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GEOMETRY AND DEFORMATION OF SPECIAL SCHUBERT VARIETIES

by

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Contents

Introduction

The problems of splitting bundles and smoothing cycles, as presented in [1], lead to the general study of a bundle N of rank n on a quasi-projective, nonsingular variety X over an algebraically closed field k, such that $N(-1)$ is generated by its global sections. In particular, ([1], (5.6) and (5.8)), the Chern classes $c_i(N)$ are represented by subvarieties Z_i, which are nonsingular if $i = 1$ or $i > \frac{1}{2}(\dim X-2)$. This is obtained by choosing an appropriate k-vector subspace E of $\Gamma(X, N)$, and embedding X as a 'twisted' subvariety of $\mathrm{Grass}_n(E)$, the Grassmannian of n-quotients of E. Then, for a general subspace A of E, of dimension $a = (n-i+1)$, $c_i(N)$ is represented by the subvariety $Z_i = \sigma_1(A) \cap X$, where $\sigma_1(A)$ is the first Special Schubert subvariety of $\mathrm{Grass}_n(E)$ defined by A. The sets $S_p = \sigma_p(A) \cap X$, for $p = 1, \cdots, (n-i+2)$, stratify $S_1 = Z_i$; in particular, $S_p = \emptyset$ for $i = 1$ or $i > \frac{1}{2}(\dim X-2)$.

Our principal aim is to carry out an analysis of this stratification and of the 'generic' singularities of the subvarieties Z_i. A natural approach is to study the Special Schubert varieties, and 'induce' their properties in the S_p's; thus we carry out an extension and elaboration of the results in [1]. Proceeding thus, one shows that S_p is irreducible if $p \geqq 1$; and that the monoidal transformation $f: \sum \to S_p$ of S_p with center $S_{(p+1)}$ restricts over $U = (S_{(p+r)} - S_{(p+r+1)})$ to an algebraic fiber bundle of the form $\mathrm{Grass}_r(B) \times \mathrm{Grass}_{(n-a+p)}(C)$, where B is a $(p+r)$-bundle, and C is an $(n-a+p+r)$-bundle. Moreover, $\sum$ is nonsingular. (See § 7).

A further analysis is made possible by Theorem (2.3.2). Modelled after [2], Section 2, Corollary 2, our result differs in that here $gr(A)$ is assumed rigid in the category of all k-algebras, not in that of filtered k-algebras. In geometric terms, the theorem asserts that if the normal cone of a scheme X along a subscheme Y is rigid, then the completion of X along Y is isomorphic to the completion of the normal cone at the vertex. In our context, $(\sigma_p(A) - \sigma_{(p+z)}(A))$ is locally (in the canonical affine open covering of $\mathrm{Grass}_n(E)$) isomorphic to the product of a linear space and a certain determinantal variety D_z; hence the normal cone of $(S_p - S_{(p+z)})$ along $(S_{(p+z-1)} - S_{(p+z)})$ is an algebraic fiber bundle with fiber D_z. However, D_2 is the projecting cone C over $\boldsymbol{P}^{(a-1)} \times \boldsymbol{P}^{(n-1)}$ in the Segre embedding, whose vertex is the rigid singularity of Thom-Grauert-Kerner-Schlessinger; hence the normal cone of $(S_p - S_{(p+2)})$ along $(S_{(p+1)} - S_{(p+2)})$ is rigid. Thus the completion of $(S_p - S_{(p+2)})$ along $(S_{(p+1)} - S_{(p+2)})$ is locally isomorphic to the product of $(S_{(p+1)} - S_{(p+2)})$ and the completion of C at the vertex.

A general theory of algebraic deformations, including the criterion for rigidity employed above, was developped by Schlessinger in his (partly published) dissertation; § 1 and § 2, (2.1) and (2.2), contain some of the definitions and basic results. Since the wider scope of Schlessinger's theory is not needed in our context, we found it possible to make our distillation of it self-contained by providing some unpublished proofs; in particular, we present his beautiful approach to the verification of the rigidity of the vertex of the projecting cone over $\boldsymbol{P}^n \times \boldsymbol{P}^m$ for $n+m \geqq 3$.

Definitions and basic facts on monoidal transformations are recalled in § 3, which also contains proofs of two technical lemmas on the functoriality of normal cones. The definitions and some fundamental properties of Grassmannians can be found in § 4, which is a sequel to § 1 and § 2 of [1]. Here the properties of duality are especially emphasized, and the connection of determinantal varieties with Special Schubert varieties is made explicit.

In particular, the determinantal varieties $D_z(n, a)$ are shown to be nonsingular in codimension one and irreducible. Since, as shown by Hochster and Eagon in [5], $D_z(n, a)$ is Cohen-Macaulay for all z, then $D_z(n, a)$ is normal for $z > 1$; and it follows that the subvarieties Z_i representing the Chern classes $c_i(N)$ (see above) are all normal and Cohen-Macaulay. Moreover, one finds that $D_2(n, a)$ is the projecting cone over $\boldsymbol{P}^{(n-1)} \times$

$P^{(a-1)}$, hence it is rigid for $n+m \geqq 5$; and we believe $D_z(n, a)$ to be rigid for all $z \geqq 1$, unless $z = n = a$. The 2-codimensional Schubert subvarieties of the 9-dimensional Grassmannian ($\neq P^9$) are, locally along their singular locus, of the form $V \times D_2(3, 2)$, where V is a linear space of dimension 3; we conjecture that they cannot be deformed into a smooth cycle by rational equivalence.

The monoidal transformation of $\sigma_p(A)$ with center $\sigma_{(p+1)}(A)$ is studied in § 5. In § 6, the concept of a standard modification is developed, and it is shown that the monoidal transformation of $\sigma_p(A)$ with center $\sigma_{(p+1)}(A)$ can be written as the composition of a standard modification and a 'dual' standard modification. The analysis of the stratification $\{S_p\}$ can be found in § 7; in particular, the proof of the irreducibility of S_p for $p \geqq 1$ referred to in the introduction of [1] is presented.

Some of the above results are contained in Landolfi's doctoral dissertation at Brandeis University, and have appeared with an outline of their proof in [8].

We would like to extend our thanks to David Lieberman and Heisuke Hironaka for their kindness in reading a preliminary version of this work, and for their several apposite remarks.

1. Extensions and deformations

Let A be a ring, B an A-algebra, and M a B-module.

(1.1) A (one term) *extension* of B/A by M is an exact sequence

$$0 \to M \xrightarrow{i} E \xrightarrow{j} B \to 0$$

where E is an A-algebra, j is a surjective homomorphism of A-algebras, $i(M)$ is a square-zero ideal of E, and the B-module structure on M induced by i coincides with the given one.

Two extensions E and E' of B/A by M are said to be *equivalent* if there exists an A-algebra homomorphism $u : E \to E'$, inducing a commutative diagram

$$\begin{tikzcd} & E \arrow[dr] \arrow[dd, "u"] & \\ M \arrow[ur] \arrow[dr] & & B \\ & E' \arrow[ur] & \end{tikzcd}$$

(u must then be an isomorphism). The set of equivalence classes of extensions of B/A by M is denoted $\mathrm{Ex}^1(B/A, M)$.

(1.2.1) Express B as a quotient $B = P/I$ of a polynomial ring over A by an ideal I; the exact sequence $0 \to I \to P \to B \to 0$ is called a *presentation* of B. Consider the usual exact sequence of B-modules involving the Kähler differentials

$$(\mathfrak{a}) \qquad I/I^2 \to \Omega_{P/A} \otimes_P B \to \Omega_{B/A} \to 0.$$

Define $T^1(B/A, M)$ as the cokernel making the following sequence exact

$$(\mathfrak{b}) \quad 0 \to \mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, M) \to \mathrm{Hom}_B(I/I^2, M) \to T^1(B/A, M) \to 0.$$

EXAMPLE (1.2.2). Assume A is noetherian, and B is of finite type. Then B is smooth over A if and only if $T^1(B/A, M) = 0$ for all B-modules M.

Indeed, B is smooth over A if and only if $(\mathfrak{a})$ is split, and $(\mathfrak{a})$ is split if and only if $T^1(B/A, M) = 0$ for all B-modules M by a general lemma of commutative algebra.

PROPOSITION (1.2.3). *There is a bijection* $\alpha : \mathrm{Ex}^1(B/A, M) \to T^1(B/A, M)$.

Indeed, consider an extension $0 \to M \to E \to B \to 0$. Since P is a polynomial ring, the canonical surjection $g : P \to B$ factors through a ring homomorphism $f : P \to E$. Since $g(I) = 0$, then $f(I) \subset M$. However, M is a square-zero ideal in E; hence f induces a homomorphism $h : I/I^2 \to M$, and thus an element of $T^1(B/A, M)$. If $f' : P \to E$ is a second choice of lifting of g, then $i^{-1}(f-f') : P \to M$ is an A-derivation; thus α is well defined.

Conversely, given an element of $T^1(B/A, M)$, choose a representative homomorphism $h : I \to M$. Let $E = P \oplus M/\{x - h(x) | x \in I\}$. Then the composition $P \oplus M \to P \to B$ gives a surjection of A-algebras $E \to B$, and the sequence $0 \to M \to E \to B \to 0$ is exact. If $d : P \to M$ is the A-derivation giving rise to a second choice $h' = (h+d) : I \to M$, then the automorphism of $P \oplus M$ given by $\phi(x-m) = x-(m+d(x))$ induces an equivalence between the extensions E and E' defined by h and h'; thus α has a well-defined inverse.

COROLLARY (1.2.4). *$T^1(B/A, M)$ is independent of the choice of presentation* $0 \to I \to P \to B \to 0$.

(1.3) Let A' be a ring, and J an ideal of A' such that $A \cong A'/J$.

(1.3.1) A deformation of B/A to A' is an A'-algebra B', with a homomorphism $B' \to B$, inducing an isomorphism $B'/JB' \xrightarrow{\sim} B$. If J is a nilpotent ideal (resp., square-zero), then the deformation is said to be *infinitesimal* (resp., square-zero). If $J \otimes_{A'} B' \to JB'$ is an isomorphism (resp., B' is A'-flat), the deformation is said to be *admissible* (resp., flat); in view of the local criterion of flatness, ([4], V, 3.2), a deformation is flat if and only if it is admissible and B is A-flat.

Two deformations B' and B'' of B/A to A' are said to be *equivalent* if there is a homomorphism of A'-algebras $B' \to B''$, inducing a commutative diagram

$$\begin{tikzcd}
B & & B'' \arrow[ll] \\
& B' \arrow[ul] \arrow[ur] & \\
A \arrow[uu] & & A' \arrow[ll] \arrow[uu] \arrow[ul]
\end{tikzcd}$$

The set of equivalence classes of admissible infinitesimal deformations of B/A to A' is denoted Def $(B/A, A')$.

Suppose B' is an admissible square-zero deformation of B/A to A'. Then $J \otimes_A B \cong J \otimes_{A'} B'$, and there is an exact sequence

$$0 \to J \underset{A}{\otimes} B \to B' \to B \to 0.$$

Thus there is a natural map

$$\operatorname{Def}(B/A, A') \to \operatorname{Ex}^1(B/A', J \underset{A}{\otimes} B)$$

which is clearly injective. It fails to be surjective when there is an extension B' of B/A' by $J \otimes_A B$ not satisfying $J \otimes_A B \xrightarrow{\sim} JB'$.

PROPOSITION (1.3.2). *Assume* $J = \operatorname{Ker}(A' \to A)$ *is square-zero. If* $T^1(B/A, J \otimes_A B) = 0$, *then* Def $(B/A, A')$ *consists of at most one class.*[1]

Indeed, fix a presentation $0 \to I \to P \to B \to 0$, and let $P' = A' \otimes_A P$. Then the sequence $0 \to JP' \to P' \to P \to 0$ is exact, and by composition, there is a surjection $g : P' \to B$. Let $K = \operatorname{Ker}(g)$. Then $0 \to K \to P' \to B \to 0$ is a presentation of B as an A'-algebra.

Let B_1' and B_2' be any two extensions of B/A' by $J \otimes_A B$. Since P' is a polynomial ring, g lifts to homomorphisms $f_i : P' \to B_i'$, whose restriction to K induces homomorphisms $h_i : K \to J \otimes_A B$. Hence there are commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & J \otimes_A B \arrow[r, "\alpha_i"] & B_i' \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u, "h_i"] & P' \arrow[r] \arrow[u, "f_i"] & B \arrow[r] \arrow[u, "id"'] & 0
\end{tikzcd}$$

Suppose both extensions arise from deformations. Then $f_i(j \cdot p') = \alpha_i(j \otimes g(p'))$, for $j \in J$, $p' \in P'$, and for $i = 1, 2$. Set $h = h_1 - h_2$; then

[1] THEOREM ([6], 4.3.3). Assume J is square-zero. If $\operatorname{Def}(B/A, A') \neq \phi$, then $\operatorname{Def}(B/A, A')$ is a principal homogeneous space under $\operatorname{Ex}^1(B/A, A')$.

Indeed, continuing the lines of reasoning of (1.3.2) establishes this stronger assertion, which, however, will not be needed in the sequel.

$h(j \cdot p') = 0$. By construction, there is an exact sequence $0 \to JP' \to K \to I \to 0$. Hence h induces a homomorphism $\bar{h} : I \to J \otimes_A B$. Since $J \otimes_A B$ is a square-zero ideal in B_i', then $\bar{h}$ induces a homomorphism $k : I/I^2 \to J \otimes_A B$. Let $\bar{k}$ be the image of k in $T^1(B/A, J \otimes_A B)$.

Suppose $\bar{k} = 0$. Then k is the image of a homomorphism $d : \Omega_{P/A} \otimes_P B \to J \otimes_A B$. By composition, d gives rise naturally to a homomorphism $d' : \Omega_{P'/A'} \otimes_{P'} B' \to J \otimes_A B$, whose image in $\mathrm{Hom}_{B'}(K/K^2, J \otimes_A B)$ is clearly h. Hence the image of h in $T^1(B/A, J \otimes_A B)$ is zero. It follows that B_1 and B_2 are equivalent extensions.

In particular, if $T^1(B/A, J \otimes_A B) = 0$, then any two deformations of B/A to A' are equivalent, and (1.3.2) is proved.

(1.4) Some general lemmas

LEMMA (1.4.1). *Let C be an A-algebra, $D = B \otimes_A C$, and N a D-module. Assume one of the following conditions holds*:

(i) *B is A-flat*

(ii) *C is A-flat*

(iii) *There exists a homomorphism $C \to A$ inducing an isomorphism $N \to N \otimes_D B$.*

Then $T^1(D/C, N) = T^1(B/A, N)$.

Indeed, let $0 \to I \to P \to B \to 0$ be a presentation. Tensoring it with C over A yields a presentation $0 \to J \to P \otimes_A C \to D \to 0$ and a surjective homomorphism $u : I \otimes_A C \to J$.

If (i) or (ii) holds, then u is bijective and induces a bijection $(I/I^2) \otimes_A C \to J/J^2$. If (iii) holds, then tensoring the presentation of B with A over C yields a surjection $v : J \otimes_C A \to I$, and the composition $v \circ (u \otimes C)$ is the identity; since $(u \otimes C)$ is surjective, v is bijective and induces a bijection $(J/J^2) \otimes_C A \to I/I^2$. Since $\Omega_{(P \otimes C/C)} = \Omega_{P/A} \otimes_A C$, the assertion follows directly from the definition (1.2.1).

LEMMA (1.4.2). *For $i = 1, 2$, let B_i be an A-algebra, and M_i a B_i-algebra. Then*

$$T^1(B_1 \underset{A}{\otimes} B_2/A, M_1 \underset{A}{\otimes} M_2) = (T^1(B_1/A, M_1) \underset{A}{\otimes} B_2) \oplus (B_1 \underset{A}{\otimes} T^1(B_2/A, M_2)).$$

Indeed, let $0 \to I_i \to P_i \to B_i \to 0$ be presentations, let $P = P_1 \otimes_A P_2$, and $I = (I_1 \otimes_A P_2) + (P_1 \otimes_A I_2)$. Then $0 \to I \to P \to B_1 \otimes_A B_2 \to 0$ is a presentation, $\Omega_{P/A} = (\Omega_{P_1/A} \otimes_A P_2) \otimes (P_1 \oplus_A \Omega_{P_2/A})$ and $(I/I^2) = ((I_1/I_1^2) \otimes_A B_2) \oplus (B_1 \otimes_A (I_2/I_2^2))$. The assertion now follows directly from the definition (1.2.1).

LEMMA (1.4.3). *Assume* Spec (B) *is A-smooth at each of its generic points, and M has no embedded primes. Then $T^1(B/A, M) = \mathrm{Ext}^1(\Omega_{B/A}, M)$.*

Indeed, let $0 \to I \to P \to B \to 0$ be a presentation, and consider the exact sequences

$$0 \to H \xrightarrow{u} \Omega_{P/A} \underset{P}{\otimes} B \to \Omega_{B/A} \to 0$$

$$0 \to K \longrightarrow I/I^2 \xrightarrow{v} H \longrightarrow 0$$

in which $u \circ v$ is the natural map. By ([7], O_{IV}, 20.5.14), there is an open dense subset U of Spec (B) such that $K|U = 0$. Since M has no embedded primes, then $\mathrm{Hom}_B(K, M) = 0$. Hence v induces an isomorphism $\mathrm{Hom}_B(H, M) \xrightarrow{\sim} \mathrm{Hom}_B(I/I^2, M)$. Since $\Omega_{P/A}$ is a free module, there is an exact sequence

$$\mathrm{Hom}_B(\Omega_{P/A} \underset{P}{\otimes} B, M) \to \mathrm{Hom}_B(H, M) \to \mathrm{Ext}^1(\Omega_{B/A}, M) \to 0.$$

The assertion now follows directly from the definition (1.2.1).

2. Rigidity

(2.1) Definition (2.1.1). Let A be a ring, and B an A-algebra. If any admissible infinitesimal deformation B' of B/A to an A-algebra A' is necessarily equivalent to the *product family* $B \otimes_A A'$, then B (resp., Spec (B)) is said to be *rigid* over A.

Theorem (2.1.2) (*Schlessinger*). *Let A be an Artin local ring, and B a flat A-algebra, and let k be the residue class field of A. Then B is rigid over A if and only if $T^1(B/A, k \otimes_A B) = 0$.*

Indeed, let B' be a flat infinitesimal deformation of B/A to an A-algebra A'. Then A' is an Artin ring, and we may clearly assume it is local. Set $J = \mathrm{Ker}\,(A' \to A)$, let M be the maximal ideal of A', and n the largest integer such that $J' = M^n J$ is nonzero. Set $A'' = A'/J'$, and $B'' = B' \otimes_{A'} A''$. Then B'' is a flat infinitesimal deformation of B/A to A'', and B' is a flat square-zero deformation of B''/A'' to A'.

$$\begin{tikzcd} B & B' \arrow[l] \arrow[r] & B'' \arrow[ll, bend right] \\ A \arrow[u] & A' \arrow[l] \arrow[r] \arrow[u] & A'' \arrow[u] \arrow[ll, bend left] \end{tikzcd}$$

By induction on n, we may assume that B'' is equivalent to $B \otimes_A A''$. Since $N = J' \otimes_{A''} B''$ is A''-isomorphic to a finite direct sum of copies of $k \otimes_A B$, then $T^1(B/A, N) = 0$ by (1.4.1). By (1.3.3), therefore the two deformations B' and $B \otimes_A A'$ of B''/A'' to A' are equivalent.

Conversely, the extensions E of B/A by $k \otimes_A B$ are easily seen to be the admissible deformations of B/A to $A' = A \oplus (k \otimes_A B)$.

(2.2) LEMMA (2.2.1). *Let A be a ring, let k, M, and N be A-modules. There exist two spectral sequences*

$$''E_2^{pq} = \mathrm{Ext}_A^q(\mathrm{Tor}_p^A(k, M), N), \text{ and } 'E_2^{pq} = \mathrm{Ext}_A^p(k, \mathrm{Ext}_A^q(M, N)),$$

with the same limit H^n.

Indeed, let $k^{\cdot}$ and $M^{\cdot}$ be projective resolutions of k and M, and let $N^{\cdot}$ be an injective resolution of N. Let $T^{\cdot}$ be the simple complex associated to the double complex $k^{\cdot} \otimes_A M^{\cdot}$, and consider the spectral sequences $'E_2^{pq}$ and $''E_2^{pq}$ of the double complex $\mathrm{Hom}_A(T^{\cdot}, N^{\cdot})$. We have

$$''E_1^{pq} = \mathrm{Hom}_A(\mathrm{Tor}_p^A(k, M), N),$$

and

$$''E_2^{pq} = \mathrm{Ext}_A^q(\mathrm{Tor}_p^A(k, M), N).$$

On the other hand,

$$'E_1^{pq} = \mathrm{Ext}_A^p(T^q, N).$$

Since T^q is projective, this spectral sequence degenerates, and it converges to the homology of the double complex

$$\mathrm{Hom}_A(k^{\cdot} \underset{A}{\otimes} M^{\cdot}, N) = \mathrm{Hom}_A(k^{\cdot}, \mathrm{Hom}_A(M^{\cdot}, N)).$$

Finally, the first spectral sequence of this complex is

$$'E_2^{pq} = \mathrm{Ext}_A^p(k, \mathrm{Ext}_A^q(M, N)).$$

LEMMA (2.2.2). *Let A be a noetherian local ring with residue class field k, and let M and N be A-modules of finite type. If $depth_A(N) \geqq 2$, then $depth_A(\mathrm{Hom}_A(M, N)) \geqq 2$; and, if $depth_A(N) \geqq 3$, then there exists a natural isomorphism*

$$\mathrm{Hom}_A(k, \mathrm{Ext}_A^1(M, N)) \xrightarrow{\sim} \mathrm{Ext}_A^2(k, \mathrm{Hom}_A(M, N)).$$

Indeed, consider the spectral sequences $''E_2^{pq}$ and $'E_2^{pq}$ in (2.2.1). By ([4], III, 3.13), if $\mathrm{depth}_A(N) \geqq d$, then $''E_2^{pq} = 0$ for $q < d$, hence $H^q = 0$ for $q < d$.

Now the exact sequence of terms of low degree of $'E_2^{pq}$ is

$$0 \to \mathrm{Ext}_A^1(k, \mathrm{Hom}_A(M, N)) \to H^1 \to \mathrm{Hom}_A(k, \mathrm{Ext}_A^1(M, N)) \to \mathrm{Ext}_A^2(k, \mathrm{Hom}_A(M, N)) \to H^2$$

hence the result.

PROPOSITION (2.2.3) (*Schlessinger*). *Let A be a noetherian local ring, $\mathbf{m}$ its maximal ideal, and k its residue class field. Let M and N be A-modules of finite type, and assume that M is locally free on* $(\mathrm{Spec}(A) - \{\mathbf{m}\})$, *and that $depth_A(N) \geqq 3$. Then $\mathrm{Ext}_A^1(M, N) = (0)$ if and only if*

$$depth_A(\mathrm{Hom}_A(M, N)) \geqq 3.$$

Indeed, $E = \mathrm{Ext}_A^1(M, N)$ has support in $\{\boldsymbol{m}\}$ because M is locally free off $\{\boldsymbol{m}\}$; hence $E = (0)$ if and only if $\mathrm{Hom}_A(k, E) = (0)$. Since $\mathrm{depth}_A(N) \geqq 3$, (2.2.2) applies; hence $\mathrm{Ext}_A^q(k, \mathrm{Hom}_A(M, N))$ is zero for $q = 0, 1$, and it is zero for $q = 3$ if and only if $E = (0)$.

(2.2.4) Let A be a field, and $S = \oplus_{n=0}^{\infty} S_n$ a graded A-algebra generated by S_1. Let $\boldsymbol{m} = \oplus_{n=1}^{\infty} S_n$ denote the irrelevant ideal of S, and set $R = S_{\boldsymbol{m}}$. Let N be a graded S-module of finite type. Set $X = \mathrm{Proj}\,(S)$.

Proposition (*Grothendieck*). *Let* $\phi : N \to \oplus_{k=-\infty}^{+\infty} H^0(X, \tilde{N}(k))$ *be the canonical homomorphism. Let* $d = \mathit{depth}_R(N_{\boldsymbol{m}})$. *Then*

(i) $d \geqq 1$ *if and only if* ϕ *is injective*;

(ii) $d \geqq 2$ *if and only if* ϕ *is bijective*;

(iii) $d \geqq 3$ *if and only if* ϕ *is bijective, and* $H^1(X, \tilde{N}(k)) = (0)$ *for all* k;

. . . . .

(p) $d \geqq p$ *if and only if* ϕ *is bijective, and* $H^i(X, N(k)) = (0)$ *for all* $1 \leqq i \leqq (p-2)$ *and for all* k.

Indeed, by definition, $d = 0$ if and only if there exists an $x \in N$ such that $sx = 0$ for all $s \in \boldsymbol{m}$; i.e., such that $\phi(x) = 0$. Thus (i) holds.

Suppose every $s \in S_1$ is a zero-divisor in N. Then

$$\boldsymbol{m}R \subset \left(\bigcup_{P \in Ass(N_{\boldsymbol{m}})} P\right)\boldsymbol{m}^2 R.$$

Hence $\boldsymbol{m}R \subset P$ for some associated prime P, and they must coincide, because $\boldsymbol{m}$ is maximal.

Assume $d \geqq 1$. There must then exist an $s \in S_1$ which is not a zero-divisor in N. Set $X' = \mathrm{Proj}\,(S/sS)$, and $N' = N/sN$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N_{k-1} \arrow[r] \arrow[d, "\phi_{k-1}"'] & N_k \arrow[r] \arrow[d, "\phi_k"'] & N'_k \arrow[r] \arrow[d, "\phi'_k"'] & 0 \arrow[d] \\
0 \arrow[r] & H^0(X, \tilde{N}(k-1)) \arrow[r] & H^0(X, \tilde{N}(k)) \arrow[r] & H^0(X, \tilde{N}'(k)) \arrow[r] & H^1(X, \tilde{N}(k-1)).
\end{tikzcd}$$

Suppose $d \geqq 2$. Then, by (i), all the vertical arrows are injections. By Serre's theorem, ϕ_k is bijective for all $k \gg 0$. Hence, by descending induction, it follows that ϕ_k is bijective for all k. Conversely, if ϕ_k is bijective for all k, then ϕ'_k is injective for all k. Thus (ii) holds.

Assume that $p \geqq 2$, that (p) holds, and that $d \geqq p$. Then the following sequence is also exact on the left:

$$0 \to H^{p-2}(X', \tilde{N}'(k)) \to H^{p-1}(X, \tilde{N}(k-1)) \to H^{p-1}(X, \tilde{N}(k)).$$

Thus, since (p) holds for N', $d \geqq (p+1)$ if $H^{p-1}(X, \tilde{N}(k)) = (0)$ for all k; and the converse follows by descending induction from Serre's theorem, completing the proof.

(2.2.5) Let A be a ring, $S = \oplus_{n=0}^{\infty} S_n$ a graded A-algebra generated by S_1, and set $X = \mathrm{Proj}\,(S)$. Grade the Kähler differentials $\Omega_{S/A}$ by $\deg(ds) = n$ for $s \in S_n$.

PROPOSITION. *There exists a canonical exact sequence of* 0_X*-modules*

$$0 \to \Omega_{X/A} \to \tilde{\Omega}_{S/A} \to 0_X \to 0.$$

Indeed, let U be the complement in Spec (S) of Spec (A), embedded by the augmentation $S \to A$, and let $f: U \to X$ be the canonical map. Since f is smooth, the following sequence is exact and split

$$0 \to f^*\Omega_{X/A} \to (\Omega_{S/A})|U \xrightarrow{u} \Omega_{U/X} \to 0.$$

Let D denote the canonical derivation of the graded algebra S given by $D(s) = ns$ for $s \in S_n$, and $w: \Omega_{\mathrm{Spec}(S)/A} \to 0_{\mathrm{Spec}(S)}$ the corresponding homomorphism. Since $D|S_{(s)}$ is zero for all $s \in S_1$, w factors into $v \circ u$, where $v: \Omega_{U/X} \to 0_U$. Since w is surjective, and $\Omega_{U/X}$ is invertible, v is an isomorphism. Finally, it is easy to see that the grading on Ker (w) defined by the induced isomorphism $f^*\Omega_{X/A} \xrightarrow{\sim} \mathrm{Ker}\,(w)|U$ coincides with its grading as a submodule of $\Omega_{X/S}$.

THEOREM (2.2.6) (*Schlessinger*). *Let k be a field, $S = \oplus_{n=0}^{\infty} S_n$ a graded k-algebra; set $X = \mathrm{Proj}\,(S)$, and $T_X = \mathrm{Hom}_{0_X}(\Omega_{X/k}, 0_X)$. Assume that S is normal, that X is k-smooth, and that $H^1(X, 0_X(n)) = (0)$ for all n. If $H^1(X, T_X(n)) = (0)$ for all n, then S is rigid over k.*

Indeed, in view of (2.1.2), it suffices to show $T^1 = T^1(S/k, S) = 0$. By (1.4.3), $T^1 = \mathrm{Ext}^1_S(\Omega_{S/k}, S)$. Since X is smooth, $\Omega_{S/k}$ is locally free on $(\mathrm{Spec}\,(S) - \{\boldsymbol{m}\})$, where $\boldsymbol{m} = \oplus_{n=1}^{\infty} S_n$. Therefore $T^1 = 0$ if and only if $\mathrm{Ext}^1_R(\Omega_{R/k}, R) = 0$, where $R = S_{\boldsymbol{m}}$.

By (2.2.4), $\mathrm{depth}_R(R) \geqq 3$. Hence (2.2.3) applies to $\Omega_{R/k}$. Thus $T^1 = 0$ if and only if $\mathrm{depth}_R(N_{\boldsymbol{m}}) \geqq 3$, where $N = \mathrm{Hom}_S(\Omega_{S/k}, S)$.

By (2.2.2), $\mathrm{depth}_R(N_{\boldsymbol{m}}) \geqq 2$. By (2.2.5), there is an exact sequence $0 \to 0_X \to \tilde{N} \to T \to 0$. Hence $H^1(X, \tilde{N}(n)) = 0$ for all n. Therefore, by (2.2.4), $\mathrm{depth}_R(N_m) \geqq 3$; thus $T^1 = 0$.

REMARK (2.2.7). In the above theorem, if also $H^2(X, \tilde{N}(n)) = 0$ for all n, then the proof yields the converse.

THEOREM (2.2.8) (*Thom, Grauert-Kerner, Schlessinger*). *Let k be a field. Let $X = \boldsymbol{P}^n_k x \boldsymbol{P}^m_k$, and embed X projectively by the Segre morphism. If $n \geqq 1$ and $m \geqq 2$, then the projecting cone C of X is rigid over k.*

Indeed, $0_X(p) = 0_{\boldsymbol{P}^n}(p) \otimes 0_{\boldsymbol{P}^m}(p)$ for all p. So, by the Künneth formula, $H^0(X, 0_X(p))$ is the set of bihomogeneous polynomials of degree p, and $H^i(X, 0_X(p)) = 0$ for $1 \leqq i < (n+m)$ and for all p. Let S be the homogeneous coordinate ring of X. By construction, S is reduced, because X is. Thus, by explicit computation, the canonical map $\phi : S \to \oplus_{p=-\infty}^{+\infty} H^0(X, 0_X(p))$ is bijective. Since X is regular, it follows from (2.2.4) that X is normal (and Cohen-Macaulay).

For any k-scheme Y, set $T_Y = \underline{\mathrm{Hom}}_{0_Y}(\Omega_{Y/k}, 0_Y)$. It follows from (2.2.5) that there is an exact sequence

$$0 \to 0_{\boldsymbol{P}^n} \to (0_{\boldsymbol{P}^n}(1))^{\oplus(n+1)} \to T_{\boldsymbol{P}^n} \to 0$$

Hence, $H^0(\boldsymbol{P}^n, T_{\boldsymbol{P}}(p)) = 0$ if $p \leqq -2$, and $H^1(\boldsymbol{P}^n, T_{\boldsymbol{P}}(p)) = 0$ if $p \geqq -2$. In general, for any k-scheme Z, $T_{Y\times Z} = (T_Y \otimes 0_Z) \oplus (0_Y \otimes T_Z)$. So, the Künneth formula yields $H^1(X, T_X(p)) = 0$ for all p, because $m \geqq 2$. The conclusion then follows from (2.2.6).

(2.3)

(2.3.1) Let A be a ring, B a graded A-algebra, and M a graded B-module. An extension $0 \to M \to E \to B \to 0$ is called *homogeneous* if E is a graded A-algebra, and the maps are homogeneous (of degree zero). The set of classes of homogeneous extensions E, E' under equivalence defined by homogeneous maps $u : E \to E'$ is denoted $\mathrm{Ex}_0^1(B/A, M)$.

Choose a presentation $0 \to I \to P \to B \xrightarrow{g} 0$, and define $\deg(p) = \deg(g(p))$ for $p \in P$. Then I is homogeneous, and the usual exact sequence

$$I/I^2 \to \Omega_{P/A} \underset{P}{\otimes} B \to \Omega_{B/A} \to 0$$

consists of graded B-modules and homogeneous maps of degree zero. Define $T_0^1(B/A, M)$ as the cokernel of $\mathrm{Hom}_0(\Omega_{P/A} \otimes_P B, M) \to \mathrm{Hom}_0(I/I^2, M)$, where $\mathrm{Hom}_0(-, -)$ denotes the set of homogeneous maps of degree zero. Then there exists a bijection $\mathrm{Ex}_0^1(B/A, M) \simeq T_0^1(B/A, M)$, whose construction is analogous to that in the inhomogeneous theory, (cf. (1.2.3)).

It is clear from the definitions that $T_0^1(B/A, M)$ is a direct summand of $T^1(B/A, M)$. In particular, the following result has been proved.

LEMMA. *The canonical map* $\mathrm{Ex}_0^1(B/A, M) \to \mathrm{Ex}^1(B/A, M)$ *is injective.*

In other words, if there is any equivalence $u' : E \to E'$ between two homogeneous extensions of B/A by M, there is a second one $u : E \to E'$ which is homogeneous.

Alternately, the Lemma can be proved by defining $u(e)$ for $e \in E_n$ to be the component of degree n of $u'(e)$, and by directly verifying that u preserves multiplication.

(2.3.2) Let k be a ground ring, A a k-algebra with an ascending filtration $\{A_i\}_{i=-\infty}^{+\infty}$, and set $gr(A) = \oplus_{-\infty}^{\infty} (A_i/A_{i-1})$. (For example, let I be an ideal of A, and set $A_i = A$ for $i \geqq 0$, and $A_i = I^{-1}$ for $i < 0$.)

THEOREM (*Gerstenhaber*). *If* $T^1(gr(A)/k, gr(A)) = 0$, *then there exists an isomorphism of separated completions* (*in fact, of pro-objects*): $\hat{A} \simeq gr(A)^\wedge$.

Indeed, let $B = \oplus_{-\infty}^{\infty} A_i t^i$, where t is an indeterminate. Then $B/t^n B = \oplus_{-\infty}^{\infty}(A_i/A_{i-n})$ for all n. Starting with the natural map $u_1 : B/t^n B \to gr(A)$, and proceeding by induction on n, construct as follows a $k[t]$-algebra homomorphism $u_n : B/t^n B \to B_n$, where $B_n = gr(A) \otimes (k[t]/t^n)$, such that:

(i) u_n reduces to u_{n-1}

(ii) the submodule (A_{i-m}/A_{i-n}) of A_i/A_{i-n} is carried onto $\oplus_{j=m}^{n}(A_{i-j}/A_{i-j-1})\bar{t}^j$, where $\bar{t}$ is the residue class of t. Suppose u_{n-1} has been constructed, and consider the following diagram, whose rows are the natural homogeneous admissible extensions:

$$\begin{tikzcd}
0 & B/t^{n-1}B \arrow[l] \arrow[d, "u_{n-1}"'] & B/t^n B \arrow[l] \arrow[d, "u_n"'] & B/tB \arrow[l] \arrow[d, "u_1"'] & 0 \arrow[l] \\
0 & B_{n-1} \arrow[l] & B_n \arrow[l] & B_1 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

By (1.4.1), (iii), $T^1(B_{n-1}/k[t]/t^{n-1}, gr(A)) = 0$. It therefore follows from (1.3.3) that there exists a $k[t]$-algebra isomorphism u_n which renders the above diagram commutative; moreover, by (2.2.1), we may assume u_n is homogeneous. It is then easily seen that u_n satisfies (ii).

Finally, (ii) implies that the u_n form a coherent family of filtration-preserving isomorphisms

$$A_i/A_{i-n} \simeq (A_i/A_{i+1}) \oplus \cdots \oplus (A_{i-n+1}/A_{i-n})$$

whence the assertion.

3. Monoidal transformations

Let X be a scheme, Y a closed subscheme of X, and I the ideal of Y. Assume I is of finite type.

(3.1) (See [7], II, 8.1). Let $Z = \operatorname{Proj}(\oplus_{n=0}^{\infty} I^n)$. The structure morphism $f : Z \to X$ is called the *monoidal transformation* (or blowing-up) of X with center Y.

Clearly, Z is reduced (resp., integral), if X is. The morphism f is projective, and is an isomorphism precisely at those points where I is invertible; in particular,

$$(Z - f^{-1}(Y)) \simeq (X - Y).$$

The (scheme-theoretic) fiber $E = f^{-1}(Y)$ is called the *exceptional divisor*, and, indeed, it enters into an exact sequence

$$0 \to 0_Z(1) \to 0_Z \to 0_E \to 0.$$

It has the form $E = \mathrm{Proj}\,(\oplus_{n=0}^{\infty} I^n/I^{n+1})$, and thus comes equipped with a relatively projective embedding over Y, and a cone,

$$C_Y(X) = \mathrm{Spec}\,(\oplus_{n=0}^{\infty} I^n/I^{n+1}),$$

called the *normal cone* of X along Y.

LEMMA (3.1.1). *With the above notation, if C is reduced, then X is reduced.*

Indeed, for any ring A and ideal I of A, if $gr(A)$ is reduced, then A is.

(3.2) Let M be a quasi-coherent 0_X-module, and $u : M \to 0_X$ a homomorphism such that $u(M) = I$. Then:

(i) the surjections $\mathrm{Sym}^n(M) \to I^n$ define a closed immersion $i : Z \hookrightarrow \mathbf{P}(M)$

(ii) the restriction $u|(X-Y)$ defines a section $s : (X-Y) \to \mathbf{P}(M)$.

PROPOSITION (3.2.1). *Under the above conditions i embeds Z as the (scheme-theoretic) closure T of the image $s(X-Y)$ in $\mathbf{P}(M)$.*

Indeed, clearly $s(X-Y) = i(Z-E)$, and $T \subseteq i(Z)$. Hence the question is local on X and Z; we may then assume $X = \mathrm{Spec}\,(A)$, and I is an ideal of A.

As g runs through I, the homogeneous part of degree one of the graded algebra $S = \oplus_{n=0}^{\infty} I^n$, Z is covered by the affines $\mathrm{Spec}(B)$, where B is the component $S_{(g)}$ of degree zero of the localization S_g; symbolically, $B = A[I/g]$. Now, T corresponds to an ideal J of B, and $J(B_{(g/1)})$ is zero. However, the natural map $B \to B_g(= A_g)$ is injective. Hence $J = (0)$, and $T = i(Z)$.

(3.3) Let X' be a closed subscheme of X, and $Y' = X' \cap Y$. Let I' be the ideal of Y' on X', and let $f' : Z' \to X'$ be the monoidal transformation of X' with center Y'. Then the surjection $I \to I'$ defines a closed immersion $j : Z' \hookrightarrow Z$.

PROPOSITION (3.3.1). *Under the above conditions j embeds Z' as the closure of $(X'-Y)$ in Z, and it induces a closed immersion $C_{Y'}(X') \hookrightarrow C_Y(X)$.*

Indeed, the first assertion results immediately from (3.2), in view of the commutative diagram

$$\begin{tikzcd} Z' \arrow[rr, "i'"] \arrow[dr, "j"'] & & P(I) \\ & Z. \arrow[ur, "i"'] & \end{tikzcd}$$

The second assertion is an immediate consequence of the definitions.

(3.4) Fix an algebraically closed field k, and assume that X is an integral algebraic k-scheme, and that Y is an equidimensional closed subscheme of X. Let X' be an irreducible closed subscheme of X. With the notation of (3.1) and (3.3), let $E = f^{-1}(Y)$ (resp., $E' = (f')^{-1}(Y')$) be the exceptional divisor of the monoidal transformation $f: Z \to X$ (resp., $f': Z' \to X'$).

(3.4.1) Suppose that E (resp., $C_Y(X)$) is an algebraic fiber bundle over Y with integral fiber F (resp., with integral fiber the projecting cone over F). Then, necessarily, $\dim(F) = (\operatorname{codim}(Y, X) - 1)$. Assume Y' not equal to X', locally integral and equidimensional, and assume that $\operatorname{codim}(Y, X) = \operatorname{codim}(Y', X')$.

PROPOSITION (3.4.2): *Under the above conditions $E' = E \times_Y Y'$ (resp., $C_{Y'}(X') = C_Y(X) \times_Y Y'$), and Z' has the same underlying space as $Z \times_Y X'$ (resp., and X' is integral).*

Indeed, consider Z' as the closure of $(X' - Y')$ in Z, and f' as the restriction $f|Z'$. Let $\{Y_i'\}$, for $i = 1, \cdots, r$, be the irreducible components of Y'. Then $(f')^{-1}(Y_i')$ is a closed subscheme of $f^{-1}(Y_i')$ which is not empty, because X' is irreducible and f' is a monoidal transformation. We show next that they have the same dimension, and hence they coincide. Since E is a fiber bundle over Y, then $\dim(f^{-1}(Y_i')) = (\operatorname{codim}(Y,X) - 1) + \dim(Y_i')$. However, $\operatorname{codim}(Y, X) = \operatorname{codim}(Y', X')$, and Y' is equidimensional, hence $\dim(f^{-1}(Y_i')) = \dim X' - 1$. On the other hand, since E' is a Cartier divisor on Z', $\dim (f')^{-1}(Y_i') = \dim X' - 1$. Hence $f^{-1}(Y_i')$ and $(f')^{-1}(Y_i')$ have the same dimension. Moreover, $f^{-1}(Y_i')$ is integral, since F and Y_i' are integral, and $f^{-1}(Y_i')$ is a fiber bundle over Y_i' with fiber F. Hence $E' = Ex_Y Y'$.

The argument for the normal cones is essentially the same, and X' is integral by (3.1.1).

(3.4.3) Let $Y_0 \supset Y_1 \cdots \supset Y_{n+1}$ be a sequence of equidimensional closed subschemes of X, with $Y_0 = X$, and $Y_{n+1} = \emptyset$. For $0 \leqq p \leqq n$, let $U_p = (Y_p - Y_{p+1})$, let $E_p = f^{-1}(U_p)$, and assume it is a fiber bundle over U_p with integral fiber. Let $W_p = (Y_p - Y_{p+2})$, and let $f_p : Z_p \to W_p$ be the monoidal transformation with center U_{p+1}. Assume that the exceptional divisor of Z_p is a fiber bundle over U_{p+1} with integral fiber, and that there exists a fiber bundle P_p over Z_p with integral fiber, and a surjective X-morphism $g_p : P_p \to f^{-1}(W_p)$.

Let

$$W_p' = W_p \cap X'; \qquad U_p' = U_p \cap X'; \qquad E_p' = (f')^{-1}(U_p');$$

and assume

(i) W_p' is equidimensional and locally irreducible;

(ii) U'_p is equidimensional and locally integral;

(iii) codim $(U'_{p+1}, W'_p) = $ codim (U_{p+1}, W_p).

Let $f'_p : Z'_p \to W'_p$ be the monoidal transformation with center U'_{p+1}.

PROPOSITION (3.4.4). *Under the above conditions* $E'_p = E_p \times_{Y_p} Y'_p$, *for* $0 \leqq p \leqq n$.

Indeed, for $p = 0$ the assertion is trivially true. Assume $p \geqq 0$. Proposition (3.4.2), applied to f_p and f'_p restricted to the irreducible components of W'_p, asserts that $Z_p \times_X X'$ and Z'_p have the same underlying space. Since $P_p \times_X X' = P_p \times_{Z_p} (Z_p \times_X X')$, then $P_p \times_X X'$ and $P_p \times_{Z_p} Z'_p$ have the same underlying space.

Now Z'_p is the closure of U'_p in Z_p, hence $P_p \times U'_p$ is dense in $P_p \times_{Z_p} Z'_p$. Since the base extension $g_p \times_X X' : P_p \times_X X' \to f^{-1}(W'_p)$ is surjective, it follows that $f^{-1}(U'_p)$ is topologically dense in $f^{-1}(W'_p) = E_p \times_{Y_p} Y'_p$.

Let $\bar{E}'_p$ be the closure of E'_p in E'. Then we have

$$(\bar{E}'_p x_{Y_{p+1}} Y'_{p+1}) \subseteq E'_{p+1} \subseteq (E_{d+1} \times_{Y_{p+1}} Y'_{p+1}).$$

By induction, $f^{-1}(U'_p) = E'_p$, hence $\bar{E}'_p$ has the same underlying space as $E_p \times_{Y_p} Y'_p$. Therefore, E'_{p+1} is a closed, topologically dense subscheme of $E_{p+1} \times_{Y_{p+1}} Y'_{p+1}$. Since the latter is reduced, they coincide.

4. Fundamentals

Fix a ground scheme S, an e-bundle[2] E on S, and an integer n, with $0 \leqq n \leqq e$.

(4.1) The *Grassmann scheme* $X = \mathrm{Grass}_n(E)$, equipped with its *universal n-quotient*[3] Q of E_X[4], represents the functor whose points t with values in an S-scheme T are the n-quotients N of E_T; i.e., the morphisms $t : T \to \mathrm{Grass}_n(E)$ and the n-quotients N of E_T are in bijective correspondence by $N = t^* Q$.

(4.2) There is a natural commutative diagram

$$\begin{tikzcd} \mathrm{Grass}_n(E) \arrow[r, hook, "\pi"] \arrow[d, "d"', "\wr"] & \boldsymbol{P}(\bigwedge^n E) \\ \mathrm{Grass}_m(E^*) \arrow[r, hook, "\pi^*"] & \boldsymbol{P}(\bigwedge^m E^*) \arrow[u, "i"', "\wr"] \end{tikzcd}$$

in which:

$m = (e-n)$;

d is the isomorphism induced by the natural correspondence between

[2] Shorthand for locally free 0_S-module of rank e.
[3] Shorthand for locally free quotient of rank n.
[4] Notation for the pull-back of E via the structure map $X \to S$.

n-quotients N of E_T, m-subbundles[5] M of E_T, and m-quotients M^* of E_T^* [6];

π (resp., π^*) is the *Plücker embedding*, which is defined by mapping an n-quotient N of E_T to the 1-quotient $\bigwedge^n N$ of $\bigwedge^n E_T$ (resp., similarly for π^*).

i is the isomorphism defined by mapping a 1-quotient L of $\bigwedge^m E_T^*$ to the 1-quotient $L \otimes (\bigwedge^e E)$ of $(\bigwedge^m E_T^*) \otimes (\bigwedge^e E)$, and identifying the latter with $\bigwedge^n E$ via the following canonical maps, which are perfect dualities, since E is locally free:

$$(\overset{m}{\bigwedge} E) \otimes (\overset{m}{\bigwedge} E^*) \to 0_S, \text{ by } (e_1 \wedge \cdots \wedge e_m) \otimes (e_1^* \wedge \cdots \wedge e_m^*) \mapsto \det((e_i, e_j)),$$

and

$$(\overset{m}{\bigwedge} E) \otimes (\overset{n}{\bigwedge} E) \to \overset{e}{\bigwedge} E, \text{ by } (e_1 \wedge \cdots \wedge e_m) \otimes (e_{m+n} \wedge \cdots \wedge e_e) \mapsto (e_1 \wedge \cdots \wedge e_e).$$

Indeed, the commutativity holds because the following diagram of canonical maps (arising from an exact sequence $0 \to M \to E_T \to N \to 0$ on T) is itself commutative:

$$\begin{tikzcd}
\bigwedge^n E_T \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(\bigwedge^m E_T, \bigwedge^e E_T) \arrow[d] & (\bigwedge^m E_T^*) \otimes (\bigwedge^e E_T) \arrow[l, "\sim"'] \arrow[d] \\
\bigwedge^n N \arrow[r, dashed, "\sim"] & \mathrm{Hom}(\bigwedge^m M, \bigwedge^e E_T) & (\bigwedge^m M^*) \otimes (\bigwedge^e E_T) \arrow[l, "\sim"']
\end{tikzcd}$$

(4.3) If $E = E_1 \otimes E_2$, and $t_i : T \to \mathrm{Grass}_{n_i}(E_i)$ are two morphisms defined by n_i-quotients N_i of E_{iT}, their *Segre product* is the morphism $t_1 \otimes t_2 : T \to \mathrm{Grass}_n(E)$ defined by the n-quotient $N_1 \otimes N_2$ of E, with $n = n_1 n_2$. In particular, the projections p_i give rise to the *Segre embedding*

$$s = p_1 \otimes p_2 : \mathrm{Grass}_{n_1}(E_1) \times \mathrm{Grass}_{n_2}(E_2) \hookrightarrow \mathrm{Grass}_n(E).$$

(4.4) Fix an a-subbundle A of E, and an integer p satisfying $\max(0, a-n) \leqq p \leqq 1+\min(a, e-n)$. The *p-th special Schubert scheme* $\sigma_p(A)$ defined by A represents the subfunctor of $\mathrm{Grass}_n(E)$ whose T-points are those n-quotients N of E_T such that the induced map $\bigwedge^q A_T \to \bigwedge^q N$ is zero, where $q = (a-p+1)$. (Intuitively, this condition requires $rk(A_T \cap M \geqq p$ where $M = \mathrm{Ker}(E_T \to N)$.)

Proposition (4.5). *The map of* (4.2) *induces an isomorphism*

$$\sigma_p(A) \xrightarrow{\sim} \sigma_{(p+n-a)}((E/A)^*).$$

[5] Shorthand for the submodules which are locally direct summands of rank m.
[6] Notation for the dual of M.

Indeed, for an exact sequence $0 \to M \to E_T \to N \to 0$ defining a T-point of $\mathrm{Grass}_n(E)$, the following conditions are equivalent:

(i) the map $\bigwedge^q A_T \to \bigwedge^q N$ is zero, where $q = (a-p+1)$;

(ii) the map $\bigwedge^{q'} C \to \bigwedge^{q'} E_T$ is zero, where $q' = (a-p+e-n+1)$ and $C = A_T + M$;

(iii) the map $\bigwedge^{q^*} M \to \bigwedge^{q^*}(E/A)_T$ is zero, where $q^* = (e-n-p)$;

(iv) the map $\bigwedge^{q^*}(E/A)_T^* \to \bigwedge^{q^*} M_T^*$ is zero.

The equivalence (i) $\Leftrightarrow$ (ii) (resp., (ii) $\Leftrightarrow$ (iii)) holds in view of [1], (2.5), because M (resp., A_T) is locally a direct summand of E_T; the equivalence (iii) $\Leftrightarrow$ (iv) holds by duality (see (4.3)); whence the assertion.

(4.6) Set $P = \mathrm{Ker}\,(E_X \to Q)$; it is called the *universal subbundle* on $X = \mathrm{Grass}_n(E)$, and its dual P^* is the universal quotient on $d(X) = \mathrm{Grass}_m(E^*)$. Set $B = (E/A)_X$ and $F = (\bigwedge^b P) \otimes (\bigwedge^b B^*)$ where $b = (e-n-p)$. Let $v : \bigwedge^b B^* \to \bigwedge^b P^*$ be the natural map and define u as the composition

$$u : F \xrightarrow{1 \otimes v} (\overset{b}{\bigwedge} P) \otimes (\overset{b}{\bigwedge} P^*) \xrightarrow{\det((,))} 0_X . \tag{4.6.1}$$

PROPOSITION. *The image of* $u : F \to 0_X$ *is the ideal of* $\sigma_{p+1}(A)$.

Indeed, by (4.5), a point $t : T \to \mathrm{Grass}_n(E)$ lies in $\sigma_{p+1}(A)$ if and only if $t^*v = 0$. Certainly, $t^*u = 0$ if $t^*v = 0$; the converse holds because P is locally free.

(4.7) Let K be a k-subbundle of E. If $k \leqq (e-n)$, then $\sigma_k(K) = \mathrm{Grass}_n(E/K)$, for they have the same T-points, the n-quotients N of $(E/K)_T$, and their embedding in $\mathrm{Grass}_n(E)$ corresponds to reinterpreting N as an n-quotient of E_T. Furthermore, the following diagram of canonical closed immersions is clearly commutative:

$$\begin{tikzcd} \mathrm{Grass}_n(E) \arrow[r, hook, "\pi"] & \boldsymbol{P}(\bigwedge^n E) \\ \sigma_k(K) = \mathrm{Grass}_n(E/K) \arrow[u, hook] \arrow[r, hook] & \boldsymbol{P}(\bigwedge^n (E/K)) \arrow[u, hook] \end{tikzcd}$$

Dually, in view of (4.5) and (4.2), if $k \geqq m = (e-n)$, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Grass}_n(E) \arrow[r, hook, "\pi"] & \boldsymbol{P}(\bigwedge^n E) & \\ & & \boldsymbol{P}(\bigwedge^m E^*) \arrow[ul, "i", "\sim"'] \\ \sigma_m(K) = \mathrm{Grass}_m(K^*) \arrow[uu, hook] \arrow[r, hook] & \boldsymbol{P}(\bigwedge^m K^*) \arrow[ur, hook] & \end{tikzcd}$$

In particular, if $k = (e-n-1)$, then $\sigma_k(K) = \boldsymbol{P}((E/K)^*)$, and, if $k = (e-n+1)$, then $\sigma_{(e-n)}(K) = \boldsymbol{P}(K)$; furthermore, the Plücker morphism π of $\mathrm{Grass}_n(E)$ embeds them as linear subspaces of $\boldsymbol{P}(\bigwedge^n E)$.

(4.8) *Determinantal Varieties*

Let S_0 be an affine scheme, and denote by $D_z(n, a)$ the affine S_0-scheme of $(n \times a)$-matrices $[U_{ij}]$ whose minors of order z are all zero. Clearly, $D_1(n, a) = S_0$, and $D_z(n, a)$ is a cone. Further it is a theorem of Eagan and Northcott that $D_n(n, a)$ is Cohen-Macaulay ([5]).

PROPOSITION (4.9). *$D_2(n, a)$ is the projecting cone over $\boldsymbol{P}^{(n-1)} \times \boldsymbol{P}^{(a-1)}$ embedded by the Segre morphism.*

Indeed, $D_2(n, a)$ is the closed subscheme of $A^{na}_{S_0}$ defined by the homogeneous ideal I generated by the relations

$$\{U_{ij}\,U_{kl} - U_{il}\,U_{kj}\}, \qquad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq a.$$

in the polynomial ring $k[\{U_{ij}\}]$.

On the other hand, $\boldsymbol{P}^{(n-1)} \times \boldsymbol{P}^{(a-1)}$ is embedded by the Segre morphism in $\boldsymbol{P}^{(na-1)}$ as the closed subscheme whose homeogeneous coordinate ring is the image of the homomorphism

$$g : k[\{U_{ij}\}] \to k[\{X_i\}] \otimes k[\{Y_j\}], \qquad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq a,$$

defined by $g(U_{ij}) = X_i \otimes Y_j$.

Let

$$M = M^N(i_1, \cdots, i_N, j_1, \cdots, j_N) = (\prod_{k=1}^{N} X_{i_k}) \otimes (\prod_{k=1}^{N} Y_{j_k}),$$

and let $R(M)$ be the set of

$$P(\beta) = \prod_{\alpha \in A} U_{\alpha, \beta(\alpha)},$$

where $A = (i_1, \cdots, i_N)$, and β runs over the set of all bijections $\beta : A \to \{j_1, \cdots, j_N\}$. Then $R(M)$ consists precisely of those monomials in $k[\{U_{ij}\}]$ whose image under g is M. Since any polynomial P has the form $(\sum_M \sum_{P \in R(M)} a_P P)$, it will be sufficient to show that any two elements of $R(M)$ are congruent (modulo I). But, clearly, one need only consider elements $P(\beta)$ and $P(\gamma)$, where β and γ differ by a single transposition of $\{j_1, \cdots, j_N\}$, in which case the assertion is trivial.

THEOREM (4.10). *With the notation of* (4.1), *let* $X = \mathrm{Grass}_n(E)$, *and let* $x \in (X - \sigma_{(r+1)}(A))$. *Then there exists an affine subscheme S_0 of S, a* (*standard*) *affine neighborhood $U = A^{mn}_{S_0}$ of x, and a linear subspace V of dimension* $[mn-(n-c)(a-c)]$, *depending only on x and r, such that* $\sigma_p(A) \cap U = V \times D_z(n-c, a-c)$, *with* $p = (r+1-z)$, *and* $c = (a-r)$.

Indeed, let $A(x) = A \otimes_{0_S} k(x)$, $E(x) = E \otimes_{0_S} k(x)$, and $Q(x) = Q \otimes_{0_S} k(x)$. Then $x \in \sigma_{(r+1)}(A)$ if and only if the homomorphism $\bigwedge^c A(x) \to \bigwedge^c Q(x)$ is not zero. Hence there are elements $\{f_1(x), \cdots, f_c(x)\}$ whose images in $Q(x)$ are linearly independent. Extend $\{f_1(x), \cdots, f_c(x)\}$

to a basis of $E(x)$ so that $\{f_1(x), \cdots, f_n(x)\}$ generate $Q(x)$. Replacing S by a suitably small affine S_0, we may assume that the $f_i(x)$ are represented by global sections f_i of E which are independent at every point of S_0.

Let E' (resp., E'') be the subbundle of E generated by $\{f_1, \cdots, f_n\}$ (resp., by $\{f_{n+1}, \cdots, f_{n+m}\}$. For any S_0-scheme T, let $U(T)$ be the set of n-quotients N of E_T such that $E'_T \to N$ is surjective. Clearly $x \in U$, by construction. Since any surjective homomorphism $E'_T \to N$ is necessarily bijective, $U(T)$ can be identified with $\mathrm{Hom}_{O_T}(E''_T, E'_T)$, the set of $(m \times n)$-matrices over $\Gamma(O_T)$.

In particular, to any T-point t there corresponds a surjective homomorphism $g : E_T \to Q_T$, and hence a matrix of the form $[IM]$, where I is the identity $(n \times n)$-matrix, and $M \in \mathrm{Hom}_{O_T}(E''_T, E'_T)$. Let $g' : A_T \to Q_T$ be the composition of g with the injection of A_T into E_T. Then to g' there corresponds a submatrix M' of $[IM]$ of the form $\begin{bmatrix} I & T_1 \\ 0 & T_2 \end{bmatrix}$, where I is the identity $(c \times c)$-matrix, and T_2 is an $(n-c) \times (a-c)$-submatrix of M.

By definition, $t \in \sigma_p(A)$ if and only if $\bigwedge^{(a-p+1)}(M') = 0$. Since $\{f_1, \cdots, f_c\}$ generate a free submodule of Q_T, then ([1], (2.5), (ii)$\Leftrightarrow$(iii)) shows that this is equivalent to $\bigwedge^z T_2 = 0$. Hence $T_2 \in D_z(n-c, a-c)$. Moreover, this condition on M involves only the submatrix T_2, so that its remaining entries freely generate a linear subspace V of U, of dimension $[mn-(n-c)(a-c)]$. Hence $[\sigma_p(A) - \sigma_r(A)] \cap U = V \times D_z(n-c, a-c)$.

COROLLARY (4.11). *The normal cone of* $[\sigma_p(A) - \sigma_{(p+z)}(A)]$ *along* $[\sigma_{(p+z-1)}(A) - \sigma_{p+z}(A)]$ *is an algebraic fiber bundle over* $[\sigma_{(p+z-1)}(A) - \sigma_{(p+z)}(A)]$ *with fiber* $D_z(n-c, a-c)$.

COROLLARY (4.12). $[\sigma_p(A) - \sigma_{(p+1)}(A)]$ *is locally isomorphic to affine* $[mn - p(n-a+p)]$*-space.*

COROLLARY (4.13). (i) $D_z(n, a)$ *is irreducible, of dimension* $(na - (a-z+1)(n-z+1))$, *and nonsingular in codimension* 1.

(ii) $D_n(n, a)$ *is normal and Cohen-Macaulay.*

Indeed, (i) holds because, choosing $m = a$ in (4.10), then $D_z(n, a)$ is isomorphic to $\sigma_{(a-z+1)}(A)$, which has the required properties. (See [1], (3.3)). Moreover, $D_z(n, a)$ is Cohen-Macaulay by [5], hence it is normal, since it is nonsingular in codimension 1 by (i) above. Hence (ii) holds.

5. The monoidal transformation of $\sigma_p(A)$ with center $\sigma_{(p+1)}(A)$

(5.1) Preserve all the notation of (4). Set $Y = \mathrm{Grass}_{(n-a+p)}(E/A)$, and $Z = \mathrm{Grass}_{(a-p)}(A)$. Let R be the universal quotient on Y, and J the universal subbundle on Z. Let $K = \operatorname{Ker}(E_Y \to R)$. Then $A_Y \subset K$ and $J \subset A_Z$. Set

$$\sum = \operatorname{Grass}_{(a-p)}(K_{Y\times Z}/J_{Y\times Z})$$

Essentially by definition, a T-point of $\sum$ corresponds to the following triple:

$$(5.1.1)\quad \begin{cases} \text{a) a } p\text{-subbundle } J' \text{ of } A_T \\ \text{b) a } (b+a)\text{-subbundle } K' \text{ of } E_T \text{ containing } A_T, \text{ where} \\ \qquad b = (e-n-p) \\ \text{c) an } (e-n)\text{-subbundle } P' \text{ of } K' \text{ containing } J'. \end{cases}$$

Define a morphism $f: \sum \to \operatorname{Grass}_n(E)$ by mapping t to the T-point $f(t)$ defined by the $(e-n)$-subbundle P' of E_T; since $P' \cap A_T \supset J'$, the map $\bigwedge^q A_T \to \bigwedge^q(E_T/P')$ factors through $\bigwedge^q(A_T/J') = 0$, hence $f(t)$ lies in $\sigma_p(A)$.

THEOREM (5.2). *Then $f: \sum \to \sigma_p(A)$ is the monoidal transformation with center $\sigma_{(p+1)}(A)$. Over $U = (\sigma_{(p+z)}(A) - \sigma_{(p+z+1)}(A))$, the exceptional locus restricts to*

$$f^{-1}(U) = \operatorname{Grass}_z(P_U \cap A_U) \times \operatorname{Grass}_{(n-a+p)}(E_U/(P_U + A_U))$$

and its relatively projective embedding is given by the Plücker and Segre embeddings. (Note that $P_U \cap A_U$ and $(P_U + A_U)$ are subbundles of E_U by [1], (2.6).)

Indeed, in view of (4.6) and (3.2), it suffices to define an embedding $i: \sum \hookrightarrow \boldsymbol{P}(F)$ such that $i(\sum)$ is the closure of the section s over $U_0 = (\sigma_p(A) - \sigma_{(p+1)}(A))$ defined by $u|U_0$.

$$\begin{tikzcd} \sum \arrow[r, dashed, "i"] \arrow[d, "g"'] & \boldsymbol{P}(F) \\ \operatorname{Grass}_b(P) \times \operatorname{Grass}_b(B^*) \arrow[r, "\text{Plücker}\times 2"] & \boldsymbol{P}(\bigwedge^b P) \times \boldsymbol{P}(\bigwedge^b B^*) \arrow[u, "\text{Segre}"'] \end{tikzcd}$$

Define g by mapping the triple (5.1.1) to the pair: the p-subbundle J' of P' $(=(f \circ t)^* P)$, and the b-subbundle (K'/A_T) of B_T.

The image of $\sum(T)$ under g is contained in the set of triples:

$$(5.2.1)\quad \begin{cases} \text{a) an } (e-n)\text{-subbundle } P' \text{ of } E_T, \\ \text{b) a } p\text{-subbundle } J' \text{ of } P' \text{ which is also a subbundle of } A_T, \\ \text{c) a } (b+a)\text{-subbundle } K' \text{ of } E_T \text{ containing both } A_T \text{ and } P'. \end{cases}$$

Conversely, such a triple determines a triple of the form (5.1.1). Thus g is a monomorphism, and (5.2.1) describes its image.Therefore g is closed immersion, because it is proper [7], being complete, and $f^{-1}(U)$ has the asserted form.

Let $\sum_0 = f^{-1}(U_0)$. Then $\sum_0$ contains all points of depth 0, because $\sum$ is flat over S, its geometric fibers $\sum(s)$ are integral, and $\sum_0$ induces a dense open subset in each $\sum(s)$; hence $\sum$ is the scheme-theoretic closure of $\sum_0$. By the computation of $f^{-1}(U)$, $f|\sum_0$ is an isomorphism. It remains to prove that $i = (s \circ f)$ on $\sum_0$.

Under i, the triple (5.1.1) is mapped to the 1-quotient $L = (\bigwedge^b(P'/J')) \otimes (\bigwedge^b(K'/A_T)^*)$ of $F_T = (f \circ t)^* F$. Consider the diagram

(5.2.2)
$$\begin{tikzcd} u' : F_T \arrow[r, "1\otimes v'"] \arrow[d] & (\bigwedge^b P') \otimes (\bigwedge^b (P')^*) \arrow[r] & 0_T \arrow[d, "id"] \\ \chi : L \arrow[r, "w\otimes 1"] & (\bigwedge^b(K'/A_T)) \otimes (\bigwedge^b(K'/A_T)^*) \arrow[r] & 0_T \end{tikzcd}$$

in which w is induced by the inclusions of P' in K' and of J' in A_T; and u' is the pull-back of (4.6.1) to T. Set $T_0 = t^{-1}(\sum_0)$. Since the triple (5.1.1) restricted to T_0 corresponds under $(s \circ f)$ to the surjection $u'|T_0$, to complete the proof it suffices to prove that $\chi|T_0$ is an isomorphism and that (5.2.2) commutes.

On T_0, $(P'+A_T) = K'$, and $P' \cap A_T = J'$; hence $w|T_0$ and $\chi|T_0$ are isomorphisms. Finally, the commutativity of (5.2.2) is no more than the commutativity of the following two diagrams of canonical maps:

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P' \arrow[d] \\ (P'/J) \arrow[r] & (K'/A_T) \end{tikzcd} \qquad \begin{tikzcd} (E_T/A_T) & P' \arrow[l] \arrow[d] \\ (K'/A_T) \arrow[u] & (K'/A_T) \arrow[l] \end{tikzcd}$$

COROLLARY (5.3). *When* $p = 0$, *then* $Y = \mathrm{Grass}_{(n-a)}(E/A)$ *and* $K = \mathrm{Ker}\,(E_Y \to R)$ *is an* $(e-r+a)$*-bundle, where* R *is the universal quotient on* Y; *and* $f : \sum = \mathrm{Grass}_a(K) \to X = \mathrm{Grass}_n(E)$ *is the monoidal transformation with center* $\sigma_1(A)$.

(5.4) For $0 \leqq z \leqq (\min(a, e-n)-p)$, let $f_z : \sum_z \to \sigma_{p+z}(A)$ be the monoidal transformation with center $\sigma_{p+z+1}(A)$. In view of (5.1), $\sum_z$ carries a universal $(p+z)$-subbundle J_z of A_{Σ_z}, and a universal $(a+m-p-z)$-subbundle K_z of E_{Σ_z} containing A_{Σ_z}. Set

$$\Phi_z = \mathrm{Grass}_z(J_z) \times \mathrm{Grass}_{(n+p-a)}(E_{\Sigma_z}/K_z)$$

[7] A proper monomorphism $g : X \to Y$ is (well-known to be) a closed immersion. Indeed, each nonempty fiber of g is a monomorphism onto a field; hence an isomorphism, by considering Δ_g. Thus g is finite, and, by Nakayama's Lemma, $0_Y \to g_* 0_X$ is surjective.

PROPOSITION. *There is a natural surjective X-morphism*

$$g_z : \Phi_z \to f^{-1}\sigma_{p+z}(A).$$

Indeed, a T-point t of Φ_z corresponds to a sequence of subbundles of E_T

$$J' \subset J'_z \subset P' \subset K'_z \subset K',$$

of ranks p, $(p+z)$, m, $(a+m-p-z)$, $(a+m-p)$ respectively, such that $J'_z \subset A_T \subset K'_z$. Define $g_z : \Phi_z \to \sum$ by $g_z(t) = (J', P', K')$. Since $J'_z \subset (P' \cap A_T)$, it follows that $g_z(t)$ lies in $f^{-1}\sigma_{p+z}(A)$. Finally, if T is the spectrum of a field, and (J', P', K') is a triple of subspaces defining a T-point of $f^{-1}\sigma_{p+z}(A)$, then there exist subspaces J'_z and K'_z of ranks $(p+z)$ and $(m-p-z)$ such that $J' \subset J_z \subset (P' \cap A_T)$ and $(P'+A_T) \subset K'_z \subset K'$; hence g_z is onto $f^{-1}\sigma_{p+z}(A)$.

6. Standard modifications

(6.1) As in (4.1), let $X = \mathrm{Grass}_n(E)$, let $Y = \mathrm{Grass}_{(n-a+p)}(E/A)$, let R be the universal quotient on Y, and let $K = \mathrm{Ker}(E_Y \to R)$. Necessarily, rank $(K) = (e-n+a-p)$.

Let $X' = \mathrm{Grass}_{(a-p)}(K)$. For any S-scheme T, a T-point of X' corresponds to an $(e-n+a-p)$-subbundle K' of E_T containing A, and an $(e-n)$-subbundle P' of E_T contained in K'. In particular, to P' there corresponds a T-point s of X, which is easily seen to lie in $\sigma_p(A)$. Thus there is a natural morphism $f: X' \to \sigma_p(A)$, defined on T-points by $f(t) = s$; call f the *standard modification* of $\sigma_p(A)$.

Let P be the universal subbundle on X, and, for any non-negative integer r, set $U = [\sigma_{(p+r)}(A) - \sigma_{(p+r+1)}(A)]$.

PROPOSITION (6.2). *The standard modification* $f: X' \to \sigma_p(A)$ *is an isomorphism off* $\sigma_{(p+1)}(A)$, *and* $f^{-1}U = \mathrm{Grass}_{(n-a+p)}(E_U/P_U + A_U)$.

Indeed, by [1], (2.6), an $(e-n)$-subbundle P' of E_T determines a T-point of U if and only if $(P'+A_T)$ is an $(a+e-n-p-r)$-bundle. Fix a T-point s of U, defined by an $(e-n)$-subbundle P' of E_T. Then a T-point of $f^{-1}(s)$ corresponds to an $(e-n+a-p)$-subbundle K' of E_T containing P', hence to an $(n-a+p)$-quotient of $(E_T/P_T + A_T)$.

Finally, if $r = 0$, then $(P'+A_T)$ is an $(e-n+a-p)$-bundle, so determines a unique T-point t of X'. The morphism $s \mapsto t$ is clearly an inverse for f.

COROLLARY (6.3). *If* $p \geqq 1$, *then* $\sigma_{(p+1)}(A)$ *is the full singular locus of* $\sigma_p(A)$.

Indeed, by (5.9), $[\sigma_p(A) - \sigma_{(p+1)}(A)]$ is nonsingular. Conversely, the codimension of $f^{-1}\sigma_{(p+1)}(A)$ in X' is $(p+1)$. However, the restriction $\bar{f}$ of f to any of the geometric fibers over S is a birational morphism. Hence, if $\sigma_p(A)$ were non-singular at a geometric point of $\sigma_{(p+1)}(A)$, then the exceptional locus of $\bar{f}$ would necessarily have codimension one ([8], p. 415, Prop. 1), whence the assertion.

(6.4) Set $Z = \mathrm{Grass}_{(a-p)}(A)$, and let J be the universal subbundle on Z. Let

$$F^* : \mathrm{Grass}_{(e-p+n)}((E_Z/J)^*) \to \sigma_{(p+n-a)}((E/A)^*)$$

be the standard modification. Define the *dual standard modification* to be the morphism

$$F : \mathrm{Grass}_n(E_Z/J) \to \sigma_p(A)$$

induced from F^* by the duality isomorphisms (1.2) and (1.5). Explicitly, a T-point of $\mathrm{Grass}_n(E_Z/J)$ corresponds to an $(e-n)$-subbundle $\bar{B}$ of (E_T/J_T), hence to an $(e-n)$-subbundle B of E_T containing J_T, thence to a T-point of $\sigma_p(A)$. It follows from (6.2) that F is an isomorphism off $\sigma_1(R)$, where R is the universal quotient on Z, and that $F^{-1}U = \mathrm{Grass}_r(P_U \cap A_U)$.

(6.5) Let $f: \mathrm{Grass}_{(a-p)}(K) \to \sigma_p(A)$ be the standard modification. Let C be an $(a-p)$-subbundle of A, let $Y' = \mathrm{Grass}_{(n-a)}(E/C)$, and let $f' : \mathrm{Grass}_a(K') \to X$ be the standard modification. By (5.3), f' is the monoidal transformation of X with center $\sigma_1(C)$.

LEMMA (6.5.1). *f is the pull-back of f' via the canonical immersion of Y in Y'.*

Indeed, let R (resp., R') be the universal quotient on Y (resp., Y'). Then $R = R'_Y$. Pulling back to Y the canonical exact sequence $0 \to K' \to E_{Y'} \to R' \to 0$ then yields $K = K'_Y$, whence the assertion.

THEOREM (6.5.2). *The standard modification $f: \mathrm{Grass}_{(a-p)}(K) \to \sigma_p(A)$ is the monoidal transformation of $\sigma_p(A)$ with center $\sigma_p(A) \cap \sigma_1(C)$.*

Indeed, by (3.3.1), it will suffice to show that $\mathrm{Grass}_{(a-p)}(K)$ is the scheme-theoretic closure of $(f')^{-1}[\sigma_p(A) - \sigma_1(C)]$.

Let $i' : \mathrm{Grass}_a(K') \to Y'$ be the structure map, and let U' be the complement in $\mathrm{Grass}_a(K')$ of the exceptional divisor $\sigma_1(C_Y, K')$ of f'. Then there is a commutative diagram

$$\begin{tikzcd} & U' \arrow[dl, "i'|U'"'] \arrow[dr, "f'|U'"] & \\ Y' & & {[X-\sigma_1(C)]} \arrow[ll, "p"'] \end{tikzcd}$$

where $f'|U'$ is an isomorphism, and $p = (f'|U')^{-1} \circ i'|U'$. Moreover if t is a T-point of $[\sigma_p(A)-\sigma_1(C)]$, it follows easily by [1], (2.5) that $p(t)$ lies on $\sigma_p(A/C)$. Then

$$(f')^{-1}[\sigma_p(A)-\sigma_1(C)] = [(i')^{-1}\sigma_p(A/C)] \cap U',$$

and it will suffice to show that $\mathrm{Grass}_{(a-p)}(K)$ is the scheme-theoretic closure of $[(i')^{-1}\sigma_p(A/C)] \cap U'$ in any of the geometric fibers of i', since the latter induces an open dense subset in each of them.

However, i' is a flat morphism by construction, and its geometric fibers are integral. Hence the scheme-theoretic closure of $[(i')^{-1}\sigma_p(A)] \cap U'$ in any geometric fiber of i' can have no embedded components, which completes the proof.

(6.6) With the notation of (6.1), let $\sum = \mathrm{Grass}_{(a-p)}(K_{Y\times Z}/J_{Y\times Z})$, and let $f : \sum \to \sigma_p(A)$ be the monoidal transformation of $\sigma_p(A)$ with center $\sigma_{(p+1)}(A)$. (cf. (5.2)).

Set $Z' = \mathrm{Grass}_{(a-p)}(A_Y)$, and let J' be the universal subbundle on Z'. Let $g : \mathrm{Grass}_{(a-p)}(K) \to \sigma_p(A)$ be the standard modification, and let $h : \mathrm{Grass}_{(a-p)}(K_{Z'}/J') \to \mathrm{Grass}_{(a-p)}(K)$ be the dual standard modification. Since $Z' = Z \times Y$, it is clear that $\mathrm{Grass}_{(a-p)}(K_{Z'}/J') = \sum$. Thus there is a diagram

$$\begin{tikzcd} \sum \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathrm{Grass}_{(a-p)}(K) \arrow[dl, "g"] \\ & \sigma_p(A) & \end{tikzcd}$$

For any integer $r \geqq 0$, let $U = [\sigma_{(p+r)}(A)-\sigma_{(p+r+1)}(A)]$. Set $G_1 = \mathrm{Grass}_r(P_U \cap A_U)$, and $G_2 = \mathrm{Grass}_{(n-a+p)}(E_U/(P_U+A_U))$. Recall that $f^{-1}U = G_1 \times G_2$. Let $p_2 : G_1 \times G_2 \to G_2$ be the projection.

Proposition (6.7). *Under the above conditions* $f = g \circ h$, *and* $h|f^{-1}U = p_2$.

Indeed, recall that a T-point of $\sum$ corresponds to a triple of bundles (J', K', P'), where J' is a p-subbundle of A_r, K' is an $(e-n+a-p)$-subbundle of E_T, and P' is an $(e-n)$-subbundle of K' containing J'. By definition, $h(t)$ corresponds to the pair of subbundles (P', K') of E_T, and $g(t)$ to P'; hence $g \circ h = f$.

Assume that $f(t)$ lies in U. Then $(P'+A_T)$ is an $(a+e-n-p-r)$-subbundle of E_T, $P' \cap A_T$ is a $(p+r)$-bundle containing J', and t corresponds to the T-point of $G_1 \times G_2$ determined by the pair of bundles $(J', K'/(P'+A_T))$. However, $h(t)$ corresponds to the T-point of G_2 determined by $K'/(P'+A_T)$, whence the assertion.

7. Geometry of twisted subvarieties

Let k be an algebraically closed field, $X = \mathrm{Grass}_n(V)$ a Grassmann variety, where V is a k-vector space of dimension v, and X' a nonsingular, irreducible subvariety of X. Assume X' is twisted in X; i.e., the universal quotient Q on X induces a bundle $Q(-1)|X'$ which is generated by its global sections. Fix a "sufficiently general" subspace A of V of dimension a, and set $S_p = X' \cap \sigma_p(A)$.

THEOREM (7.1). *Under the above conditions:*
(i) *Codim* $(S_p, X) = (v-a+p)p$
(ii) $S_{(p+1)}$ *is the full singular locus of* S_p.

Indeed, (i) is a corollary of [1], (3.3). Let $x \in S_p$, and assume $\sigma_p(A)$ is nonsingular at x. Then S_p is nonsingular at x by [1], (3.3). The converse is an application of the following general fact.

LEMMA (7.2). *Let X be a scheme, X' and Y subschemes, $Y' = X' \cap Y$, and $x \in Y'$. Suppose that X, Y, and Y' are regular at x, that X' is irreducible at x, and that $\mathrm{codim}_x(Y', X') = \mathrm{codim}_x(Y, X)$. Then X' is also regular at x.*

Indeed, there are functions $f_1, \cdots, f_n$, where $n = \mathrm{codim}_x(Y, X)$, which cut out Y at x. Let I (resp., I'; resp., J) be the ideal of Y in X (resp., of Y' in Y; resp., of X' in X) at x. Then $I' = J/I$; hence choosing functions $\bar{g}_1, \cdots, \bar{g}_m$, with $m = \mathrm{codim}_x(Y', Y)$, which cut Y' out of Y at x gives functions $g_1, \cdots, g_m$ which vanish on X', and whose restrictions to Y cut out Y'. Then $f_1, \cdots, f_n, g_1, \cdots, g_m$ cut Y' out of X at x, so form part of a regular system of parameters at x. Hence $g_1, \cdots, g_m$ cut out a scheme which is regular at x, lies in X', and has the same codimension in X at x as X', whence the assertion.

THEOREM (7.3). S_p *is irreducible if* $\dim (S_p) \geqq 1$.

Indeed, in view of the constructibility of the property, it suffices to extend the ground field k to a universal domain Ω, and to consider the case where A is generic over k and $\dim (S_p) \geqq 1$. We will employ the approach of [1], (3.3). Then $(X - \sigma_{(p+1)}(A))$ is covered by certain affines U, and we shall show that the $S_p \cap U$ are geometrically irreducible and all contain a common point x. Then $(S_p - S_{(p+1)})$ is irreducible. Now every component of S_p has dimension at least $d = [\dim (X') + \dim (\sigma_p(A)) - \dim (X)]$, because X is nonsingular. But $S_p = (S_p - S_{(p+1)}) \cup S_{(p+1)}$, while $\dim (S_p - S_{(p+1)}) = d$, and $d > \dim (S_{(p+1)})$. Therefore S_p is irreducible.

In [1], (3.3), a certain k-basis (e_i) of V is fixed, and A is taken as the vector space spanned by independent, k-generic linear combinations f_i of the e_i's. Each affine U arises from an ordering of the e_i's and the f_i's as the standard affine corresponding to the two sets $f_1, \cdots, f_{(a-p)}, e_{(a-p+1)}, \cdots, e_n$, and $e_{(n+1)}, \cdots, e_v$.

Let $g_1, \cdots, g_p$ be independent generic linear combinations of the f_i's, and let $g_{(p+1)}, \cdots, g_{(v-n)}$ be generic linear combinations of the e_i's which are independent modulo A. Then the point x of X corresponding to the subspace spanned by the g_i's is a generic point of $\sigma_p(A)$ which is easily seen to lie in every U.

Finally, an analysis of the coordinates on U like that done in [1], (3.3) easily shows that, for the irreducibility of $S_p \cap U$, the following form of Bertini's Theorem is sufficient.

LEMMA (7.4). *Let k be a field, $U = \operatorname{Spec}(k[t_1, \cdots, t_N])$ affine N-space, and V a subscheme. Let $L = k(s_1, \cdots, s_m)$, and $K = k(s)$ be pure transcendental extensions, $f = s_1 t_1 + \cdots + s_m t_m + f_1$, with $f_1 \in k[t_{(m+1)}, \cdots, t_N]$, and $W = V_K \cap \{f = s\}$. Assume that the projection of V onto the coordinates $t_1, \cdots, t_m$ is an embedding. If V is geometrically integral of dimension $d \geqq 2$, then W is geometrically integral of dimension $(d-1)$.*

Indeed, let Ω be a universal domain for k. Let $(y_1, \cdots, y_N) \in V(\Omega)$ be a generic point over L, and set $s' = f(y_i)$. Then s' lies in $L(y_i)$, which is a regular extension of L. If s' were algebraic over L, then $s' \in L$, and the elements $s', s_1, \cdots, s_m, 1 \in L$ would become linearly dependent over $k(y_i)$. Since $k(y_i)$ and L are linearly disjoint over k there would exist elements $a_1, \cdots, a_m, a \in k$ such that $s' = s_1 a_1 + \cdots + s_m a_m + a$. Since $s_1, \cdots, s_m$ are linearly independent over k, then $y_1 = a_1, \cdots, y_m = a_m$, and $f(y_i) = a$. Hence $d = 0$, a contradiction. Therefore s' is transcendental over L and an L-automorphism of Ω which maps s' to s will map (y_i) to a point in $W(\Omega)$; replacing (y_i) by it, we may assume that the generic point of $V(\Omega)$ over L lies in $W(\Omega)$.

Let (x_i) be an arbitrary point in $W(\Omega)$. Since it lies in $V(\Omega)$, there is a specialization $(y_i) \mapsto (x_i)$ over L. Since $s = s_1 x_1 + \cdots + s_m x_m + f_1(x_i)$, the specialization leaves s fixed, so is actually over K. Thus $W(\Omega)$ is the locus of (y_i) over K. Therefore W is irreducible, of dimension $(d-1)$. It is reduced, because, by [1], (3.4), it is generically smooth, and does not contain any of the finite number of points of V of depth 1 and codimension $\geqq 2$.

It remains to prove that $K(y_i)$ is a regular extension of K. Reordering $y_1, \cdots, y_m$, we may assume $y_1 \notin L(y_i^p)$, where $p = \operatorname{char}(L)$. Let $y = s_1 y_1 + \cdots + s_m y_m + f_1(y_i)$. Then $s = y + s_1 y_1$. Hence $K(y_i) = L(y_i)$ is a regular extension of $K = L(y + s_1 y_1)$ by the lemma of Zariski-Matsusaka.

THEOREM (7.5). *S_p is Cohen-Macaulay and normal.*

Indeed, this is an immediate consequence of (4.13).

THEOREM (7.6). *The completion of* $(S_p - S_{(p+2)})$ *along* $(S_{(p+1)} - S_{(p+2)})$ *is locally isomorphic to the product of* $(S_{(p+1)} - S_{(p+2)})$ *with the completion at the vertex of the projecting cone* C *over* $\mathbf{P}^{(v-a+p)} \times \mathbf{P}^p$, *projectively embedded by the Segre morphism.*

Indeed, by (4.11) and (4.9), the normal cone of $(\sigma_p(A) - \sigma_{(p+2)}(A))$ along $(\sigma_{(p+1)}(A) - \sigma_{(p+2)}(A))$ is an algebraic fiber bundle whose fiber is C. Hence, by (3.4) and (7.3), every point of $(S_{(p+1)} - S_{(p+2)})$ has an affine neighborhood U such that the normal cone of $(S_p - S_{(p+2)})$ along U is isomorphic to $U \times C$. In view of (2.1.2), since U is smooth by (7.1), it is rigid by (1.2.2); and, since C is rigid by (2.2.8), then $U \times C$ is rigid by (1.4.1). Therefore, after an algebraic reformulation, the conclusion results from Gerstenhaber's theorem (2.3.2).

THEOREM (7.7). *Let* $f' : \sum' \to S_p$ *be the monoidal transformation with center* $S_{(p+1)}$, *and set* $F'_r = (f')^{-1}(S_{(p+r)} - S_{(p+r+1)})$. *Then* $\sum'$ *is nonsingular, and* F'_r *is an algebraic fiber bundle of the form*

$$F'_r = \mathrm{Grass}_r(B') \times \mathrm{Grass}_{(n-a+p)}(C'),$$

where B' *is a* $(p+r)$*-bundle, and* C' *an* $(n-a+p-r)$*-bundle.*

Indeed, let $F : \Sigma \to \sigma_p(A)$ be the monoidal transformation with center $\sigma_{(p+1)}(A)$, and set $F_r = f^{-1}(\sigma_{(p+r)}(A) - \sigma_{(p+r+1)}(A))$; consider $\sum'$ as the closure of $(S_p - S_{(p+1)})$ in $\sum$, and $f' = f | \sum'$. (See (3.3)). In view of (5.2), (5.4), (7.1) and (7.3), the conditions of (3.4.2) are met, and it implies that F'_r has the asserted form, that $\mathrm{codim}(F'_r, \sum') = \mathrm{codim}\,(F_r, \sum)$, and that F'_r is nonsingular. Since $\sum$ and F_r are nonsingular, and $F'_r = F_r \cap \sum'$, the nonsingularity of $\sum'$ results from (7.2).

THEOREM (7.8). *Let* $g : \Delta \to \sigma_p(A)$ *be the standard modification,* Δ' *the closure of* $(S_p - S_{(p+1)})$ *in* Δ, *and* $g' = g | \Delta'$. *Let* $G_r = g^{-1}(\sigma_{(p+r)}(A) - \sigma_{(p+r+1)}(A))$, *and let* $G'_r = (g')^{-1}(S_{(p+r)} - S_{(p+r+1)})$. *Then* Δ' *is nonsingular, and* G'_r *is the algebraic fiber bundle induced by* G_r, *so has the form*

$$G'_r = \mathrm{Grass}_{(n-a+p)}(C'),$$

where C' *is an* $(n-a+p-r)$*-bundle.*

Indeed, by (6.7), the monoidal transformation $f : \sum \to \sigma_p(A)$ factors as $f = gh$ in such a way that h acts on $F_r = f^{-1}(\sigma_{p+r}(A) - \sigma_{p+r+1}(A))$, which has the form $\mathrm{Grass}_r(B') \times \mathrm{Grass}_{(n-a+p)}(C')$, as projection onto the second factor. In the proof of (7.7) it is shown that $F'_r = (f')^{-1}(S_{(p+r)} - S_{(p+r+1)})$ is the fiber bundle induced by F_r. Hence G'_r is dense in the bundle induced by G_r; since the former is closed, and the latterreduced, they coincide. Finally, since Δ, G_r, and G'_r are nonsingular by (6.2) and (7.1), and since $G'_r = G_r \cap \Delta'$ by definition, Δ' is nonsingular by (7.2).

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