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ON THE PURITY OF THE BRANCH LOCUS

by

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Let $f: X \to Y$ be a quasi-finite morphism of schemes and U the open subset of X where f is étale. The various theorems about the purity of the branch locus give conditions for U to be all of X. We offer a simple elementary proof that $U = X$ in the rather useful case when Y is smooth over a locally noetherian scheme S and U contains every point of depth $\leqq 1$ and is dense in the fibers over S. The proof is inspired by Zariski's original method [5] for characteristic 0. After the usual sort of reductions, Y becomes the spectrum of the ring of formal power series in a vector T of variables. Zariski proved that the functions on X also form a power series ring by expanding them in Taylor series in T. The appropriate differential operators $(1/i!)(\partial^i g/\partial T^i)$ first lift canonically over U and then extend over all of X because of the depth condition. However, lifting these operators amounts to constructing formal descent data for X (cf. [2] and [4]) and we present the proof from this point of view without mentioning differential operators, characteristics, or descent (and without using deeper results from formal geometry [1]). The various standard results we need have been collected in [1] and all our references below are to this source.

THEOREM (Purity of the branch locus; cf. VI, 6.8). *Let S be a locally noetherian scheme, $g: Y \to S$ a smooth morphism and $f: X \to Y$ a quasi-finite morphism. Let U be the open subset of X where f is etale. Assume U contains every point x where* $\mathrm{depth}(O_x) \leqq 1$ *and that* $(U \cap X(s))$ *is dense in the fiber $X(s)$ for all s in S. Then $U = X$.*

NOTE. (i) If $g(Y) = S$ and $f(U)$ contains every point y of Y where $\dim(O_y) \leqq \dim(S)$, then automatically $(U \cap X(s))$ is dense in $X(s)$ for all s in S.

(ii) If $(g \circ f)(U) = S$, the following conditions are equivalent:

(a) X satisfies S_2 (resp. X is normal) and U contains every point x where $\dim(O_x) \leqq 1$.

[1] However, using such results, Grothendieck [3] has also proved that $U = X$ when Y is locally a complete intersection and U contains every point of dimension $\leqq 2$.

(b) U satisfies S_2 (resp. U is normal) and U contains every point x where $\operatorname{depth}(O_x) \leqq 1$.

(c) S satisfies S_2 (resp. S is normal) and U contains every point x where $\operatorname{depth}(O_x) \leqq 1$.

Indeed, the equivalence of (a) and (b) results directly from the definitions (resp. and Serre's criterion). The equivalence of (b) and (c) holds by (VII, 4.9) because $U \to S$ is smooth and surjective.

(iii) In view of (i) and (ii), the theorem (applied to X minus the components of codimension one of the branch locus) implies that if X satisfies S_2 (e.g., X normal) and $\dim(S) \leqq 1$, then the branch locus of f has pure codimension 1.

PROOF. By way of contradiction, assume $U \neq X$. Let x be a generic point of an irreducible component of $(X-U)$. We shall prove f is étale at x.

Let $y = f(x)$ and $s = g(y)$. Consider the flat base change $\operatorname{Spec}(k) \to S$ where $k = O_y$. The hypotheses clearly hold for $f \otimes k$ and $g \otimes k$; by (VII, 5.11), $U \otimes k$ is the open set on which $f \otimes k$ is etale; the depth condition holds by virtue of (VII, 4.2); and clearly $U \otimes k$ is dense in the fibers over $\operatorname{Spec}(k)$. Thus we may assume that S is the spectrum of a local ring k and that there exists a section $h : S \to Y$ such that $h(s) = y$.

Note that O_x/O_y is étale if (and only if) $\hat{O}_x/\hat{O}_y$ is, that Y is an étale extension of a polynomial ring $k[T_1, \cdots, T_n]$ with y lying over (T), that $\operatorname{depth}(\hat{O}_x) = \operatorname{depth}(O_x)$ by (VII, 4.2) and that $\hat{O}_x$ is a localization of $O_x \otimes_{O_y} \hat{O}_y$. Replace X by $\operatorname{Spec}(\hat{O}_x)$, Y by $\operatorname{Spec}(\hat{O}_y)$ and k by $\hat{k}$. While g is no longer of finite type, now $O_y \cong k[[T_1, \cdots, T_n]]$, f is finite and $U = X - \{x\}$. Furthermore, clearly U contains every point of depth $\leqq 1$ and $(g \circ f)(U) = S$. Let $V = (Y - \{y\})$. Then f is étale over V and since $\operatorname{depth}_{O_y}(B) = \operatorname{depth}_B(B)$ where $B = O_x$ by (III, 3.16), the open set V contains every point $z \in Y$ such that $\operatorname{depth}_{O_z}(B_z) \leqq 1$.

Finally, it suffices to construct an isomorphism $X_O \times_S Y \xrightarrow{\sim} X$ where $X_O = X \times_Y S$. For then, by (VII, 5.11), X_O/S is étale because $U = X_O \times_S V$ is étale over V and $V \to S$ is surjective and flat; whence X/Y is etale because $Y \to S$ is surjective and flat. Thus it suffices to prove the following theorem (whose proof will be presented after several preliminary lemmas).

THEOREM. *Let k be a noetherian ring and $A = k[[T_1, \cdots, T_n]]$ a formal power series ring. Let B be a finite A-algebra which is étale over every prime p of A where* $\operatorname{depth}(B_p) \leqq 1$. *Then there exists a (canonical) isomorphism $A \otimes_k B_O \xrightarrow{\sim} B$ where $B_O = k \otimes_A B$.*

DEFINITION. Let k be a ring, R a k-algebra. The *module of mth principal* *parts of R over k*, denoted $P^m(R)$, is defined as $(R \otimes_k R)/I^{m+1}$ where I is the diagonal ideal. It is naturally filtered by the powers of I.

LEMMA 1. *Let k be a ring, R a noetherian k-algebra and S an étale extension of R.*

(i) *The natural* $(R \otimes_k R)$*-algebra homomorphism* $v_m : P^m(R) \otimes_R S \to P^m(S)$ *sending* $(\overline{a_1 \otimes a_2}) \otimes s$ *to* $\overline{a_1 \otimes sa_2}$ *(resp. to* $\overline{sa_1 \otimes a_2}$*) is an isomorphism (where* $P^m(R)$ *is regarded as an R-module from the right (resp. left)).*

(ii) *The induced map* $gr^i(P^m(R)) \otimes_R S \to gr^i(P^m(S))$ *is an isomorphism.*

PROOF. In (i), both filtered modules are separated and complete; so it suffices to show that the $gr^i(v_m)$ are isomorphisms. Since S/R is flat, $gr^i(P^m(R) \otimes_R S)$ is isomorphic to $gr^i(P^m(R)) \otimes_R S$. Thus (i) follows from (ii).

Let I (resp. J) be the diagonal ideal of (R/k) (resp. (S/k)), and set $K = \ker(S \otimes_k S \to S \otimes_R S)$. As in (VI, 4.9 and 4.10), $K \cong I \otimes_{(R \otimes_k R)} (S \otimes_k S)$ since S/R is flat. Hence $(K^i/K^{i+1}) \cong (I^i/I^{i+1}) \otimes_{(R \otimes_k R)} (S \otimes_k S)$. Also, $(K^i/K^{i+1}) \otimes_{(S \otimes_k S)} S \cong (J^i/J^{i+1})$ since S/R is unramified. Therefore, $(I^i/I^{i+1}) \otimes_{(R \otimes_k R)} S \cong (J^i/J^{i+1})$. Since the $(R \otimes_k R)$-module structure of (I^i/I^{i+1}) coincides with the left (resp. right) R-module structure of (I^i/I^{i+1}), this isomorphism coincides with the induced map.

LEMMA 2. *Let R be a noetherian local ring; P, N two finite R-modules. If* depth$(N) \geqq 2$, *then depth* $(\mathrm{Hom}_R(P, N)) \geqq 2$.

PROOF. An N-regular sequence (x_1, x_2) is easily seen to be $\mathrm{Hom}_R(P, N)$-regular.

LEMMA 3 (cf. VII, 2.10). *Let R be a noetherian ring, M a finite R-module and V an open subset of* Spec(R).

(i) *Suppose V contains every point p where* depth$(M_p) = 0$; *(e.g., V contains every generic point of* Supp(M) *and M satisfies* S_1*). Then the restriction* $M \to \Gamma(V, \tilde{M})$ *is injective.*

(ii) *Suppose V contains every point p where* depth$(M_p) \leqq 1$; *(e.g., V contains every point of codimension* $\leqq 1$ *in* Supp(M) *and M satisfies* S_2*). Then* $M \to \Gamma(V, \tilde{M})$ *is bijective.*

PROOF. To prove (i), let $x \in M$ go to zero in $\Gamma(V, \tilde{M})$. Assume $x \neq 0$. Then there exists a prime p in $\mathrm{Ass}(Ax)$. Then $pA_p \in \mathrm{Ass}(A_p x) \subset \mathrm{Ass}(M_p)$, so depth$(M_p) = 0$. Hence $p \in V$, so $A_p x = 0$; this contradicts $pA_p \in \mathrm{Ass}(A_p x)$.

To prove (ii), let $f \in \Gamma(V, \tilde{M})$. Let E be the ideal of elements $s \in A$ such that sf extends to an element x of M. For every prime p in V the image of f in M_p is a fraction x/s, and it follows that $E \not\subset p$. By (III, 1.5), there exists therefore an element s of E not in any prime p where $\operatorname{depth}(M_p) = 0$. Let x be an element of M extending sf.

Since s is M-regular, V contains every prime p where $\operatorname{depth}((M/sM)_p) = 0$. Since the image of x in (M/sM) is zero on V, it is zero by (i). Thus there exists a g in M such that $x = sg$. Then $s(g-f)$ is zero over V. Since s is M-regular, $g = f$ on V, and the proof is complete.

PROOF OF THEOREM. Let $P = \varprojlim(P^m(A))$. It will suffice to construct a P-isomorphism $u : P \otimes_A B \to B \otimes_A P$ where in $P \otimes_A B$ (resp. $B \otimes_A P$), P is regarded as an A-module via the second (resp. first) factor. Namely, define $w : A \otimes_k A \to A$ by $w(a_1 \otimes a_2) = a_2(0)a_1$ where $a_2(0)$ denotes the constant term of a_2. Then $w(I)$ is contained in $m = T_1 A + \cdots + T_n A$, so w defines an A-homomorphism $\hat{w} : P \to A$. Since the diagram

$$\begin{array}{ccc} A & \xleftarrow{\hat{w}} & P \\ \uparrow & & \uparrow{\scriptstyle j_2} \\ k = (A/m) & \leftarrow & A \end{array}$$

is commutative, $A \otimes_P (P \otimes_A B) = A \otimes_k (k \otimes_A B)$. Hence, $(A \otimes_P u) : (A \otimes_k B_O) \xrightarrow{\sim} B$ is the required isomorphism.

Since $A = k[[T_1, \cdots, T_n]]$, the $(A \otimes_k A)$-module $P^m(A)$, regarded as an A-module on the left (resp. right) is isomorphic to $A^{\oplus r}$ for some r. Therefore $(B \otimes_A P^m(A)) \cong B^{\oplus r}$. Thus, the open set V of $\operatorname{Spec}(A)$ over which B is étale, contains all p where $\operatorname{depth}((B \otimes_A P^m(A))_p) \leqq 1$.

Regarding the two $(A \otimes_k A)$-modules $P^m(A) \otimes_A B$ and $B \otimes_A P^m(A)$ as A-modules on the left, consider $M = \operatorname{Hom}_A(P^m(A) \otimes_A B, B \otimes_A P^m(A))$. By lemma 1, M has a natural section over V. By lemma 2, V contains every point p where $\operatorname{depth}(M_p) \leqq 1$. So by lemma 3, this section extends to an A-homomorphism $u_m : P^m(A) \otimes_A B \to B \otimes_A P^m(A)$. In fact, u_m is an $(A \otimes_k A)$-homomorphism since it is on V and we may apply 3(i). Similarly, we obtain an $(A \otimes_k A)$-homomorphism $B \otimes_A P^m(A) \to P^m(A) \otimes_A B$ which is an inverse to u_m on V; hence, it is a global inverse. The isomorphisms u_m clearly form a compatible system of maps, inducing the required P-isomorphism $u : P \otimes_A B \to B \otimes_A P$.

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