho$; put $\tilde{\Delta} = \rho^{-1}(\Delta)$. We have an *involution* $\tilde{\sigma}$ of $\tilde{N}$ over $N$ as follows: if $K_T \cdot l = S_1 + S_2$ then

$$\tilde{T} = (T, S_1) \mapsto \tilde{\sigma}(\tilde{T}) = (T, S_2).$$

Since $\tilde{X} = B_l(X) \times_N \tilde{N}$, we have also an involution $\tilde{\tau} = (\text{id}) \times \tilde{\sigma}$ of $\tilde{X}$ over $B_l(X)$; note that $\tilde{\sigma}$ and $\tilde{\tau}$ are everywhere regular.

5.4. *Coordinates in $B_l(X)$ and $\tilde{X}$*. As before, we assume that (14) is the equation of $l$ and (16) the equation for $X$. Then we can use as coordinates on $B_l(X)$ a bihomogeneous system $(x : y : z : u : v; \xi : \eta : \zeta)$ with

$$x : y : z = \xi : \eta : \zeta. \tag{39}$$

Now $\xi : \eta : \zeta$ can also be used as coordinates in $N$ (see 1.20) and the equation of $K_T$ is given by (24).

Finally for $\tilde{P} \in \tilde{X}$ we can use coordinates $\tilde{P} = (x : y : z : u : v; \xi : \eta : \zeta; \tilde{u} : \tilde{v})$, where the equations are (16), (39) and

$$\xi \tilde{u}^2 + 2\eta \tilde{u}\tilde{v} + \zeta \tilde{v}^2 = 0. \tag{40}$$

Note that (40) gives the equation for $K_T \cap l$.

LEMMA (5.5). *$\tilde{N}$ is (biregular with) a non-singalar quadric surface.*

PROOF. From (24) we get for $K_T \cap l$ the equation $\xi u^2 + 2\eta uv + \zeta v^2 = 0$, hence $S_i = (0, 0, 0, u, v)$ with

$$u : v = (-\eta \pm \beta) : \xi \tag{41}$$

with

$$\beta^2 = \eta^2 - \xi\zeta \tag{42}$$

and $\tilde{T} = (T, S) = (\xi : \eta : \zeta; \tilde{u} : \tilde{v})$ on $\tilde{N}$ corresponds with the point $(\xi : \eta : \zeta : \beta)$ on the quadric surface (42).

5.6. *Singular points of* $\tilde{X}$. Introduce the ('bad') set $\tilde{\mathscr{B}} \subset \tilde{X}$:

(43) $\tilde{\mathscr{B}} = \{\tilde{P} \text{ s.t. } T = \pi \cdot \tilde{\rho}(\tilde{P}) \in H \cap \Delta \text{ and } i \cdot \tilde{\rho}(\tilde{P}) = S = K_T \cap l\}.$

Geometrically this means that $K_T$ degenerates into $l_T'$ and $l_T''$ and $l_T'$ and $l_T''$ *intersect on* $l$ (in $S$) and $\tilde{P}$ is a point above $S$. Introduce also on $N$ the set

$$\mathscr{B} = \{T; T \in H \cap \Delta\}. \tag{44}$$

Lemma (5.7). (i) $\mathscr{B} = \pi \cdot \tilde{\rho}(\tilde{\mathscr{B}})$. *Furthermore the points of* $\tilde{\mathscr{B}}$ *and* $\mathscr{B}$ *correspond one to one. For* $T \in \mathscr{B}$ *put* $\tilde{\mathscr{B}}_T = \{\tilde{P}; \tilde{P} \in \mathscr{B}$ *such that* $\pi \cdot \tilde{\rho}(\tilde{P}) = T\}$.

(ii) $\tilde{\mathscr{B}}$ *is a non-empty finite set and is point-wise invariant under* $\tilde{\tau}$.

Proof. (i) $\mathscr{B} \supset \pi \cdot \tilde{\rho}(\tilde{\mathscr{B}})$ is clear. Furthermore if $T \in \mathscr{B}$, then there is a unique $\tilde{P} \in \tilde{\mathscr{B}}$ such that $T = \pi \cdot \tilde{\rho}(\tilde{P})$. For $\tilde{P} = (P, \tilde{T})$ with $\tilde{T} = (T, S)$ where $S$ is the unique point in $K_T \cap l$ and also $P$ is uniquely determined, namely it is the point in $i^{-1}(S)$ corresponding with the 2-plane spanned by $l_T'$ and $l_T''$ in the tangent hyperplane $\mathscr{T}_S^*$ at $X$ in $S$.

(ii) Clearly $\mathscr{B}$ is finite as the intersection of $H$ and $\Delta$. Finally if $\tilde{P} \in \tilde{\mathscr{B}}$, then with the notations of (i) we have that $S$ is the unique point in $K_T \cap l$, therefore $\tilde{\sigma}(\tilde{T}) = \tilde{T}$ and hence $\tilde{\tau}(\tilde{P}) = \tilde{P}$.

Lemma (5.8). $\tilde{P} \in \tilde{X}$ *singular* $\Leftrightarrow \tilde{P} \in \tilde{\mathscr{B}}$.

Proof. Since $B_l(X)$ is non-singular it suffices to restrict our attention to $\tilde{P} \in \tilde{X}$ with $T = \pi \cdot \tilde{\rho}(\tilde{P}) \in \Delta$ because otherwise $\rho$ is étale.

Claim (*a*) If $\tilde{P}_0 \in \tilde{X}$ is such that $T_0 = \pi \cdot \tilde{\rho}(\tilde{P}_0) \in \Delta$, but $P_0 = i \cdot \tilde{\rho}(\tilde{P}_0) \notin l$, then *not both* $\partial F/\partial u(P_0) = 0$ and $\partial F/\partial v(P_0) = 0$.

Proof of claim (*a*): If $\partial F/\partial u(P_0) = \partial F/\partial v(P_0) = 0$ then the tangent hyperplane $\mathscr{T}_{P_0}^*$ of $X$ in $P_0$ contains $l$. Therefore for $T_0 = \pi \cdot \tilde{\rho}(P_0)$ the 2-plane $L_{T_0}$ is contained in $\mathscr{T}_{P_0}^*$. If $P_0 \notin l$ the following possibilities occur:

1. $K_{T_0}$ non-degenerate and non-tangent to $l$,
2. $K_{T_0}$ non-degenerate and tangent to $l$,
3. $K_{T_0}$ degenerates into $l' + l''$, intersection point $l' \cap l''$ not on $l$ and different from $P_0$,
4. $K_{T_0}$ degenerates into $l' + l''$, intersection point $l' \cap l''$ is $P_0$ (hence not on $l$),
5. $K_{T_0}$ degenerates into $l' + l''$, intersection point $l' \cap l''$ on $l$ (hence different from $P_0$).

In our present situation only case 4 is possible because in all other cases there are lines in $L_{T_0}$ through $P_0$ and not tangent to $X$ in $P_0$, contrary to our remark made above. Finally in case 4 we have $T_0 \notin \Delta$ because $l' \cap l \neq l'' \cap l$. This proves claim (a).

CLAIM (b) If $\widetilde{P}_0 \in \widetilde{X}$ is such that $P_0 = i \cdot \tilde{\rho}(\widetilde{P}_0) \notin l$ then $\widetilde{P}_0$ is non-singular.

PROOF. We have only to consider $\widetilde{P}_0 \in \widetilde{X}$ with $T_0 = \pi \cdot \tilde{\rho}(\widetilde{P}_0) \in \Delta$. For such a $P_0$ either $x_0 \neq 0$ or $z_0 \neq 0$ (or both), say $x_0 \neq 0$. Then $P_0 = (1, y_0, z_0, u_0, v_0; 1, \eta_0, \zeta_0; \tilde{u}_0, 1)$. For $X$ we have the following (affine) equations:

$$\begin{aligned} &F(1, y, z, u, v) = 0 \\ &\eta = y \\ &\zeta = z \\ &\tilde{u}^2 + 2\tilde{u}y + z = 0. \end{aligned}$$

Using claim (a) it is now easily checked that the corresponding equations for the tangent space to $\widetilde{X}$ in $\widetilde{P}_0$ have rank 4 (for instance: to solve $z - z_0$, $\zeta - \zeta_0$, $\eta - \eta_0$ and either $u - u_0$ or $v - v_0$). This proves claim (b).

5.9. It follows from claim (b) that we can restrict our attention – in proving 5.8 – to points $\widetilde{P}_0 \in \widetilde{X}$ with $T_0 = \pi \cdot \tilde{\rho}(\widetilde{P}_0) \in \Delta$ and $P_0 = i \cdot \tilde{\rho}(\widetilde{P}_0) \in l$. For such a point $\widetilde{P}_0$ we have to prove: $\widetilde{P}_0$ singular $\Leftrightarrow T_0 \in H$.

Since $T_0 \in \Delta$ we have either:

(i) $K_{T_0}$ non-degenerate and tangent to $l$ in $P_0 (= S_0)$.

(ii) $K_{T_0}$ degenerates into $l' + l''$ and $l' \cap l''$ is on $l$, therefore $l' \cap l'' \cap l = P_0 = S_0$.

We take suitable coordinates as follows:

(a) $P_0 = S_0 = (0, 0, 0, 0, 1)$,

(b) such that we still have equation (16) for $X$,

(c) as in 5.4.

Then $\widetilde{P}_0 = (0, 0, 0, 0, 1; 1, 0, 0; 0, 1)$. For, if $\xi_0 : \eta_0 : \zeta_0$ is the triple occurring in $\widetilde{P}_0$, then we have by (24) as equation for $K_{T_0} \cap l$:

$$u^2 \xi + 2uv\eta_0 + v^2 \zeta_0 = 0.$$

We know that we have a double root (0, 1), hence $\xi_0 : \eta_0 : \zeta_0 = 1 : 0 : 0$.

Consider now the term $Q_2(x, y, z)$ occurring in (16):

$$Q_2(x, y, z) = q_2 x^2 + x(\gamma y + \partial z) + Q_2'(y, z)$$

with constant $q_2$, $\gamma$ and $\partial$. Evaluation of the equation (25) of $H$ in the point $(1 : 0 : 0)$ gives:

$$\det \begin{pmatrix} \xi & \eta & Q_1(\xi, \eta, \zeta) \\ \eta & \zeta & Q_2(\cdots) \\ Q_1(\cdots) & Q_2(\cdots) & C(\cdots) \end{pmatrix} (1, 0, 0)$$

$$= \det \begin{pmatrix} 1 & 0 & Q_1(1, 0, 0) \\ 0 & 0 & q_2 \\ Q_1(1, 0, 0) & q_2 & C(1, 0, 0) \end{pmatrix} = -q_2^2.$$

Therefore $T_0 = \pi \cdot \tilde{\rho}(\tilde{P}_0) \in H \Leftrightarrow q_2 = 0$; i.e. in order to prove 5.8 we have to prove $\tilde{P}_0$ singular $\Leftrightarrow q_2 = 0$.

Consider the equations for $\tilde{X}$:

$$\tilde{u}^2 + 2\tilde{u}\eta + \zeta = 0$$
$$y = \eta x$$
$$z = \zeta x$$
$$\frac{1}{x} F(x, \eta x, \zeta x, u, 1) = \zeta + 2q_2 x + \cdots \text{higher terms} \cdots = 0.$$

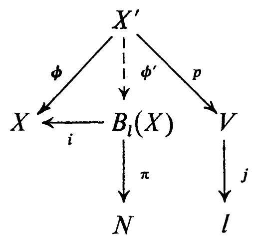
Eliminating $\zeta$ by means of the first equation gives in the last equation

$$\psi(x, \eta, u, \tilde{u}) = 2q_2 x + \cdots \text{terms of degree at least } 2 \cdots = 0.$$

Clearly $\tilde{X}$ is in $\tilde{P}_0$ biregular with the hypersurface $\psi = 0$, and $\tilde{P}_0$ corresponds with the origin $(x = 0, \eta = 0, u = 0, \tilde{u} = 0)$ on $\psi = 0$. Therefore $\tilde{P}_0$ is non-singular on $\tilde{X} \Leftrightarrow \psi$ starts with a linear term $\Leftrightarrow q_2 \neq 0$.

## 6. The birational transformation $\tilde{\phi} : X' \to \tilde{X}$

6.1. From the rational transformation $\phi : X' \to X$ of 4.1 we get the rational transformation $\phi' : X' \to B_l(X)$ as indicated; i.e. $\phi' = i^{-1} \cdot \phi$:




PROPOSITION (6.2). *$\phi'$ is everywhere regular.*

PROOF. By 4.2 $\phi$ is everywhere regular and $i^{-1}$ is an isomorphism outside $l$, so we have only to consider points $P'_0 \in X'$ such that $P_0 = \phi(P'_0) \in l$. Take a generic point $P' \in X'$, put $S = j \cdot p(P')$, $m = p(P')$, $P = \phi(P')$. Let $L$ be the linear 2-dim. space spanned by $l$ and $m$; we have $X \cdot L = l + K$ with a conic $K$. In $L$ we have the picture

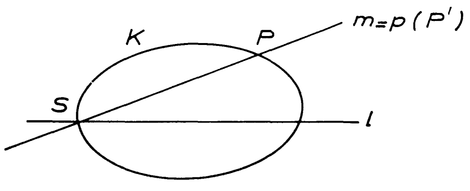



Figure 3

Let $P_0' \in X'$ be as described above and $Q_0 \in B_l(X)$ such that $(P_0', Q_0)$ is on the graph of the rational transformation $\phi'$. By Zariski's main theorem it suffices to see that $Q_0$ is uniquely determined by $P_0'$. Extend the specialization $(P', \phi'(P')) \to (P_0', Q_0)$ to a specialization $(P', \phi'(P'), m, P, L, K, S) \to (P_0, Q_0, m_0, P_0, L_0, K_0, S_0)$, where by $L$, $K$, etc. we mean – of course – Chow point of $L$, etc. We note that $m_0 = p(P_0')$, $P_0 = \phi(P_0')$ and $S_0 = j \cdot p(P_0')$ are uniquely determined by $P_0'$.

We can assume that the equation of $X$ is (16); introduce on $B_l(X)$ coordinates as in 5.4. We have $Q_0 = (0:0:0:0:u_0:v_0; \xi_0:\eta_0:\zeta_0)$. The ratio $u_0:v_0$ is determined by $P_0 = (0:0:0:u_0:v_0)$; i.e. we have to see that $T_0 = \pi(Q_0) = (\xi_0:\eta_0:\zeta_0)$ is uniquely determined, i.e. that $L_0 = L_{T_0}$ is uniquely determined.

In case $m_0 \neq l$ then $L_0$ is the span of $l$ and $m_0$. Assume therefore that $m_0 = l$. From $L \cdot X = l + K$ we get $L_0 \cdot X = l + K_0$. Furthermore from $K \cdot m = S + P$ and since $l$ is not a component of $K_0$ (1.25), we get $K_0 \cdot l = K_0 \cdot m_0 = S_0 + P_0$. Therefore $K_0 \cap l$ is uniquely determined by $P_0'$; from (24) we see that this intersection is given by $u^2\xi_0 + 2uv\eta_0 + v^2\zeta_0 = 0$; hence $\xi_0:\eta_0:\zeta_0$ is uniquely determined.

6.3. *Definition of* $\psi$ *and* $\tilde{\phi}$. Let $P_0' \in X'$ and $T_0 = \pi \cdot \phi'(P_0')$, consider also the corresponding 2-plane $L_{T_0}$ and $X \cdot L_{T_0} = l + K_{T_0}$. Put $S_0 = j \cdot p(P_0')$; by construction of $\phi'$ (or rather of $\phi$) we have $S_0 \in K_{T_0} \cap l$ (cf. also fig. 3). Therefore $(T_0, S_0) \in \tilde{N}$ (see (38)). Introduce $\psi : X' \to \tilde{N}$ by $\psi(P_0') = (\pi \cdot \phi'(P_0'), j \cdot p(P_0'))$ and $\tilde{\phi} : X' \to \tilde{X}$ by $\tilde{\phi} = \phi' \times \psi$. We have the following commutative diagram:

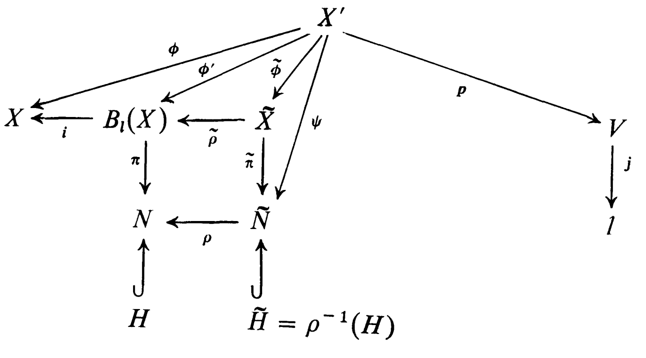



Diagram A

**Proposition** (6.4). *$\tilde{\phi}$ is an everywhere regular birational transformation.*

**Proof.** By 6.2 the $\psi$ and $\phi'$, and hence also $\tilde{\phi}$, are everywhere regular. In order to see that $\tilde{\phi}$ is birational, take $\tilde{P} \in \tilde{X}$ generic and let $P = i \cdot \tilde{\rho}(\tilde{P})$. By 4.6 there are two points $P_i' \in X'$ such that $\phi(P_i') = P$ $(i = 1, 2)$. Furthermore if $T = \pi \cdot \tilde{\rho}(\tilde{P})$ then the points $S_i = K_T \cap l$

are the points $j \cdot p(P_i')$ ($i = 1, 2$; see the proof of 4.6). Let $\tilde{P} = (\tilde{\rho}(\tilde{P}), \tilde{T})$ with $\tilde{T} = (T, S_1)$ (see (38)). From the definition of $\psi$ we see that $P_1'$ is then the unique point of $X'$ such that $\tilde{\phi}(P_i') = \tilde{P}$.

6.5. *Investigation of* $\tilde{\phi}^{-1}$.

LEMMA (6.5). *$\tilde{\phi}^{-1}$ is regular outside $\tilde{\mathscr{B}}$, where $\tilde{\mathscr{B}}$ is the set on $\tilde{X}$ introduced in* 5.6. (*Remark*: *$X'$ is a desingularization of $\tilde{X}$; by* 5.8 *$\tilde{\mathscr{B}}$ is the set of singular points of $\tilde{X}$, therefore* 6.5 *is the best we can expect from the point of 'equality' of $X'$ and $\tilde{X}$.*).

PROOF. Let $\tilde{P}_0 \in \tilde{X}$, $\tilde{P}_0 \notin \tilde{\mathscr{B}}$ and $P_0' \in X'$ such that $\tilde{\phi}(P_0') = \tilde{P}_0$. Since $\tilde{P}_0$ is non-singular by 5.8, it *suffices*, by Zariski's main theorem, *to see that there are only finitely many points $P_0'$ possible.*

Start again with a generic point $\tilde{P} \in \tilde{X}$. Introduce $P' = \tilde{\phi}^{-1}(\tilde{P})$, $P = i \cdot \tilde{\rho}(\tilde{P}) = \phi(P')$, $m = p(P')$, $T = \pi \cdot \tilde{\rho}(\tilde{P})$ and $S = j \cdot p(P')$ and also the 2-plane $L_T$ and $L_T \cdot X = l + K_T$. Extend the specialization $(\tilde{P}, P') \to (\tilde{P}_0, P_0')$ to a specialization $(\tilde{P}, P', P, m, S, T, L_T, K_T) \to (\tilde{P}_0, P_0', P_0, m_0, S_0, T_0, L_{T_0}, K_{T_0})$. Note that $P_0 = i \cdot \tilde{\rho}(\tilde{P}_0)$, $T_0 = \pi \cdot \tilde{\rho}(\tilde{P}_0)$, $L_{T_0}$, $K_{T_0}$ and $S_0$ (via $\tilde{\pi}(\tilde{P}_0)$) are uniquely determined by $\tilde{P}_0$. Furthermore:

$$L_{T_0} \subset \text{tangent hyperplane to } X \text{ at } S_0,$$
$$m_0 \subset L_{T_0}.$$

CASE 1: $P_0 \neq S_0$. Since $j \cdot p(P_0') = S_0$, we have $P_0' \in X'_{S_0}$ (see 4.3). Since $\phi(P_0') = P_0 \neq S_0$ we have that $P_0'$ is unique by lemma 4.5(i).

CASE 2: $P_0 = S_0$. From the construction of $\phi$ in 4.1 we have

$$m \cdot K_T = S + P,$$

hence either $m_0 \cdot K_{T_0} = S_0 + P_0 = 2S_0$ or $m_0 \subset K_{T_0}$. This gives the following possibilities (keeping in mind that now $m_0$ is tri-tangent to $X$ in $S_0$):

(a) $K_{T_0}$ non-degenerated:

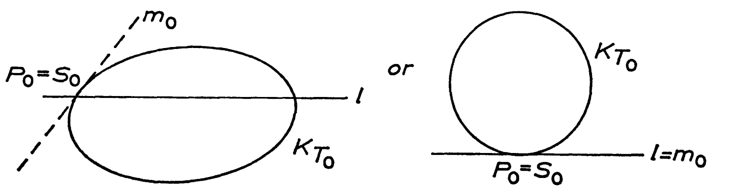
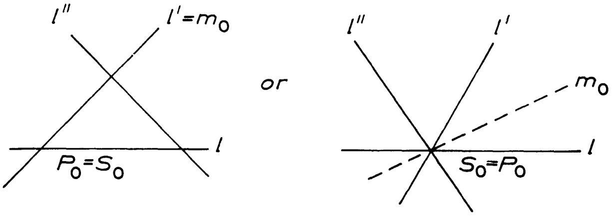



Figure 4

(b) $K_{T_0}$ degenerated:




Case $b_1$ Case $b_2$

Figure 5

However case ($b_2$) does not occur because then $\tilde{P}_0 \in \tilde{\mathscr{B}}$ (see 5.6). In the cases (a) and ($b_1$) we see that $m_0$ is *unique*. Finally we have also $P'_0 \in p^{-1}(m_0)$ and, using the notation $\phi_{S_0}$ of 4.3, $P'_0 \in \phi_{S_0}^{-1}(S_0)$. Therefore we get by 4.5 (ii)

$$P'_0 \in p^{-1}(m_0) \cap \phi_{S_0}^{-1}(S_0) = p^{-1}(m_0) \cap p^{-1}[W_{S_0}].$$

Now the right hand side clearly is a finite set and this completes the proof of 6.5.

6.6. *The set $\mathscr{B}'$ on $X'$.* Put $\mathscr{B}' = \tilde{\phi}^{-1}(\tilde{\mathscr{B}})$. More particularly, recall (see 5.7) that every point $\tilde{P} \in \tilde{\mathscr{B}}$ is uniquely determined by the point $T = \pi \cdot \tilde{\rho}(\tilde{P})$ and $T \in \mathscr{B} \subset N$.

For every $T \in \mathscr{B}$ put $\mathscr{B}'_T = \tilde{\phi}^{-1}(\tilde{P})$, i.e., $\mathscr{B}'_T = \tilde{\phi}^{-1}(\tilde{\mathscr{B}}_T)$, then

$$\mathscr{B}' = \bigcup_{T \in B} \mathscr{B}'_T.$$

Recall that $\mathscr{B}$ and $\tilde{\mathscr{B}}$ are finite sets (but $\mathscr{B}'$ is not!).

Lemma 6.7. *Every component of $\mathscr{B}'_T$, and hence of $\mathscr{B}'$ itself, has dimension at most one.*

Proof. Let $P'_0 \in X'$ be such that $\tilde{\phi}(P'_0) = \tilde{P}_0 \in \tilde{\mathscr{B}}$. Put $T_0 = \pi \cdot \tilde{\rho}(\tilde{P}_0)$ and $S_0 = j \cdot p(P'_0)$, then $\phi(P'_0) = i \cdot \tilde{\rho} \cdot \tilde{\phi}(P'_0) = S_0$ by the definition of $\tilde{\mathscr{B}}$. Note that $S_0$ is determined by $T_0$ alone because $K_{T_0} = l'_{T_0} + l''_{T_0}$ and $S_0 = l \cap l'_{T_0} \cap l''_{T_0}$. With the notation of 4.3 we have $P'_0 \in \phi_{S_0}^{-1}(S_0)$. By 4.5 (ii) we have

$$\tilde{\phi}^{-1}(\tilde{P}_0) = \mathscr{B}'_{T_0} \subset \phi_{S_0}^{-1}(S_0) = p^{-1}[W_{S_0}] \tag{45}$$

and $p^{-1}[W_{S_0}]$ has dimension one.

Remarks (6.8) (i) Contrary to what happens in 4.5 (ii) we don't have equality in (45). It is possible, however, to make a more precise statement. Namely in the present case $W_{S_0}$ is degenerated; let $W^*_{S_0}$ denote that component of $W_{S_0}$ containing $l$ and $l'_{T_0}$ (and hence also containing $l''_{T_0}$). Then:

$$\tilde{\phi}^{-1}(\tilde{P}_0) = p^{-1}[W^*_{S_0}].$$

We don't need this fact.

(ii) We know from 6.4 that $X'$ is a *desingularization* of $\tilde{X}$. However it is not a desingularization obtained by blowing up the singular points, because then $\tilde{\phi}^{-1}(\tilde{P}_0)$ would have dimension 2. [5]

## 7. The involutions on $X'$ and $\tilde{X}$

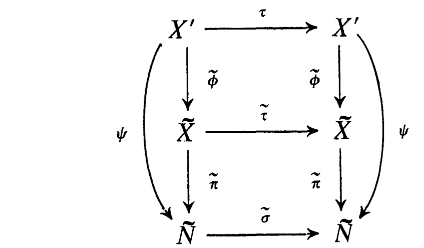
7.1. By 4.6 the rational transformation $\phi : X' \to X$ is generically two to one. Let $P$ be a generic point of $X$, put $\phi^{-1}(P) = P'_1 \cup P'_2$. We get a *birational transformation*

$$\tau : X' \to X'$$

by putting $\tau(P'_1) = P'_2$ and clearly $\tau$ is an *involution*.

Geometrically $\tau$ can be described as follows. Start with (a non necessarily generic) point $P' \in X'$, $P' \notin \phi^{-1}(l)$. Then $P = \phi(P')$ determines a 2-plane $L$ through $l$. Through $P$ there are two lines $m_1$ and $m_2$ tangent to $X$ in points $S_1$ and $S_2$ (see fig. 2 in 4.6). Then $p(P')$ is either $m_1$ or $m_2$; say $p(P') = m_1$. By 4.5 (i) we get a unique point $P'_2 \in X'$ such that $j \cdot p(P'_2) = S_2$ and $\phi(P'_2) = P$. Then $\tau(P') = P'_2$.

7.2. By construction of $\tau$, $\tilde{\tau}$ and $\tilde{\sigma}$ (see 5.3) we have commutative diagrams




Proposition (7.3). (i) *The birational transformation $\tau$ is biregular outside $\mathscr{B}'$ (see 6.6).*

(ii) *For $T \in \mathscr{B}$ on $N$ (see (44) and 6.6):*

$$\tau^{-1}(\mathscr{B}'_T) = \mathscr{B}'_T.$$

Proof. Let $P'_1$ be generic on $X'$ and $\tau(P'_1) = P'_2$. Let $(P'_1, P'_2) \to (P'_{1*}, P'_{2*})$ be a specialization. Put $\tilde{P}_i = \tilde{\phi}(P'_i)$, $i = 1, 2$; extend the specialization to $(P'_1, P'_2, \tilde{P}_1, \tilde{P}_2) \to (P'_{1*}, P'_{2*}, \tilde{P}_{1*}, \tilde{P}_{2*})$. Since $\tilde{\phi}$ is everywhere regular we get $\tilde{P}_{i*} = \tilde{\phi}(P'_{i*})$ for $i = 1, 2$. Furthermore $\tilde{\tau}(\tilde{P}_1) = \tilde{P}_2$; since $\tilde{\tau}$ is regular we get $\tilde{\tau}(\tilde{P}_{1*}) = \tilde{P}_{2*}$.

[5] See footnote 4.

(i) Follows from the fact that $\tilde{\mathscr{B}}$ is invariant under $\tilde{\tau}$ (5.7), from $\tilde{\phi}^{-1}$ regular outside $\tilde{\mathscr{B}}$ (6.5), from Zariski's main theorem and from $\tilde{\tau}$ biregular.

(ii) Follows from the fact (5.7) that $\tilde{P} \in \tilde{\mathscr{B}}_T$ is invariant under $\tilde{\tau}$ and from the definition of $\mathscr{B}'_T$.

## 8. Behaviour of some cycles on $X'$ under the involution

8.1. Let $\tilde{T} \in \tilde{N}$ and consider the *cycle*

$$\psi^{-1}(\tilde{T}) = \mathrm{pr}_{X'}(\Gamma_\psi \cdot (X' \times \tilde{T})).$$

For the notations in the next lemma see diagram A in 6.3 for the various morphisms, see 1.25 for $H$, $L_T$, $K_T$, see 5.3 for the ramification divisor $\Delta$ and for $\tilde{\Delta}$ and recall that $\phi^{-1}[\cdots]$ etc. means *proper* transform.

LEMMA (8.1).

(i) $\psi^{-1}(\tilde{T})$ *is defined for all* $\tilde{T} \in \tilde{N}$.

(ii) $\psi^{-1}(\tilde{T}) \sim \psi^{-1}(\tilde{T}_*)$ *with rational equivalence; all* $\tilde{T}$ *and* $\tilde{T}_*$ *on* $\tilde{N}$.

(iii) *Let* $T \in N$, $T \notin H \cap \Delta$ *and* $\rho^{-1}(T) = \tilde{T}_1 + \tilde{T}_2$ *(of course it may happen that* $\tilde{T}_1 = \tilde{T}_2$*). Then as point sets:*

$$\psi^{-1}(\tilde{T}_1) \cup \psi^{-1}(\tilde{T}_2) = \phi^{-1}[K_T].$$

(iv) *Let* $\tilde{T} \in \tilde{N} - \tilde{H}$, *then* $\psi^{-1}(\tilde{T})$ *consists of one absolutely irreducible variety, counted with multiplicity one.*

(v) *For* $T \in \tilde{H} - \tilde{H} \cap \tilde{\Delta}$ *the cycle* $\psi^{-1}(\tilde{T})$ *consists of two absolutely irreducible components* $\Gamma(\tilde{T})$ *and* $\Omega(\tilde{T})$ *each counted once, i.e. (as cycles):*

$$\psi^{-1}(\tilde{T}) = \Gamma(\tilde{T}) + \Omega(\tilde{T})\,[6] \tag{46}$$

*Moreover if* $\tilde{T} = (T, S)$ *with* $T \in N$ *and* $S \in K_T \cap l$, *then* $S \in \phi(\Gamma(\tilde{T}))$ *and* $S \notin \phi(\Omega(\tilde{T}))$.

(vi) *For* $\tilde{T} \in \tilde{H} - \tilde{H} \cap \tilde{\Delta}$, *put* $\tilde{T} = (T, S)$, $K_T = l'_T + l''_T$ *and let* $S \in l'_T \cap l$. *Then as varieties* $\Gamma(\tilde{T}) = p^{-1}(l'_T)$*(where* $l'_T$ *should be considered as a point on* $\hat{H}$*).*

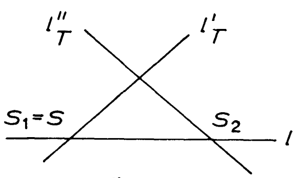



Figure 6

[6] We use the symbols $\Gamma$ and $\Omega$ only to distinguish the two components; the distinction is given by the last property in 8.1 (v) Actually $\Gamma(\tilde{T})$ and $\Omega(\tilde{T})$ are both rational over the field $k(\tilde{T})$(see 5.2 (ii)) and therefore it is possible to introduce cycles $\Gamma$ and $\Omega$ on $\tilde{H} \times X'$, but we don't need this point of view.

PROOF. (i) We have to check that every component of the *set* $\psi^{-1}(\tilde{T})$ has (at most) dimension one. As point sets

$$\psi^{-1}(\tilde{T}) = \tilde{\phi}^{-1}(\tilde{\pi}^{-1}(\tilde{T})).$$

Now $\tilde{\pi}^{-1}(\tilde{T})$ has only components of dimension one (see 5.2 and 5.1). Then the assertion follows from 6.5, from the fact that $\tilde{\mathscr{B}}$ is a finite set (5.7) and from 6.7.

(ii) Obvious since $\tilde{N}$ is a quadric surface (5.5).

(iii) If we exclude $\tilde{T} \in \tilde{H} \cap \tilde{\Delta}$ then $\tilde{\phi}$ is an isomorphism (6.5). Therefore we can replace in assertion (iii) the $X'$ by $\tilde{X}$, $\psi$ by $\tilde{\pi}$ and $\phi$ by $i \cdot \tilde{\rho}$. Then as point sets

$$\tilde{\pi}^{-1}(\tilde{T}_1) \cup \tilde{\pi}^{-1}(\tilde{T}_2) = (\rho \cdot \tilde{\pi})^{-1}(T) = \tilde{\rho}^{-1}(\pi^{-1}(T)).$$

By 5.1, $\pi^{-1}(T) = i^{-1}[K_T]$ and from this we get, since $\tilde{\rho}$ is finite:

$$\tilde{\rho}^{-1}(\pi^{-1}(T)) = \tilde{\rho}^{-1}(i^{-1}[K_T]) = (i \cdot \tilde{\rho})^{-1}[K_T].$$

(iv) Replace again $X'$ by $\tilde{X}$ and use 5.2 (i).

(v) Replace again $X'$ by $\tilde{X}$. Let $\tilde{T} = (T, S)$ with $T \in N$, then $T = \rho(\tilde{T})$, $K_T = l'_T + l''_T$, $S = S_1 = l'_T \cap l$ and $S_2 = l''_T \cap l$ (see fig. 6). Put $\tilde{T}_1 = \tilde{T} = (T, S_1)$ and $\tilde{T}_2 = (T, S_2)$.

By (iii) we have as sets:

$$\tilde{\pi}^{-1}(\tilde{T}_1) \cap \tilde{\pi}^{-1}(\tilde{T}_2) = \tilde{\rho}^{-1}(i^{-1}[l'_T]) \cup \tilde{\rho}^{-1}(i^{-1}[l''_T]).$$

Put

$$\begin{aligned} \Gamma(\tilde{T}) &= \tilde{\rho}^{-1}(i^{-1}[l'_T]) \cap \tilde{\pi}^{-1}(\tilde{T}), \\ \Omega(\tilde{T}) &= \tilde{\rho}^{-1}(i^{-1}[l''_T]) \cap \tilde{\pi}^{-1}(\tilde{T}). \end{aligned} \tag{47}$$

This gives the required decomposition of $\psi^{-1}(\tilde{T})$ as point sets. The fact that both are irreducible varieties and have multiplicity one in the cycle has already been remarked in 5.2 (ii). Finally the assertion that $S \in \phi(\Gamma(\tilde{T}))$ and $S$ not in $\phi(\Omega(\tilde{T}))$ follows directly from (47)(cf. figure 6).

(vi) From the description of $\Gamma(\tilde{T})$ in (47) and the definition of $\phi$ in 4.1) follows that for $P'_0 \in X'$ we have

$$P'_0 \in \Gamma(\tilde{T}) \Leftrightarrow p(P'_0) = l'_T.$$

REMARK (8.2). For use later on we restate (47) on $X'$ itself (instead of on $\tilde{X}$). First note that $\tilde{\rho}^{-1}(i^{-1}[l'_T]) = (i \cdot \tilde{\rho})^{-1}[l'_T]$ and similar for $l''_T$. For $\tilde{T} = (T, S) \in \tilde{H} - \tilde{H} \cap \tilde{\Delta}$ we have, if $S = l'_T \cap l$:

$$\begin{aligned} \Gamma(\tilde{T}) &= \phi^{-1}[l'_T] \cap \psi^{-1}(\tilde{T}) \\ \Omega(\tilde{T}) &= \phi^{-1}[l''_T] \cap \psi^{-1}(\tilde{T}) \end{aligned} \tag{47'}$$

LEMMA 8.3. *If* $\tau$ *(resp.* $\tilde{\sigma}$*) is the involution on* $X'$ *(resp. on* $\tilde{N}$*; see* 7.1 *and* 5.3*) then*:

(i) $\tau^{-1}(\psi^{-1}(\tilde{T})) = \psi^{-1}(\tilde{\sigma}(\tilde{T}))$ *for all* $\tilde{T} \in \tilde{N}$ *with* $\tilde{T} \notin \tilde{H} \cap \tilde{\Delta}$.

(ii) *For* $\tilde{T} \in \tilde{H} - \tilde{H} \cap \tilde{\Delta}$ *we have*:

$$\tau^{-1}(\Gamma(\tilde{T})) = \Omega(\tilde{\sigma}(\tilde{T}))$$
$$\tau^{-1}(\Omega(\tilde{T})) = \Gamma(\tilde{\sigma}(\tilde{T})).$$

PROOF. For both assertions we can replace $X'$ by $\tilde{X}$ since $\tilde{T} \notin \tilde{H} \cap \tilde{\Delta}$; then $\psi$ is replaced by $\tilde{\pi}$ and $\tau$ by $\tilde{\tau}$. Note also that $\tilde{\sigma}^{-1} = \tilde{\sigma}$.

(i) Follows at once from the commutative diagram of *morphisms*:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\tau}} & \tilde{X} \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ \tilde{N} & \xrightarrow{\tilde{\sigma}} & \tilde{N} \end{array}$$

(ii) From the above commutativity and (46) in 8.1 (v) follows that $\tau^{-1}(\Gamma(\tilde{T}))$ either equals $\Gamma(\tilde{\sigma}\tilde{T})$ or $\Omega(\tilde{\sigma}\tilde{T})$. Let $\tilde{T} = (T, S_1)$ with $T \in N$ and $S_1 = l'_T \cap l$, then $\tilde{\sigma}\tilde{T} = (T, S_2)$ with $S_2 = l''_T \cap l$ (see fig. 6). By (47) we have (on $\tilde{X}$):

$$\Gamma(\tilde{T}) = \{(P, (T, S_1)); P \in i^{-1}[l'_T]\}$$
(48)
$$\Omega(\tilde{T}) = \{(P, (T, S_1)); P \in i^{-1}[l''_T]\}$$

Now

$$\tilde{\tau}^{-1}(\Gamma(\tilde{T})) = \{(P, (T, S_2)); P \in i^{-1}[l'_T]\}$$

because $\tilde{\tau}$ operates only on the second factor of $\tilde{X} = B_l(X) \times_N \tilde{N}$.

By (48) we have

$$\Omega(\tilde{\sigma}\tilde{T}) = \{(P, (T, S_2)); P \in i^{-1}[l'_T]\}$$

and hence

$$\tau^{-1}(\Gamma(\tilde{T})) = \Omega(\tilde{\sigma}\tilde{T}).$$

## 9. Relations between the curves $H$, $\mathscr{H}$, $\hat{H}$ and $\tilde{H}$

9.1. Recall that, after fixing a line $l$ (see 1.20 and 2.1), we have introduced the curves:

(i) $H = \{T; T \in N$ such that $K_T = l'_T + l''_T\}$, see 1.21.

(ii) $\mathscr{H}$: the curve on the Fano surface $\mathscr{F}$ consisting of all lines $l'$ on $X$ such that $l' \cap l \neq \emptyset$, see 1.23.

(iii) $\hat{H}$: a curve on $V$ obtained via the lines $l'$ on $X$ meeting $l$, see 2.3.

(iv) $\tilde{H} = \rho^{-1}(H)$, curve on $\tilde{N}$ (see 5.2).

9.2. We have the following relations between these curves:

(a) $\mathscr{H}$ and $\hat{H}$ are non-singular curves (see 1.25 iv and 2.5); there is a (canonical) birational transformation between these curves (see 2.4).

In the following we identify $\mathscr{H}$ and $\hat{H}$ by means of this biregular transformation.

(b) $\mathscr{H}$, and hence $\hat{H}$, is an étale covering of $H$ of degree 2 (see 1.25 (iv)). Let $\sigma$ be *the involution on* $\hat{H}$, *or* $\mathscr{H}$, *over* $H$. Let $L$ be the 2-plane spanned by $l$ and $l'$ and $L \cdot X = l+l'+l''$. Then $\sigma$ was defined (see 1.23) as follows: $\sigma$ interchanges $l'$ and $l''$.

(c) There is a birational transformation:

$$\lambda : \hat{H} \to \tilde{H}.$$

Namely if $l' \in \hat{H} = \mathscr{H}$ then $\lambda(l') = (T, S)$ where $L_T = \text{span}\ (l, l')$ and $S = l \cap l'$.

It is clear that $\lambda$ is a canonical birational transformation; it is regular and commutes with the involutions $\sigma$ (see 9.2b) and $\tilde{\sigma}$ (see 5.3).

Note however that $\tilde{H}$ may have (and has!) singularities in $\tilde{H} \cap \tilde{\Delta}$ and $\lambda$ is not *bi*regular. Finally put

$$B = \lambda^{-1}(\tilde{H} \cap \tilde{\Delta}).$$

9.3. Let $h \in \hat{H}$, $h \notin B$. Consider $\lambda(h) \in \tilde{H}$. Instead of writing $\Gamma(\lambda h)$ and $\Omega(\lambda h)$ we write shortly $\Gamma(h)$ and $\Omega(h)$. Then 9.2(c) and lemma 8.3 (ii) give immediately:

LEMMA (9.4). *For* $h \in \hat{H}$, $h \notin B$ *we have*:

$$\tau^{-1}(\Gamma(h)) = \Omega(\sigma h)$$
$$\tau^{-1}(\Omega(h)) = \Gamma(\sigma h).$$

## 10. Algebraic equivalence modulo rational equivalence

10.1. Let $A(X)$ (resp. $A(X'), \cdots$) denote the *Chow ring* of $X$ (resp. $X', \cdots$) in the sense of Chow [2]. From $\phi : X' \to X$ (see 4.1) we get homomorphisms

$$A(X) \xrightarrow{\phi^*} A(X')$$

and

$$A(X') \xrightarrow{\phi_*} A(X).$$

Note that $\phi^*$ is obtained via inverse images of cycles and $\phi^*$ is a homomorphism of *rings* ([2], th. 4, p. 471), whereas $\phi_*$ is obtained via images of cycles and $\phi_*$ is a homomorphism of *additive groups* ([2], th. 3, p. 468; cf. also [10], Chap VIII, § 4). From the definitions of $\phi^*$ and $\phi_*$ follows at once:

$$\phi_* \cdot \phi^* = \text{multiplication by 2.} \tag{49}$$

Next we want to investigate the behaviour of the involution $\tau : X' \to X'$ on the Chow ring $A(X')$. This is complicated by the fact that $\tau$ is only a birational *transformation* and not a morphism. Nevertheless from the correspondence class of the graph $\Gamma_\tau$ of $\tau$ on $X' \times X'$ we get:

LEMMA (10.2). (i) $\Gamma_\tau$ *induces a homomorphism of additive groups*

$$\tau^* : A(X') \to A(X').$$

(ii) $\tau^*$ *is an involution, i.e.* $(\tau^*)^2 = id$.

PROOF. (i) From [2], th. 3, p. 468.

(ii) By the theory of composition of correspondence classes in the Chow ring we have: $(\tau^*)^2$ is in $A(X' \times X')$ given by the correspondence class

$$\mathrm{pr}_{13}\{\mathrm{Class}\,(\Gamma_\tau \times X') \cdot \mathrm{Class}\,(X' \times \Gamma_\tau)\}.$$

Now we claim that

$$(\Gamma_\tau \times X') \cap (X' \times \Gamma_\tau)$$

intersects properly (i.e. has 'correct' dimensions) in $X' \times X' \times X'$. In fact let

$$(\Gamma_\tau \times X') \cap (X' \times \Gamma_\tau) = \nabla \cup D_1 \cup \cdots \cup D_h,$$

where $\nabla$ is the locus of $(P, \tau(P), P)$ over $k$, with $P$ generic on $X'$ over $k$, and where the $D_i$ are extra components due to the fact that $\tau$ is not regular. By general intersection theory dim $D_i \geqq 3$. To prove: dim $D_i \leqq 3$. If $(P_i, P_i', P_i'')$ is a generic point of $D_i$ then it follows from prop. 7.3 that *and* $P_i$ *and* $P_i'$ *and* $P_i''$ are in $\mathscr{B}'_T$ for a certain $T \in \mathscr{B}$ on $N$ (5.6); furthermore we have of course $(P_i, P_i') \in \Gamma_\tau$ and $(P_i', P_i'') \in \Gamma_\tau$. The fact that dim $D_i \leqq 3$ follows now from lemma 6.7. From this lemma *follows moreover* that for such a generic point of $D_i$ the $P_i$, $P_i'$ and $P_i''$ are independent transcendental over $k$, each of transcendence degree 1. But then, by definition of projection, $\mathrm{pr}_{13} D_i = 0$. Therefore $(\tau^*)^2$ is given by $\mathrm{pr}_{13}$ (Class $\nabla$), hence $(\tau^*)^2 = \mathrm{id}$.

10.3. Since $X' = B_{\hat{H} \cup I}(V)$ we have (see [9], prop 13, p. 482) for the additive structure:

$$A(X') = A(V) + {}^*A(\hat{H}) + {}^*A(I) \tag{50}$$

where $^*A$ means that the codimension has been increased by one, which in our present case amounts to increasing the dimension by one.

The map $^*A(\hat{H}) \to A(X')$ is obtained in the following way:

$$\begin{array}{ccc} D & \longrightarrow & X' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \hat{H} & \longrightarrow & V \end{array}$$

If $Z$ is a subvariety on $\hat{H}$, i.e., $\hat{H}$ itself or a point on $\hat{H}$, then

$$\text{class } Z \mapsto \text{class } p^{-1}(Z),$$

where $p^{-1}(Z)$ is the corresponding subvariety of $X'$. By linearity this map is extended to cycles. Now *note that this is the map occuring in lemma* 8.1 (vi), where $l'_T$ should be considered as a *point* on $\hat{H}$.

10.4. From now on we consider only the cycles *algebraically equivalent to zero modulo rational equivalence*

$$A^{\mathrm{alg}}(\cdots) \hookrightarrow A(\cdots);$$

we have also ([9], prop. 13, p. 482):

$$A^{\mathrm{alg}}(X') = A^{\mathrm{alg}}(V) + {}^*A^{\mathrm{alg}}(\hat{H}) + {}^*A^{\mathrm{alg}}(I). \tag{50'}$$

By [4], condition *I*-11 on pages 4–14 and 4–35 we have $A^{\mathrm{alg}}(V) = 0$. Also since I is birational equivalent with a line (lemma 2.4) we have $A^{\mathrm{alg}}(I) = 0$.

Hence the only non-trivial part of $A^{\mathrm{alg}}(X')$ is $A_1^{\mathrm{alg}}(X')$: *the* 1-*dimensionnal cycles* (the subscript means dimension) and (50′) reads:

$$A_1^{\mathrm{alg}}(X') \underset{\alpha}{\approx} A_0^{\mathrm{alg}}(\hat{H}) = J_0(\hat{H}), \tag{51}$$

where $J(\hat{H})$ is the Jacobian of $\hat{H}$ and $J_0$ means the part belonging to the cycles of degree 0.

Proposition (10.5). *Let $Z(t)$ be a family of* 1-*dimensional cycles on $X$, algebraically equivalent to zero, parametrized by an irreducible variety $T$ and obtained, as the notation indicates, by means of a cycle $Z$ on $T \times X$. Then there exists a rational transformation $\rho : T \to J_0(\hat{H})$, such that for every non-singular point $t_0 \in T$ we have*

$$\alpha \cdot \rho(t_0) = \phi^*\{\text{Class } Z(t_0)\},$$

*whenever $Z(t_0)$ is defined. Moreover if $k$ is a field of definition as in* 2.1, *such that also $T$ is defined over $k$ and $Z$ rational over $k$, then $\rho$ is defined over $k$.*

Proof. See section 11.

Proposition (10.6). *Let $\eta \in J_0(\hat{H})$; consider $\xi = \alpha(\eta)$ in $A(X')$. Furthermore $\sigma$ denotes the involution of $\hat{H}$ over $H$ (see* 9.2 *(b)) and $\tau$ is the involution on $X'$ (see* 7.1*). Then*

$$\tau^*(\xi) = -\alpha(\sigma^*\eta). \tag{52}$$

Proof. The statement does not depend on the groundfield; we can assume that $k$ is algebraically closed. We have $\eta = \text{class } (\sum_i h_i)$, where the

$h_i$ are points on $\hat{H}$ and deg $(\sum_i h_i) = 0$. We can assume that the cycle $\sum_i h_i$ is choosen within its *linear equivalence class* in such a way that $h_i \notin B$, where $B$ is the set on $\hat{H}$ introduced in 9.2 (c). According to 10.3 we have $\xi =$ class $(\sum_i p^{-1}(h_i))$, where $p : X' \to V$ is the morphism of 3.1. Using the birational transformation between $\hat{H}$ in $V$ and $\tilde{H}$ in $\tilde{N}$ (see 9.2 (c)), which is biregular outside $B$, we can write by lemma 8.1 (vi) $\xi =$ class $(\sum_i \Gamma(h_i))$ where $\Gamma$ is the symbol introduced in 8.1 (v). Therefore $\tau^*(\xi) =$ class $\{\tau^{-1}(\sum_i \Gamma(h_i))\}$. Furthermore by lemma 9.4 $\tau^{-1}(\Gamma(h_i)) = \Omega(\sigma h_i)$. Now take a fixed $\tilde{T}_0 \in \tilde{N} - \tilde{H}$, then by lemma 8.1 (i):

$$\Omega(\sigma h_i) \sim \psi^{-1}(\tilde{T}_0) - \Gamma(\sigma h_i) \text{ (\textit{rational} equivalence).}$$

Hence

$$\tau^*(\xi) = \text{class } \{\tau^{-1}(\sum_i \Gamma(h_i))\} = \text{class } \{\sum_i \psi(\tilde{T}_0) - \sum_i \Gamma(\sigma h_i)\},$$

and finally using deg $(\sum_i h_i) = 0$:

$$\tau^*(\xi) = \text{class } (-\sum_i \Gamma(\sigma h_i)) = -\alpha(\sigma^* \eta)$$

10.7. For étale coverings $\hat{H} \to H$ of degree 2 Mumford has developed a *theory of Prym varieties* ([5]). From this theory we use the following facts. The Prym variety Prym $(\hat{H}/H)$ is an abelian subvariety of $J_0(\hat{H})$. If $\sigma$ denotes, as before, the involution of $\hat{H}$ over $H$, then Prym $(\hat{H}/H)$ is the *connected component*, of the zero element, of the set

$$\{\eta; \eta \in J_0(\hat{H}), \sigma^*(\eta) = -\eta\}. \tag{53}$$

In fact the above set consists of Prym $(\hat{H}/H)$ and one coset of a point of order 2 on the Jacobian.

The following theorem, together with its corollary, is the main result of the paper. By lemma 2.4 (and see also 9.2 (a)) we identify the curve $\mathscr{H}$ on the Fano surface and the curve $\hat{H}$ in $V$ (see 9.1 for the description of these curves). Also, by (51), we identify by means of the isomorphism $\alpha$ the group $A_1^{\text{alg}}(X')$ with the Jacobian $J_0(\mathscr{H})$.

THEOREM (10.8). *With the above identifications we have for the morphism* $\phi : X' \to X$ *of section* 4:

(i) $\phi^*(A_1^{\text{alg}}(X)) =$ Prym $(\mathscr{H}/H)$.

(ii) *The kernel of* $\phi^*$ *consists of elements of order* 2.

PROOF. By (51) we have $\phi^*(A_1^{\text{alg}}(X)) \subset J_0(\hat{H}) = J_0(\mathscr{H})$.

CLAIM A. Im $(\phi^*) \subset$ Prym $(\hat{H}/H)$.

(a) Let $\xi \in A_1^{\text{alg}}(X)$. Then $\tau^*(\phi^*(\xi)) = \phi^*(\xi)$. This follows essentially from the commutative diagram of *rational* maps

$$\begin{array}{ccc} X' & \xrightarrow{\ \tau\ } & X' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & X & \end{array}$$

but there is a complication since $\tau$ is not a morphism. However, let $Z$ be a cycle on $X$ representing the class $\xi$, then it suffices to see that $\phi^{-1}(Z) = \tau^{-1}(\phi^{-1}(Z))$ for *suitable* $Z$. Consider the set $\mathscr{B}'$ of 6.6 on $X'$ and put $X'' = X' - \mathscr{B}'$, then $\tau|X''$ is a morphism by lemma 7.3 (i). Since the set $\tilde{\phi}(\mathscr{B}') = \tilde{\mathscr{B}}$ is finite on $\tilde{X}$ (5.7), certainly $\phi(\mathscr{B}')$ is finite on $X$. Therefore we can choose, by the 'moving lemma', $Z$ within its rational equivalence class in such a way that the support $|Z|$ does not meet $\phi(\mathscr{B}')$. Then $\phi^{-1}(Z) = \tau^{-1}(\phi^{-1}(Z))$ by [10], chap VIII, th. 13a. Hence $\tau^*(\phi^*(\xi)) = \phi^*(\xi)$.

(b) By 10.4 we have $\phi^*(\xi) = \alpha(\eta)$ with $\eta \in J_0(\hat{H})$. Using 10.6 we get

$$\alpha(\eta) = \phi^*(\xi) = \tau^*\phi^*(\xi) = -\alpha(\sigma^*\eta),$$

hence $\sigma^*\eta = -\eta$, i.e. $\eta$ *is in the set* (53). In order to prove claim $A$ it only remains to show, by 10.7, that $\eta$ is in the connected component of zero in that set.

(c) Since $\xi \in A_1^{\mathrm{alg}}(X)$, there exists an irreducible non-singular curve $C$, a cycle $Z$ on $C \times X$ and two points $t_0$, $t_1$ on $C$ such that $Z(t_0)$, $Z(t_1)$ are defined, class $Z(t_0) = 0$ and class $Z(t_1) = \xi$. By proposition 10.5 there is a rational transformation $\rho : C \to J_0(\hat{H})$ such that for all $t \in C$ we have

$$\alpha \cdot \rho(t) = \phi^*\{\text{class } (Z(t))\}.$$

By what we have seen under (b) above we have that $\rho(t)$ is in the set (53) for all $t \in C$; since $\rho(t_0) = 0$ we have $\rho(t) \in \mathrm{Prym}\,(\hat{H}/H)$ for all $t \in C$. Hence $\mathrm{Im}\,(\phi^*) \subset \mathrm{Prym}\,(\hat{H}/H)$.

Claim B. $\mathrm{Im}\,(\phi^*) \supset \mathrm{Prym}\,(\hat{H}/H)$.

By the usual tricks with group varieties, and using 10.5, it suffices to prove that there exists a family $Z(u)$ of 1-cycles on $X$, algebraically equivalent to zero, parametrized by a variety $U$ such that if we take sufficiently many independent generic points $u_i \in U$ then

$$\sum_i \phi^*(\text{Class } Z(u_i))$$

is generic on the Prym $(\hat{H}/H)$.

Now take $U = \hat{H} - B$ (see 9.2 (c)). Using the birational transformation $\lambda : \hat{H} \to \tilde{H}$ we have

$$h \rightleftarrows \tilde{T} \mapsto T = \rho(\tilde{T}),$$

with $h \in U$, $\tilde{T} \in \tilde{H} - \tilde{H} \cap \tilde{\Delta}$ and $T \in H - H \cap \Delta$. Consider the corresponding 2-plane $L_T$ and $L_T \cdot X = l + l'_T + l''_T$. Put $Z(h) = l'_T - l''_T$. Note that this cycle is rational over the field $k(h)$. Furthermore this cycle is clearly algebraically equivalent to zero. We have:

$$\phi^{-1}(l'_T - l''_T) = \phi'^{-1}\{i^{-1}[l'_T] - i^{-1}[l''_T]\} + \phi'^{-1}\{i^{-1}(S_1) - i^{-1}(S_2)\}$$

where $S_1 = l'_T \cap l$, $S_2 = l''_T \cap l$ (see figure 6 in 8.1). Since $S_1$ and $S_2$ are on a line we have:

$$\begin{aligned}\phi^*\{\text{class}\,(l'_T - l''_T)\} &= \text{class}\,\phi'^{-1}\{i^{-1}[l'_T] - i^{-1}[l''_T]\}\\ &= \text{class}\,\{\phi^{-1}[l'_T] - \phi^{-1}[l''_T]\}.\end{aligned}$$

Next using for the right hand side the expressions (47′) in 8.2 we get

$$\phi^*\{\text{class}\,(l'_T - l''_T)\} = \text{class}\,\{\Gamma(h) + \Omega(\sigma h) - \Omega(h) - \Gamma(\sigma h)\}.$$

Fix a point $\tilde{T}_0 \in \tilde{N} - \tilde{H}$; by lemma 8.1 (ii) and (iv) we have with rational equivalence:

$$\begin{aligned}\Omega(h) &\sim \psi^{-1}(\tilde{T}_0) - \Gamma(h)\\ \Omega(\sigma h) &\sim \psi^{-1}(\tilde{T}_0) - \Gamma(\sigma h).\end{aligned}$$

Therefore we finally obtain

$$\phi^*(\text{Class}\ Z(h)) = \phi^*(\text{Class}\,(l'_T - l''_T)) = 2\ \text{Class}\,\{\Gamma(h) - \Gamma(\sigma h)\},$$

i.e., $\phi^*(\text{Class}\ Z(h))$ corresponds with the point $2\{h - \sigma(h)\}$ on $J_0(\hat{H})$. Let $h_i \in \hat{H}$ be independent generic points on $\hat{H}(i = 1, \cdots, q)$ and consider the point on $J_0(\hat{H})$ corresponding with the cycle

$$\sum_{i=1}^{q} \{h_i - \sigma(h_i)\}$$

By the theory of Prym varieties ([5]) this is, for large $q$, a generic point on Prym $(\hat{H}/H)$. Since this remains true after multiplication by 2, this completes the proof of claim *B*.

10.9. From claim *A* and *B* follows the assertion (i) of theorem 10.8. The assertion (ii) follows immediately from (49).

COROLLARY (10.10). *The exact sequence*

$$0 \to \text{Ker}\,(\phi^*) \to A_1^{\text{alg}}(X) \overset{\phi^*}{\to} \text{Prym}\,(\mathscr{H}/H) \to 0$$

*splits, i.e. we have*

$$A_1^{\text{alg}}(X) \simeq \text{Prym}\,(\mathscr{H}/H) \oplus T$$

*and every element of T has order* 2.

REMARK. My original version of 10.10 was somewhat weaker, namely $A_1^{\text{alg}}(X) \simeq B \oplus T$, where $B$ is an abelian variety isogenous to the Prym and differing from it by 2-torsion elements at most. I owe Mumford the improved version, i.e. the fact that the sequence actually splits.

Proof of 10.10. As in the proof of 10.8 we tacitly identify the curves $\mathscr{H}$ and $\hat{H}$ and also $\tilde{H}$ as far as the non-singular points are concerned (i.e., the points outside $\tilde{H} \cap \tilde{\Delta}$), see 9.2.

First note that $\phi^* \cdot \phi_*$ is multiplication by 2 on the Prym $(\mathscr{H}/H)$; this follows from (49) and from the fact that $\phi^*$ is onto the Prym. Put $B = \operatorname{Im}(\phi_*)$. The corollary will follow if we prove that

$$B \cap \operatorname{Ker}(\phi^*) = (0).$$

This in turn will follow, using the remark about $\phi^* \cdot \phi_*$, from the statement:

$$(*) \qquad \operatorname{Prym}(2) \subset \operatorname{Ker}(\phi_*),$$

where Prym (2) is the subgroup of the 2-torsion points on the Prym.

Now we have to use the following fact from the theory of Prym varieties (see [5]). If $\rho : \mathscr{H} \to H$ is the covering and $\sigma$ the involution then

$$\operatorname{Im}[\rho^* : J_0(H) \to J_0(\mathscr{H})] = \{\eta \in J_0(\mathscr{H}); \sigma^*\eta = \eta\}.$$

Since the points of the Prym satisfy $\sigma^*\eta = -\eta$, we have:

$$(**) \qquad \operatorname{Prym}(2) \subset \rho^*(J_0(H)).$$

Consider $\xi \in A_1^{\mathrm{alg}}(X')$, then by (51) we have $\xi = \alpha(\eta)$ with $\eta \in J_0(\mathscr{H})$. Now *suppose* $\eta \in \rho^*(J_0(H))$, i.e., $\eta = \rho^*(\zeta)$ with $\zeta \in J_0(H)$. Let $\zeta =$ class $(\sum_j h_j)$, $h_j \in H$ and $\deg(\zeta) = 0$. Moreover we can assume that $h_j \notin H \cap \Delta$ (cf. 5.3). Looking at $\tilde{H}(= \mathscr{H})$ we have $\rho^{-1}(h_j) = h'_j + h''_j$ with $h'_j$, $h''_j \in \tilde{H}$. Then by 10.3:

$$\xi = \alpha(\eta) = \text{class}\,\{\sum_j p^{-1}(h'_j) + p^{-1}(h''_j)\},$$

where $p : X' \to V$ is the morphism of 3.1. By lemma 8.1 (vi) we have

$$\xi = \text{class}\,\{\sum_j \Gamma(h'_j) + \Gamma(h''_j)\}.$$

By (47′) we have

$$\phi_*\{\Gamma(h'_j) + \Gamma(h''_j)\} = l'_{h_j} + l''_{h_j}.$$

Hence finally we have

$$\phi_*(\xi) = \text{class}\,\{\sum_j (l'_{h_j} + l''_{h_j})\} = \text{class}\,(\sum_j K_{h_j}),$$

where $K_{h_j}$ is the degenerated conic (see prop. 1.25). Since $h_j \in N$, with $N$ a rational variety, and $\deg(\zeta) = 0$, we have $\phi_*(\xi) = 0$. Therefore:

$$(^+) \qquad \rho^*(J_0(H)) \subset \operatorname{Ker}(\phi_*).$$

The statement (*) follows from ($^+$) and (**). This completes the proof.

## 11. Algebraic families of cycles

11.1. *Our purpose is to prove proposition* 10.5. Let $Z(t)$ be a family of 1-cycles on $X$ satisfying the assumptions of prop. 10.5. In the following $k$ denotes a field as in 2.1 and such that moreover $T$ is defined over $k$ and $Z \subset T \times X$ is rational over $k$. It *suffices now*, in order to prove prop. 10.5, *to prove the following two statements* (a) and (b): [7])

(a) If $t$ is a point of $T$, generic over $k$, then the point $\phi^*(\text{Class } Z(t))$ in $J_0(\hat{H})$ is rational over $k(t)$.

From this fact we get a rational map $\rho : T \to J_0(\hat{H})$. By the theory of abelian varieties this map is defined in every non-singular point of $T$.

(b) If $t_0$ is a non-singular point of $T$ then $\rho(t_0) = \phi^*(\text{Class } Z(t_0))$. In the following $t$ (resp. $t_0$) denotes a point on $T$ generic over $k$ (resp. a non-singular point of $T$ rational over $k$).

11.2. It is necessary now to describe in some more detail the splitting (50) of the Chow rings.

Let $p : V' \to V$ be a monoidal transformation, with $V' = B_W(V)$, $V$ and $W$ non-singular and $W$ of codimension 2. Put $D = p^{-1}(W)$.

Let $C'$ be a cycle on $V'$ *such that* $C' \cdot D$ *is defined*, then $p^{-1}(p(C'))$ is also defined. Here $p$ (resp. $p^{-1}$) denotes the image (resp. the inverse image) *of the cycle*. In that case there exists a cycle $C_1$ on $W$ such that we have (compare [9], p. 481, formula (5)):

$$p^{-1}p(C') = C' + \alpha(C_1). \tag{54}$$

Here $\alpha : {}^*A(W) \to A(V')$ is the homomorphism defined in 10.3; namely as follows: for a variety $C_1$ on $W$ the $\alpha(C_1)$ is the *variety* $p^{-1}(C_1)$ and for cycles the definition is extended by linearity. The formula (54) defines the splitting (50).

Now note that if $C'$ is rational over a field $K$ containing the groundfield $k$ then $\alpha(C_1)$ is rational over $K$. Since it is well-known that locally $D = W \times P^1$ ([11], prop. I.8.2b) we can, at least in the case of a 0-dimensional $C_1$, which is the only case of interest for us, obtain the cycle $C_1$ itself from $\alpha(C_1)$ by intersection and projection. Namely take a $k$-rational point $Q$ in $\boldsymbol{P}^1$ then

$$C_1 = \mathrm{pr}_W(\alpha(C_1) \cdot (W \times Q)).$$

Therefore $C_1$ is rational over $K$ as soon as $C'$ is rational over $K$.

By a similar argument we see that if $C' \to C_*$ is a specialization of positive cycles on $X'$ *such that both* $C' \cdot D$ *and* $C'_* \cdot D$ *are defined*, then, with obvious notations, $C_{1*}$ is the unique specialization of $C_1$ over the specialization $C' \to C_*$ over $k$.

[7] We suppress in the notation of statements (a) and (b) the ismorphism $\alpha$ of (51).

11.3. Returning to the family of cycles $Z(t)$ on $X$ from proposition 10.5, write $Z = Z_1 - Z_2$ with $Z_i$ $(i = 1, 2)$ a positive cycle on $T \times X$. Now *make the following additional assumptions*:

($\alpha$) $\phi^{-1}(Z_i(t))$ is defined for $i = 1, 2$ (recall: $t$ generic on $T$),

($\beta$) no component of $\phi^{-1}(Z_i(t))$, for $i = 1, 2$, is contained in the divisor $D_1 = D + D_*$ of proposition 3.2:

$$\begin{array}{ccc} D_1 & \longrightarrow & X' = B_{\hat{H} \cup I}(V) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \hat{H} \cup I & \longrightarrow & V \end{array}$$

(recall $D = p^{-1}(\hat{H})$ and $D_* = p^{-1}(I)$). Under *these extra assumptions* it follows at once from 11.2 that $\phi^*(\text{Class } Z(t))$ is a $k(t)$-rational point on $J_0(\hat{H})$, i.e. under these extra assumptions we have proved (a) of 11.1.

11.4. Recall that $t_0$ is a non-singular $k$-rational point of $T$. Consider the *following additional assumptions*:

($\alpha'$) $\phi^{-1}(Z_i(t_0))$ is defined for $i = 1, 2$,

($\beta'$) no component of $\phi^{-1}(Z_i(t_0))$, for $i = 1, 2$, is contained in the divisor $D_1$.

Under these extra assumptions it follows, from what we have seen in 11.2, that the cycle on $\hat{H}$ defining the class of $\phi^{-1}(Z(t_0))$ in $A_0(\hat{H})$ is the *unique* specialization of the corresponding cycle for $\phi^{-1}(Z(t))$, over the specialization $t \to t_0$ over $k$. Since $\rho(t_0)$ is also the unique specialization of $\rho(t)$ and since we have by construction $\rho(t) = \phi^*$ (Class $Z(t)$), we have also $\rho(t_0) = \phi^*(\text{Class } Z(t_0))$ under *the extra assumptions* ($\alpha'$) and ($\beta'$).

Remarks (11.5). (i) It is necessary to work with $Z_i$, $i = 1, 2$, separately because in the specialization $Z(t) \to Z(t_0)$ may enter extra components cancelling against each other, so called 'latent' cycles (see [2], p. 455).

(ii) Clearly ($\alpha'$) $\Rightarrow$ ($\alpha$) and ($\beta'$) $\Rightarrow$ ($\beta$).

11.6. *Removal of the conditions* ($\alpha'$) *and* ($\beta'$).

For this we need a technical lemma from the theory of 'moving of cycles'. Let $X^n$ be a projective non-singular variety in $\boldsymbol{P}^N$ and $A$ a subvariety of $X$, all defined over a field $k$. Let $L^{N-n-1}$ be a *generic* linear space over $k$, i.e. defined by equations with independent transcendental coefficients over $k$; let $K_1$ be the field obtained by adjoining these coefficients. Let $\Gamma = \Gamma_{(A, L)}$ be the *projecting cone* through $A$ with $L$ as centre of projection. Then $\Gamma$ is defined over $K_1$. In case of a $k$-rational cycle $A$ we extend the construction by linearity; then $\Gamma$ is also $K_1$-rational. The following lemma is well-known (see for instance [7], lemma 3, p. 313):

LEMMA (11.7). (i) $\Gamma \cdot X = A + \sum n_\lambda A_\lambda$.

(ii) *If $B$ is a subvariety of $X$ then every component $C$ of $A_\lambda \cap B$, not contained in a component of $A \cap B$, is a proper component of intersection (i.e., has correct dimension) and a generic point of such a component over $K_1$ is a generic point of $B$ itself over $k$.*

(iii) *If $C$ is a subvariety of $X$, defined over $k$, then there is a point of $C$ not contained in any $A_\lambda$. In particular, if $C$ is a component of $A_\lambda \cap B$ contained in a component $C_1$ of $A \cap B$, then $C$ is strictly contained in $C_1$.*

11.8. Apply the above construction to the cycles $Z_i(t)$ above ($i = 1, 2$). For simplicity we omit the subscript $i$ and work with a *positive* cycle $Z(t)$ (i.e., we perform the construction for each $Z_i(t)$, $i = 1, 2$, separately). We get a cone $\Gamma(t)$ rational over $K_1(t)$ and we have

$$\Gamma(t) \cdot X = Z(t) + Z'(t).$$

Next we make a *generic* projective transformation $\tau$ over $K_1$; let $K$ be the field obtained by adjoining the coefficients from the transformation matrix to $K_1$. Let $\Gamma^\tau(t)$ be the image of $\Gamma(t)$ under $\tau$. Again the following lemma is well-known ([7], lemma, 1, p. 312):

LEMMA (11.9). *For every variety $B$ in $\mathbf{P}^N$, defined over $k$, $\Gamma^\tau(t) \cdot B$ is defined, rational over $K(t)$ and a generic point of a component $C$ of the intersection over $K(t)$ is a generic point of $B$ itself over $k(t)$.*

11.10. Finally we put

$$Y(t) = \Gamma^\tau(t) \cdot X - Z'(t).$$

We have with *rational* equivalence $Y(t) \sim Z(t)$ (we have only used a purely transcendental extension $K$ of $k$). This cycle is $K(t)$-rational, and we can construct in the usual way the corresponding cycle $Y$ on $T \times X$; moreover we can assume that $Y(t_0)$ is defined.

On the other hand we can perform, with the same centre of projection $L$, for $Z(t_0)$ the same construction as for $Z(t)$. This leads to a cycle $Y^*(t_0)$, and we have $Y(t_0) = Y^*(t_0)$ since both are the unique specialization of $Y(t)$ over the specialization $t \to t_0$ over $K$.

11.11. Next we claim that we can assume that $Y(t_0)$ satisfies the conditions $(\alpha')$ and $(\beta')$ of 11.4.

PROOF OF $(\alpha')$. Recall that $\phi : X' \to X$ is $2-1$ outside $\phi^{-1}(l)$ (see 4.7). Therefore $\phi^{-1}(Y(t_0))$ is defined as soon as $Y(t_0) \cap l = \emptyset$. By (iii) of lemma 11.7 and lemma 11.9 we see that $l$ is *not contained* in $Y(t_0)$. Then repeating, if necessary, the construction and using the same lemmas we see that we *can assume* $Y(t_0) \cap l = \emptyset$.

PROOF OF ($\beta'$). Part (ii) of lemma 11.7 and lemma 11.9, both applied to $X$ itself, show that every component of $Y(t_0)$ contains a generic point of $X$ itself over $k$. Since we can assume moreover that $Y(t_0) \cap l = \emptyset$ we have the same property for the components of $\phi^{-1}(Y(t_0))$. However then no component of $\phi^{-1}(Y(t_0))$ is contained in $D_1$.

11.12. PROOF OF (a) and (b) of 11.1.

(a) Since ($\alpha'$) and ($\beta'$) are fulfilled for $Y(t_0)$, ($\alpha$) and ($\beta$) are fulfilled for $Y(t)$ with $t$ generic on $T$ over $K$. We have

$$\phi^*(\text{Class } Z(t)) = \phi^*(\text{Class } Y(t))$$

and by 11.3 this determines a point on $J_0(\hat{H})$ which is rational over $K(t)$. Repeating the construction with another centre $L'$ of projection, generic over $k$ and independent with respect to $L$, we get that the point is also rational over a field $K'(t)$, with $K'$ and $K$ independent over $k$. Hence the point is rational over (i.e. has coordinates in) $K(t) \cap K'(t) = k(t)$. This proves (a).

(b) We have $\phi^*(\text{Class } Z(t_0)) = \phi^*(\text{Class } Y(t_0))$ and by 11.4 we have also $\phi^*(\text{Class } Y(t_0)) = \rho(t_0)$. This completes the proof of propostion 10.5 and hence also of theorem 10.8.

REMARK (11.13). Using now theorem 10.8 we see that we can replace in 10.5 the $J_0(\hat{H})$ by Prym $(\mathscr{H}/H)$, i.e. we have a rational map from $T$ to the Prym $(\mathscr{H}/H)$.

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