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# ON THE EXISTENCE OF DISTINGUISHED BASES IN A $V$-SPACE

by

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In a sequence of papers [1], Pierre Robert introduced, among other things, the concepts of a $V$-space and a distinguished basis and proves the existence of distinguished bases in a $V$-space. In this note, we present a completely different proof which make no use of the 'Modified Riesz's Lemma' [1, I, p. 12] [1]), of the existence of a distinguished basis in a $V$-space. For notation and definitions we refer the reader to [1].

THEOREM. *A $V$-space admits a distinguished basis.*

PROOF. Let $X$ be a $V$-space over the field $F$. Let $\Omega(X)$ [1, p. 16] be ordered according to its natural order. For $i \in \Omega(X)$ let $X_i = \{x \in X : |x| \leqq i\}$. $X_i$ is clearly a $F$-linear space. Let $\overline{X}_0 = \{0\}$, and if $i \neq 0$, let $\overline{X}_i = X_i/X_{p(i)}$, where $p(i)$ is the predecessor of $i$ ($p(i)$ could be equal to 0). Each $\overline{X}_i$ is again a $F$-linear space. Choose a $F$-basis $\overline{H}_i$ of the $F$-linear space $\overline{X}_i$ ($\overline{H}_0 = \emptyset$). For each $\bar{x} \in \overline{H}_i$ choose a unique $x \in X_i$ which is mapped to $\bar{x}$ under the natural projection $X_i \to \overline{X}_i$. (In the following, we shall consistently use – notation in this fashion.) Let $H_i$ be the set of all such $x$'s in $X_i$ ($H_0 = \emptyset$). We claim that $H = \bigcup_{i \in \Omega(X)} H_i$ is a distinguished bases of $X$. To prove this assertion, we have to (1) verify condition (i) of Definition 5.3 in [1, I], and (2) show that $[H] = X$.

(1) First of all we remark that if $\{x_1, x_2, \cdots, x_j\}$ is a finite subset of $H$ such that $|x_1| = |x_2| = \cdots = |x_j|$ and if $\alpha_1, \alpha_2, \cdots, \alpha_j$ are all non-zero elements in $F$, then $|\alpha_1 x_1 + \cdots + \alpha_j x_j| = |x_1|$. Now let $\{x_1, x_2, \cdots, x_n\}$ be a finite subset of $H$. Rename the elements so that $|x_1| = |x_2| = \cdots = |x_j| > |x_{j+1}| \geqq \cdots \geqq |x_n|$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be non-zero elements in $F$. Then $|\alpha_1 x_1 + \cdots + \alpha_j x_j + \cdots + \alpha_n x_n| = |\alpha_1 x_1 + \cdots + \alpha_j x_j| = |x_1| = \max_{1 \leqq m \leqq n} |x_m|$.

(2) Let $x \in X$. Suppose $|x| = i(1)$. Then $x \in X_{i(1)}$. If $i(1) = 0$, then $x \in [H]$. If $i(1) \neq 0$, let $\bar{x}$ be the image of $x$ under the map $X_{i(1)} \to \overline{X}_{i(1)}$. Then $\bar{x} = \alpha_{11}\bar{x}_{11} + \cdots + \alpha_{1n_1}\bar{x}_{1n_1}$ for some $\alpha_{1j} \in F$ ($1 \leqq j \leqq n_1$) and

[1, 2] I am indebted to Professor Pierre Robert for these two comments.

$\bar{x}_{1j} \in \bar{H}_{i(1)}$ $(1 \leqq j \leqq n_1)$. Clearly $|x-(\alpha_{11}x_{11}+\cdots+\alpha_{1n_1}x_{1n_1})| = i(2) < i(1)$. If $i(2) = 0$, then $x \in [H]$. If $i(2) \neq 0$, we repeat. This process either terminates after a finite number of steps or continues indefinitely. If it terminates after a finite number of steps, then $x \in [H]$. If it continues indefinitely, then we have an infinite series

$$(\alpha_{11}x_{11}+\cdots+\alpha_{1n_1}x_{1n_1})+(\alpha_{21}x_{21}+\cdots+\alpha_{2n_2}x_{2n_2})+\cdots$$

with $\alpha_{st} \in F$ and $x_{st} \in H_{i(s)}$. Since $|x-\sum_{k=1}^{s}\sum_{t}\alpha_{kt}x_{kt}|$ is a strictly decreasing function of $s$, the series generated by the above process converges to $x$ [2]. Therefore, $x \in [H]$. This completes the proof.

## REFERENCE

P. Robert

[1] On some non-Archimedean normed linear spaces, I, II, ..., VI, Compositio Mathematica, Vol 19, pp. 1–77, 1968.

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