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TRANSFORMATION GROUPS WITH NO EQUICONTINUOUS MINIMAL SET

by

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There readily come to mind at least two instances of transformation groups which contain no equicontinuous minimal set: since the universal Z-minimal set and the universal $\boldsymbol{R}$-minimal set both have homomorphic images which are not uniformly equicontinuous, e.g., the Morse minimal set, these universal minimal sets cannot themselves be equicontinuous. Since all the minimal sets in the greatest ambit are homomorphic, this means that the greatest ambits for Z and $\boldsymbol{R}$ contain no equicontinuous minimal sets. On the other hand, it's pretty patent that any compact group acting on itself is an equicontinuous minimal set; and by passage to the Weil completion, it's no more difficult to see that for totally bounded acting group the universal minimal set, indeed every ambit, is uniformly equicontinuous. The study of the middle ground – between totally bounded groups on the one hand, and Z and R on the other, opens up an area of interesting questions.

In dealing with equicontinuous minimal sets, one can pretend that one has a good rationale for limiting the actions to discrete groups – for if a minimal set is equicontinuous, then it is discretely equicontinuous [GH; 4.35]. With this as an over-riding restriction, it will be possible to find out exactly when the universal minimal set is equicontinuous – it must be finite (theorem 3.2). Of course, this occurs for finite acting groups, at least, but if T is infinite and discrete then it will follow that the universal minimal set is never equicontinuous.

An interesting sidelight on these results is a generalization of a theorem of Raimi, to the effect that if X is a compact invariant set in a discrete flow on an F-space, then it is uniformly equicontinuous if and only if it is pointwise periodic (theorem 4.1). Raimi's original version of this involved a specific discrete flow on $\beta N \setminus N$[R1].

1. Topological preliminaries

All spaces are assumed to be separated uniform spaces and the main reference for strictly topological ideas is [GJ]. References for dynamical ideas are [E1] and [GH]. A discussion of ambits is in [G], and of the universal minimal set in [E1; E2]. For discrete acting group T, the phase space of the greatest ambit is the Stone-Čech compactification βT of T, and the universal minimal set is a minimal set in the transformation group $(\beta T, T)$. This information about ambits suffices for the applications to follow.

1.1. DEFINITIONS. A space is an *F-space* provided disjoint cozero sets are completely separated [HG]. A space is *extremally disconnected* provided disjoint open sets have disjoint closures [GJ].

Evidently, every extremally disconnected space is an F-space, and every countable F-space is extremally disconnected. It is also true that every compact subset of an F-space is again an F-space, a fact which follows from theorem 2.2 of [HG]. Canonical examples of extremally disconnected spaces are the Stone-Čech compactifications of discrete spaces, hence for discrete acting group the greatest ambit is extremally disconnected. Examples of F-spaces are $\beta \boldsymbol{R} \setminus \boldsymbol{R}$ and the compact subspaces of Stone-Čech compactifications of discrete spaces [GJ], hence for T discrete the universal minimal set is an F-space. This proves:

1.2. PROPOSITION. *For discrete acting group the greatest ambit is extremally disconnected and the universal minimal set is an F-space.*

As is seen in the next proposition, it is possible for the universal minimal set to be extremally disconnected.

1.3. PROPOSITION. *Let T be a countable discrete group. Then the universal T-minimal set is extremally disconnected and is the Stone-Čech compactification of any orbit contained in it.*

PROOF. Let M be the universal T-minimal set. Since T is discrete, M is a subset of $\beta T \setminus T$ and is hence an F-space (by 1.2). Let x be in M, then xT is countable, and hence C^*-embedded in M (countable subsets of F-spaces are C^*-embedded [GJ; 14N]). This means that $M = \beta(xT)$. Since $\beta(xT)$ is an F-space, so is xT; hence xT is extremally disconnected and thus so is $\beta(xT)$. Consequently, M is the Stone-Čech compactification of every orbit contained in it [GJ; 6M].

1.4. DEFINITION. A space is *dyadic* provided it is a continuous image of a product of finite discrete spaces.

1.5. PROPOSITION. *If (X, T) is an equicontinuous transformation group, then every compact orbit closure is dyadic.*

PROOF. Let M be a compact orbit closure, then (M, T) is equicontinuous. In this situation the enveloping semigroup of (M, T) is a compact group [E1; 4.4] and [GH; 4.45], and hence dyadic [K]. Now M is dyadic because it is a continuous image of the enveloping semigroup.

The theorem of Kuz'minov used in the proof of 1.5 has not been translated, however special cases are treated in [HR; page 95 and page 423]. The totally disconnected case (page 95) suffices for dealing with the universal minimal set.

It is to be noted that dyadicity is a generalization of metrizability, in the sense that every compact metric space is dyadic. Lately there has been some question of when the greatest ambit is metrizable – the next remark shows that the acting group must be totally bounded for this to occur, in particular, for locally compact non-compact acting group the greatest ambit is never metrizable. An additional value of the remark is the characterization of precisely when the greatest ambit is a minimal set, and precisely when it is an equicontinuous(or a distal) transformation group.

1.6. REMARK. Let T be a topological group. Then the following are equivalent:

1) The greatest ambit (every ambit) is a minimal set
2) The greatest ambit (every ambit) is an equicontinuous transformation group
3) The greatest ambit (every ambit) is a distal transformation group
4) The greatest ambit (every ambit) is dyadic
5) T is totally bounded.

2. Equicontinuity in F-spaces

After an additional development in section 3, the main theorem of this section will provide a wide class of transformation groups with no equicontinuous minimal sets, namely the greatest ambits for discrete acting groups.

2.1. THEOREM. *Let X be an F-space and let (X, T) be a transformation group. If M is a compact invariant set in X and if (M, T) is uniformly equicontinuous, then (M, T) is pointwise periodic.*

PROOF. By 1.5 every orbit closure in (M, T) is dyadic and hence, if it is infinite, it contains an infinite compact metric space [EP; corollary to Lemma 1]. The fact that no F-space contains an infinite compact metric space [GJ; 14N] implies that every orbit closure in (M, T) is finite.

2.2. COROLLARY. *When T is discrete, either the universal T-minimal set is not equicontinuous, or every T-minimal set is finite.*

PROOF. The universal minimal set M is a subset of βT, hence M is finite by 2.1 whenever it is equicontinuous.

3. Fixed points of homeomorphisms

The next lemma will eventually allow 2.1 to be strengthened from 'pointwise periodic' to 'periodic' for the case of discrete flows. It has a more immediate effect in the succeeding theorem, which is the main result of the paper.

3.1. LEMMA. *Let X be an extremally disconnected space and let h be a hemeomorphism of X onto X. Then the set P of fixed points of h is open in X.*

PROOF. If $P \neq X$ there exists an x in $X \setminus P$ such that $xh \neq x$. Since P is closed there exists an open set U maximal with respect to

(3.1.1) U is disjoint from Uh.

This means that the closures of U and Uh are also disjoint, so the maximality of U implies that U is a closed set, since the closures of open sets in extremally disconnected spaces are open. Of course, Uh is also a closed set. If X is not the union of the three sets P, U and Uh, then there is some open set V containing P with V disjoint from U and Uh. As before, there exists an x in $V \setminus P$ with xh in V and $x \neq xh$. One can therefore find an open set W contained in V such that Wh is also contained in V and W is disjoint from Wh. Then the open set $U \cup W$ satisfies (3.1.1), contradicting the maximality of U. This means that P is the complement of $U \cup Uh$ in X and is thus an open set.

With little modification the proof given above works for into homeomorphisms – one shows first that P is open in Xh, hence open in X. A version of 3.1 for compact spaces is proved in [F] using a class theoretical lemma of Katětov.

3.2. THEOREM. *Let T be a discrete group. Then the following are equivalent*:

1) *M, the universal minimal set, is equicontinuous*
2) *M is finite*
3) *T is finite.*

PROOF. The implication 1 implies 2 is 2.2; to see that 2 implies 3, suppose that T is infinite. This means there is a $t \neq e$ in T such that, for some x in M, $\pi^t(x) = x$. By 3.1 the set of fixed points of π^t is open and hence there must exist points of T which are fixed under translation by t, but this is impossible, so T is finite. That 3 implies 1 is obvious.

3.3. CONJECTURE. *Let T be a topological group. Then the universal T-minimal set is equicontinuous if T is totally bounded.*

4. Discrete flows in $\beta N \setminus N$

Any homeomorphism h of βN onto βN carries $\beta N \setminus N$ onto $\beta N \setminus N$. It follows from 2.1 and 3.1 that the discrete flow so induced on $\beta N \setminus N$ cannot be uniformly equicontinuous unless (possibly) h has 'many' periodic points in N. But most homeomorphisms of $\beta N \setminus N$ arise otherwise than as restrictions of homeomorphisms of βN, and one is led to believe that the discrete flows so generated are rarely uniformly equicontinuous. That such is the case was shown by Raimi [R1; R2]. With slight additional argument a generalization of Raimi's theorem follows also from 2.1 and 3.1.

4.1. THEOREM. *Let Y be an F-space and let (Y, h) be a discrete flow. Let X be a compact invariant subset of Y. Then (X, h) is uniformly equicontinuous iff (X, h) is periodic.*

PROOF. If (X, h) is uniformly equicontinuous, then it is pointwise periodic by 2.1. It suffices to show that the cardinality of the orbits is bounded, the period is then the least common multiple. Suppose the cardinality of the orbits is unbounded, then there exist $x_1, x_2, \cdots, x_n, \cdots$ in X such that for each n the condition $|0(x_n)| < |0(x_{n+1})|$ is satisfied. Let $A = \bigcup 0(x_n)$, then A is countable, hence C^*-embedded, so βA is contained in X. The homeomorphism h takes A onto A, and thus takes βA onto βA. Recall that a countable subset of an F'-space is extremally disconnected, so that both A and βA are extremally disconnected. Since $\beta A \setminus A$ is non-empty and h is pointwise periodic on X, for any x in $\beta A \setminus A$ there exists an n such that $xh^n = x$. From 3.1 there must be infinitely many fixed points of h^n in A, but A is so constructed as to make this impossible. Hence the cardinality of the orbits is bounded, so (X, h) is periodic. The converse is clear.

4.2. COROLLARY (Raimi) *Let X be an invariant subset of the discrete flow $(\beta N \setminus N, h)$. Then (X, h) is uniformly equicontinuous iff (X, h) is periodic.*

PROOF. Since the closure of X is also invariant and $\beta N \setminus N$ is a F-space, the result follows.

Theorem 4.1 extends the conslusion of Raimi's result to many interesting spaces, in particular to $\beta R \setminus R$ and to $\beta D \setminus D$, where D is any discrete space, recalling that these spaces are F-spaces.

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