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$\kappa - R$-spaces

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$\kappa - R$-SPACES [1]

by

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Let R be a nonempty topological class of topological spaces. A space X is said to be locally in R provided every element of X has a neighborhood whose closure is in R. R/X denotes the class of closed subspaces of X that belong to R. A space X is called an R-space provided that for every $F \subset X$, if $A \cap F$ is closed for every $A \in R/X$, then F is closed. This concept, introduced in [2] and [3], generalizes such concepts as k-spaces, spaces determined by sequences (sequential spaces), spaces determined by well-ordered sets (see [3]), etc.

It turns out however that various classes of spaces have properties similar to the above mentioned, but which cannot be defined in terms of R-spaces. For instance, it has been shown in [4] that every m-adic space X has the following property:

(p) For every Q-open subset A of X, if $A \cap C$ is closed for every compact countable subset C of X, then A is closed. (A set is Q-open provided that it is a union of G_δ-sets.)

REMARK. Property (p) is equivalent to the statement that every sequentially closed Q-open subset of X is closed.

The purpose of this paper is to introduce a further generalization of R-spaces which in particular will enable one to handle property (p).

1.

Let κ be a map which assigns to each space X a nonempty class $\kappa(X)$ of subsets of X. We say that a space X is a $\kappa - R$-space if and only if for every $G \in \kappa(X)$, if $G \cap A$ is closed for every $A \in R/X$, then G is closed. It is clear that every space X is a $\kappa - R$-space if $\kappa(X)$ contains only closed subsets of X. Henceforth we shall assume that $\kappa(F) = \{G \cap F : G \in \kappa(X)\}$ for every closed subspace F of X and that $\varphi^{-1}[G] \in \kappa(X)$ for every $G \in \kappa(Y)$ where φ is a continuous map of X onto a space Y. All spaces will be assumed Hausdorff.

[1] A portion of this paper is a part of the author's doctoral dissertation directed by Professor S. Mrowka and submitted to The Pennsylvania State University in December 1969.

1.1. The property of being a $\kappa - R$-space is local (i.e., if every element of X has a neighborhood whose closure is a $\kappa - R$-space, then X is a $\kappa - R$-space).

PROOF. Suppose that $G \in \kappa(X)$ and G is not closed. Let $x \in \bar{G} \setminus G$. There is a neighborhood U of x such that $\bar{U}$ is a $\kappa - R$-space. But $\bar{U} \cap G$ is not closed in $\bar{U}$ and $\bar{U} \cap G \in \kappa(\bar{U})$. Thus there is an $A \in R/\bar{U}$ such that $A \cap \bar{U} \cap G$ is not closed in $\bar{U}$. Clearly, $A \in R/X$ and $G \cap A$ is not closed in X. Thus X is a $\kappa - R$-space.

1.2. COROLLARY. *Every space which is locally in R is a $\kappa - R$-space.*

1.3. COROLLARY. *A discrete union of $\kappa - R$-spaces is a $\kappa - R$-space.*

PROOF. This follows since a discrete union of $\kappa - R$-spaces is locally a $\kappa - R$-space.

1.4. If every member of R' is a $\kappa - R$-space, then every $\kappa - R'$-space is a $\kappa - R$-space.

PROOF. Suppose that X is a $\kappa - R'$-space, $G \in \kappa(X)$, and $G \cap A$ is closed for every $A \in R/X$. To show that G is closed it suffices to show that $G \cap B$ is closed for every $B \in R'/X$.

Let $B \in R'/X$ be arbitrary. Then B is a $\kappa - R$-space by our hypothesis. Since $G \cap A$ is closed for every $A \in R/X$ and B is closed in X, $G \cap B \cap A$ is closed in B for every $A \in R/B$. But $G \cap B \in \kappa(B)$ and B is a $\kappa - R$-space; therefore, $G \cap B$ is closed in B. Clearly, $G \cap B$ is closed in X. Thus X is a $\kappa - R$-space.

A map φ of a space X onto a space Y is called κ-quotient provided that φ is continuous and B is closed in Y for every $B \in \kappa(Y)$ for which $\varphi^{-1}[B]$ is closed in X. The map φ is called $\kappa - R$-quotient provided that φ is κ-quotient and $\varphi[A] \in R/Y$ for every $A \in R/X$.

1.5. The image of a $\kappa - R$-space under a $\kappa - R$-quotient map is a $\kappa - R$-space.

PROOF. Suppose that φ is a $\kappa - R$-quotient map of a $\kappa - R$-space X onto a space Y. Let $G \in \kappa(Y)$ and suppose that $G \cap A$ is closed in Y for every $A \in R/Y$. By our hypothesis, $\varphi^{-1}[G] \in \kappa(X)$. Let $B \in R/X$. Then $\varphi[B] \in R/Y$; consequently, $\varphi[B] \cap G$ is closed in Y. But $B \cap \varphi^{-1}[G] = B \cap \varphi^{-1}[\varphi[B] \cap G]$. Thus $B \cap \varphi^{-1}[G]$ is closed in X. Since X is a $\kappa - R$-space, $\varphi^{-1}[G]$ is closed. Finally, φ being a κ-quotient map implies that G is closed in Y. Thus Y is a $\kappa - R$-space.

1.6. COROLLARY. *If R consists of compact spaces and R is closed under continuous maps, then the property of being a $\kappa - R$-space is closed under κ-quotient maps.*

PROOF. This follows since under these hypotheses, every κ-quotient map must be $\kappa - R$-quotient.

1.7. Let X be a $\kappa - R$-space and let K be a subclass of R/X such that every member of R/X is contained in some member of K and $\bigcup K = X$. Then X is a κ-quotient of the discrete union of members of K.

PROOF. Let Y denote the discrete union of K. For convenience we consider the elements of Y to be the elements of X. Let φ map Y onto X so that the image of a point in Y is that same point in X. It follows that φ is continuous since its restriction to each space in the union is continuous.

Now let $B \in \kappa(X)$ such that $\varphi^{-1}[B]$ is closed in Y. If B is not closed in X, then there is an $A \in R/X$ such that $B \cap A$ is not closed in X. By our hypothesis, there is an $X' \in K$ such that $A \subset X'$. Thus $B \cap A$ is not closed in X'; consequently, there is an $x \in (\overline{B \cap A} \setminus (B \cap A)) \cap X'$. However, $x \in \varphi^{-1}(x) \cap X' \subset (\varphi^{-1}[\bar{B}] \cap \varphi^{-1}[A]) \cap X'$ and $x \notin (\varphi^{-1}[B] \cap \varphi^{-1}[A]) \cap X'$. Since $\varphi^{-1}[B]$ is assumed closed and $B \cap X' \subset \varphi^{-1}[B] \cap X'$, it follows that $x \notin \bar{B}$. This is a contradiction. Thus φ is κ-quotient.

1.8. THEOREM. *Let every $A \in R$ be compact and every continuous image of A be a $\kappa - R$-space. Then the following are equivalent*:

a) *X is a $\kappa - R$-space*;
b) *X is a κ-quotient of a discrete union of some members of R*;
c) *X is a κ-quotient of a space which is locally in R.*

PROOF. To show that a) implies b) we let $R' = R \cup \{\text{one-point spaces}\}$. Clearly, X is a $\kappa - R'$-space. By 1.7, X is a κ-quotient of the discrete union of R'/X; however, every one-point space is a quotient of a space in R. Thus b) follows.

The proof that b) implies c) is clear since a discrete union of members of R is locally in R.

Finally, c) implies a). Suppose that Y is a space which is locally in R (and hence a $\kappa - R$-space) and that φ is a κ-quotient map of Y onto X. Let R^* be the class of all continuous images of members of R. Then Y is a $\kappa - R^*$-space. Thus by 1.6, X is a $\kappa - R^*$-space. And by 1.4, X is a $\kappa - R$-space.

Let R denote the class of spaces homeomorphic to N^* (the one-point compactification of an infinite countable discrete space) and let $\kappa(X)$ denote the class of Q-open subsets of the space X. Then a space X is a $\kappa - R$-space if and only if every sequentially closed Q-open subset of X is closed. Furthermore, Theorem 1.8 gives a characterization of such spaces.

2.

This section contains special results which do not fit into the above. Let κ be a map as in the first section. We say that a space X has property (q) provided that every sequentially closed set belonging to $\kappa(X)$ is closed. This property is a generalization of such properties as (p) and that of a space being sequential. (For the latter property take $\kappa(X)$ to be all subsets of X.)

The following two results are generalizations of results in [1] which refer to sequential spaces. The proofs are straightforward modifications of those in [1].

2.1. Let X have property (q) and $Y \subset X$. Then Y has property (q) if and only if $\varphi|\varphi^{-1}[Y]$ is κ-quotient where φ is a κ-quotient map of the discrete union of the convergent sequences in X onto X.

2.3. The product of two spaces X and Y having property (q) has property (q) if and only if $\varphi \times \varphi_1$ is κ-quotient where φ and φ_1 are κ-quotient maps of the discrete unions of the convergent sequences in X and Y onto X and Y respectively.

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