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A HOMOTOPY THEORETIC CHARACTERIZATION OF THE TRANSLATION IN E^n

by

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Let h be an orientation preserving homeomorphism of Euclidean n-space, E^n, onto itself and let h' be the unique extension of h to the n-sphere, $S^n = E^n \cup \{\infty\}$. Let d be a metric for S^n. Kinoshita [11] [12] has shown that the following four conditions are equivalent.

1. *Sperner's condition* [22]: for each compact subset C of E^n, there exists a positive integer N such that for each $|m| > N, h^m C \cap C = \phi$.

2. *Terasaka's condition* [24]: for each compact subset C of E^n, $\lim_{m \to \pm \infty} h^m C = \infty$.

3. *Kerékjártó's condition* [10]: h' is regular at each point of E^n but not at ∞; i.e. if $x \in E^n$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(h^m x, h^m y) < \varepsilon$ for each integer m. (Note that d is the metric of S^n, not E^n!).

4. The orbit space is Hausdorff and the natural projection of E^n onto the orbit space is a covering map.

If h satisfies these conditions, h is called *quasi-translation* [24]. Sperner and Kerékjártó showed that for $n = 2$, their conditions implied that h is a *topological translation*; i.e. if $t(x) = x+1$, then there exists a homeomorphism k of E^2 such that $h = k^{-1}tk$ (h has the *same topological type* as t). Clearly a topological translation is a quasi-translation.

THEOREM: (Sperner, Kerékjártó). *If h is a homeomorphism of E^2 onto itself, h is a topological translation if and only if h is a quasi-translation.*

Kinoshita [11] has given an example of a quasi-translation in E^3 which is not a topological translation. In fact, it has been shown by Sikkema, Kinoshita and Lomonaco [20] that there exists uncountably many distinct topological types of quasi-translations of E^3.

In this paper, we prove the following.

THEOREM 1: *For each $n \geqq 4$, there exists a quasi-translation of E^n which is not a topological translation.*

THEOREM 2: *A necessary and sufficient condition that a quasi-translation h of E^n, $n > 4$, be a topological translation is that for each compact subset*

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C of E^n there exists a compact set D containing C such that each loop in $E^n - \hat{D}$ is contractible in $E^n - \hat{C}$, where $\hat{X} = \bigcup_{-\infty}^{+\infty} h^i(X)$.

If h is a diffeomorphism (a piecewise linear homeomorphism) which satisfies the hypotheses of Theorem 2, then it is possible to find a diffeomorphism (a piecewise linear homeomorphism) k such that $khk^{-1} = t$ by a slight modification of the proof below. We should also note that the homeomorphism given by Theorem 1 can be chosen so that it is either a diffeomorphism or a piecewise linear homeomorphism.

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1. Proof of Theorem 1

Recall that a map $f : X \to Y$ is *proper* if for each compact set $C \subseteq Y$, $f^{-1}(C)$ is compact. A homotopy $f_t : X \to Y$, $t \in I = [0, 1]$, is a *proper homotopy* if the induced map $F : X \times I \to Y$ is proper. $f : X \to Y$ is a *proper homotopy equivalence* if there exists a proper map $g : Y \to X$ such that fg and gf are properly homotopic to the identity maps of Y and X, respectively.

PROPOSITION 1.1. *Let $f : X \to Y$ be a proper map of Hausdorff spaces and let $i : C \to C$ be the identity map of a compactum C. If $i \times f : C \times Y \to C \times Y$ is a proper homotopy equivalence, then f is a proper homotopy equivalence.*

PROOF. Let $g : C \times Y \to C \times X$ be a proper map such that $(i \times f)g$ and $g(i \times f)$ are properly homotopic to the identity maps of Y and X, respectively. Let $F : C \times X \times I \to C \times X$ be a proper homotopy such that $F(c, x, 0) = g(i \times f)(c, x)$ and $F(c, x, 1) = (c, x)$. Let $c_0 \in C$ and define $j : Y \to C \times Y$ and $p : C \times X \to X$ by $j(x) = (c_0, x)$ and $p(c, x) = x$.

Define $g' = pgj$ and note that the homotopy $F' : X \times I \to X$ defined by $F'(x, t) = pF(s_0, x, t)$ is a proper map such that $F'(x, 0) = g'f(x)$ and $F'(x, 1) = x$. Similarly, one can show that fg' is properly homotopic to the identity of Y.

COROLLARY 1.2. *Let f, X, Y and C be as in Proposition 1.1. If $r : C \to C$ is a homotopy equivalence and if $r \times f : C \times X \to C \times Y$ is a proper homotopy equivalence, then f is a proper homotopy equivalence.*

Let $[X, Y]$ be the homotopy classes of mapping of X into Y.

PROPOSITION 1.3. *Let C be a compact Eilenberg-MacLane space $K(G, 1)$* [21; p. 424] *where G is a finitely generated Abelian group and let X and Y*

be Hausdorff spaces such that $[X, C]$ *and* $[Y, C]$ *are trivial. If there exists a proper homotopy equivalence from* $C \times X$ *to* $C \times Y$, *then there exists a proper homotopy equivalence from* X *to* Y.

PROOF. Let $p_1 : C \times X \to C$ and $p_2 : C \times Y \to C$ be the natural projections and let $f : C \times X \to C \times Y$ be a proper homotopy equivalence. Since $[X, C] = H^1(X; G) = 0 = [Y, C] = H^1(Y; G) = 0$, by the Kunneth formula it follows that $p_1^* : H^1(C; G) \to H^1(C \times X; G)$ and $p_2^* : H^1(C; G) \to H^1(C \times Y; G)$ are isomorphisms. Let $[i] \in H^1(C; G) = [C, C]$ be the class of the identity map. Since $f^* : H^1(C \times Y; G) \to H^1(C \times X; G)$ is an isomorphism, there exists a homotopy equivalence $k : C \to C$ such that $p_1^*([k]) = f^* p_2^*([i])$. Hence there exists a homotopy $k_t : C \times X \to C$, $t \in I$, such that $k_0 = p_2 f$ and $k_1 = kp_1$.

Define $h_t : C \times X \to C \times Y$ by

$$h_t(z, x) = (k_t(z, x), qf(z, x)) \qquad t \in I$$

where $q : C \times Y \to Y$ is the natural projection. Note that h_t is a proper homotopy such that $h_0 = f$ and $h_1 = k \times (qf)$. Since qf is a proper map, we can apply Corollary 1.2.

PROOF OF THEOREM 1. If $n = 4$, let W^{n-1} be Whitehead's example of a contractible 3-manifold which is not homeomorphic to E^3 [25] and if $n > 4$, let W^{n-1} be the interior of contractible $(n-1)$-manifold $\overline{W}^{n-1}$ such that *bdry* $\overline{W}^{n-1}$ is not simply-connected [14] [16] [4]. By [15], $E^1 \times W^3$ is homeomorphic to E^4 and since $I \times W^{n-1}$ is homeomorphic to I^n, $n > 4$, $E^1 \times W^{n-1}$ is homeomorphic to E^n.

Consider $S^1 \times W^{n-1}$. If $S^1 \times E^{n-1}$ were homeomorphic to $S^1 \times W^{n-1}$, then by proposition 1.3, W^{n-1} is proper homotopy equivalent to E^{n-1}. For $n \geqq 6$, then W^{n-1} is homeomorphic to E^{n-1} by Siebenmann [17]. A step in Siebenmann's proof of this fact is Lemma 2.10 of [18] which says that π_1(end of W^{n-1}) is trivial. This proof does not depend upon the dimension. If $n = 5$, $\pi_1(\text{end of } W^{n-1}) = \pi_1(\text{bdry } \overline{W}^{n-1}) \neq 1$. If $n = 4$, the fact that $W^3 \subseteq E^3$ and $\pi_1(\text{end of } W^3) = 1$ implies that W^3 is homeomorphic to E^3 [9]. These contradictions imply that $S^1 \times W^{n-1}$ is not homeomorphic to $S^1 \times E^{n-1}$. Let $p : E^n = E^1 \times W^{n-1} \to S^1 \times W^{n-1}$ be the universal covering and let h be a generator of the covering transformation group. Clearly h satisfies Sperner's condition (cf [12]) and hence is a quasi-translation of E^n but the orbit space of h is $S^1 \times W^{n-1}$ and hence h is not a topological translation.

2. Proof of Theorem 2

Let $\mathscr{U}$ be the orbit space and let $p : E^n \to \mathscr{U}$ be the natural projection.

By Kinoshita [12], p is a covering map. Hence $\mathscr{U}$ is a manifold which has the homotopy type of S^1.

PROPOSITION. *$\mathscr{U}$ is homeomorphic to $S^1 \times E^{n-1}$.*

PROOF. We shall show first that $\mathscr{U}$ is the interior of a compact manifold. We assume familiarity with [18] (Note the remark on p. 224 of [18] which allows us to work in the topological category). We shall show that $\mathscr{U}$ has one end and that π_1 is essentially constant at this end.

It follows from Theorem 12 of [6] that $\mathscr{U}$ is not compact and hence $\mathscr{U}$ has at least one end. By duality, $H_c^1(\mathscr{U}) = H_{n-1}(\mathscr{U}) = 0$ and by [18; p. 204], $\mathscr{U}$ has one end, say ε.

Let $K_1 \subset K_2 \subset \cdots$ be a sequence of compacta in $\mathscr{U}$ such that $\mathscr{U} = \bigcup K_i$. There exists a compact set L_1 in E^n such that $p(L_1) = K_1$. By hypothesis, there exists a compact set C_1 in E^n such that $L_1 \subset C_1$ and each loop in $E^n - C_1$ is contractible in $E^n - \hat{L}_1$. Note that $p(\hat{C}_1)$ is compact; for suppose $\{x_i\}$ is a sequence of points in $p(\hat{C}_1)$. Pick $\{y_i\} \subseteq C_1$ such that $p(y_i) = x_i$. $\{y_i\}$ has a convergent subsequence; therefore, so does $\{x_i\}$.

Note that $p^{-1}p(\hat{C}_1) = \hat{C}_1$. Let L_2 be a compact set in E^n such that $p(L_2) = K_2 \cup p(\hat{C}_1)$. Find C_2, compact, containing L_2 such that each loop in $E^n - \hat{C}_2$ is contractible in $E^n - \hat{L}_2$. By induction, we can find a sequence of compacta $\{C_i\}$ in E^n such that $K_i \subseteq p(\hat{C}_i) \subseteq p(\hat{C}_{i+1})$, $\mathscr{U} = \bigcup_{j=1}^{\infty} p(\hat{C}_i)$, $p^{-1}p(\hat{C}_j) = \hat{C}_j$ and each loop in $E^n - \hat{C}_{i+1}$ is contractible in $E^n - \hat{C}_i$.

Consider the following commutative diagram, where f_i, g_i and h_i are induced by inclusions.

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(E^n - \hat{C}_{i+1}) & \xrightarrow{p_*} & \pi_1(\mathscr{U} - p(\hat{C}_{i+1})) & \xrightarrow{q} & Z & \to & 1 \\ & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle h_i} & & \\ 1 & \to & \pi_1(E^n - \hat{C}_i) & \xrightarrow{p_*} & \pi_1(\mathscr{U} - p(\hat{C}_i)) & \xrightarrow{q} & Z & \to & 1 \end{array}$$

The rows are exact by the exact homotopy sequence of a covering space; clearly, h_i is an isomorphism. Suppose $f: S^1 \to \mathscr{U} - p(\hat{C}_{i+1})$ represents $[f] \in \pi_1(\mathscr{U} - p(\hat{C}_{i+1}))$. If f can be lifted to $E^n - \hat{C}_{i+1}$, then, by construction of $\hat{C}_{i+1}$, $q_i[f] = 1$. If f cannot be lifted to $E^n - \hat{C}_{i+1}$, then $q[f] \neq 1$. Hence image $g_i \cap$ image $p_* = \{1\}$ and $q|$image g_i is an isomorphism onto Z.

Since $q|$image g_{i+1} is also an isomorphism onto Z, it follows that $g_i|$image g_{i+1} is an isomorphism of image g_{i+1} onto image g_i. Therefore π_1 is essentially constant at ε and $\pi_1(\varepsilon) = Z$. Note that this implies that $H_e^1(X) = Z$. From the exact sequence

$$\cdots \to H_c^1(X) \to H^1(X) \to H_e^1(X) \to H_c^2(X) \to \cdots$$

and duality, $H^1_c(X) = H_{n-1}(X)$, we have an isomorphism induced by inclusion, $H^1(X) \to H^1_e(X)$. This implies that inclusion induces isomorphisms $H_1(\varepsilon) \to H_1(X)$ and $\pi_1(\varepsilon) \to \pi_1(X)$.

Let $\alpha : S^1 \to \mathscr{U}$ be a locally flat embedding which is a homotopy equivalence [5]. Since an orientable manifold supports a stable structure [3], [8], there exists [3] an embedding $\alpha' : S^1 \times I^{n-1} \to \mathscr{U}$ such that $\alpha'|\text{bdry}\,(S^1 \times I^{n-1})$ is locally flat and $\alpha'(S^1 \times \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) = \alpha(S^1)$. Let $V = Cl\,(U\text{-image}\,\alpha')$. By using universal coverings, relative Hurewicz theorem and excision theorem, one easily sees that $\pi_1(V, \partial V) = 0$ for all i. The proposition now follows from [18].

Proof of Theorem 2. (Continued). Consider $p^{-1}(\{x\} \times E^{n-1})$ for some $x \in s^1$. Since $p|p^{-1}(\{x\} \times E^{n-1})$ is a covering map, $p^{-1}(\{x\} \times E^{n-1})$ is a countable collection of disjoint $(n-1)$-planes $\{E_\sigma\}$ such that $p|E_\sigma$ is a homeomorphism for each σ. Note that $hE_\sigma \cap E_\sigma = \phi$ for each σ. We now proceed as in [7] to complete the proof; we include the proof for completeness.

There is a homeomorphism γ of E^n onto itself such that $\gamma(E_\sigma) = E^{n-1} \times \{0\} \subseteq E^{n-1} \times E = E^n$ and $\gamma(hE_\sigma) = E^{n-1} \times \{1\}$. Define $\delta : E^{n-1} \to E^{n-1}$ by $\gamma^{-1}h\gamma(x, 0) = (\delta(x), 1)$. Since δ is orientation-preserving, it follows from [8] [13] that there is an isotopy δ_t of E^{n-1}, $t \in I$, such that $\delta_0 = \text{identity}$ and $\delta_1 = \delta$.

Define $F_0 : \beta(E^{n-1} \times [0, 1]) \to E^n$ by $F_0(x, t) = (\delta_t(x), t)$. Extend F_0 to F, a homeomorphism of E^n, by $F(x, r) = \gamma^{-1}h^q\gamma F_0(x, z)$ where $r = q+z$, $z \in (0, 1]$. Note that if $r = q+z$, $z \in (0, 1]$,

$$\begin{aligned} F^{-1}\gamma^{-1}h\gamma F(x, r) &= F^{-1}\gamma^{-1}h\gamma\gamma^{-1}h^q\gamma F_0(x, z) \\ &= F^{-1}\gamma^{-1}h^{q+1}\gamma F_0(x, z) \\ &= F^{-1}F(x, z+q+1) \\ &= (x, r+1). \end{aligned}$$

Corollary. *The n-th suspension of a quasi-translation of E^r is a topological translation provided either $n \geqq 2$ and $n+r \geqq 5$ or $n+r \leqq 3$*; i.e., *if h is a quasi-translation of E^r, then $h' : E^r \times E^n \to E^r \times E^n$, defined by $h'(x, y) = (h(x), y)$ is a topological translation.*

Proof. Let us suppose $n+r \geqq 5$, the other case is trivial. If U is the orbit space of h, then $U \times E^n$ is the orbit space of h'. By Theorem 6.12 and the Main Theorem of [19], $U \times E^n$ is homeomorphic to the interior of a compact manifold which has the homotopy type of S^1. We proceed now as in the proof of Theorem 2 to show that $U \times E^n$ is homeomorphic to $S^1 \times E^{r+n-1}$ and to show h' is a topological translation.

REFERENCES

E. M. Brown.
[1] Unknotting in $M^2 \times I$. Trans. Amer. Math. Soc. 123 (1966), 480–505.

M. Brown
[2] Locally flat imbeddings of topological manifolds. Ann. of Math. (2) 75 (1962), 331–341.

M. Brown and H. Gluck
[3] Stable structures on manifolds: I–III. Ann. of Math. (2) 79 (1964), 1–58.

M. L. Curtis and K. W. Kwun
[4] Infinite sums of manifolds. Topology 3 (1965), 31–42.

J. Dancis
[5] Topological analogues of combinatorial techniques. Conference on the Topology of Manifolds, Prindle, Weber & Schmidt, Inc., Boston, Mass., (1968), 31–46.

D. B. A. Epstein
[6] Ends. Topology of 3-manifolds, Prentice-Hall, Inc., Englewood Cliffs, N. J., (1962), 110–117.

T. Homma and S. Kinoshita
[7] On a topological characterization of the dilatation in E^3. Osaka Math. J. 6 (1954), 135–144.

W. C. Hsiang and J. L. Shaneson
[8] Fake tori, the annulus conjecture, and the conjectures of Kirby. Proc. National Acad. Sci. U.S.A. 62 (1969), 687–691.

L. S. Husch and T. M. Price
[9] Finding a boundary for a 3-manifold: Ann. of Math. (2) 91 (1970), 223–235.

B. v. Kerékjártó
[10] Topologische Characterisierungen der linearen Abbildungen. Acta Litt. ac. Sci. Szeged 6 (1934), 235–262.

S. Kinoshita
[11] On quasi-translations in 3-space. Fund. Math. 56 (1964), 69–79.

S. Kinoshita
[12] Notes on covering transformation groups. Proc. Amer. Soc. 19 [1968), 421–424.

R. C. Kirby
[13] Stable homeomorphisms and the annulus conjecture. Ann. Math. 89 (1969), 575–582.

B. Mazur
[14] A note on some contractible 4-manifolds. Ann. of Math. (2) 73 (1961), 221–228.

D. R. McMillan, Jr.
[15] Cartesian products of contractible open manifolds. Bull. Amer. Math. Soc. 67 (1961) 510–514.

V. Poénaru
[16] Les decompositions de l'hypercube en produit topologique. Bull. Soc. Math. France 88 (1960), 113–129.

L. C. Siebenmann
[17] On detecting Euclidean space homotopically among topological manifolds. Inventiones math. 6 (1968), 245–261.

L. C. Siebenmann
[18] On detecting open collars. Trans. Amer. Math. Soc. 142 (1969), 201–227.

L. C. Siebenmann
[19] The obstruction to finding a boundary for an open manifold of dimension greater than five. Thesis (1965) Princeton University.

C. D. SIKKEMA, S. KINOSHITA AND S. J. LOMONACO, JR.
[20] Uncountably many quasi-translations of S^3. (to appear).

E. SPANIER
[21] Algebraic Topology. Mc-Graw-Hill Book Co., New York (1966).

E. SPERNER
[22] Ueber die fixpunktfreien Abbildungen der Ebene. Abh. Math. Sem. Hamburg 10 (1934), 1–47.

J. R. STALLINGS
[23] On infinite processes leading to differentiability in the complement of a point. Differential and Combinatorial Topology. Princeton University Press, Princeton, New Jersey (1965), 245–254.

H. TERASAKA
[24] On quasi-translations in E^n. Proc. Japan Acad. 30 (1954), 80–84.

J. H. C. WHITEHEAD
[25] A certain open manifold whose group is unity. Quart. J. Math. Oxford Ser. (2) 6 (1935), 364–366.

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