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FINITE GROUP SCHEMES, LOCAL MODULI FOR ABELIAN VARIETIES, AND LIFTING PROBLEMS

by

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The first draft of these notes was written by K. Lønsted at the Nordic summer school in algebraic geometry, Oslo 1970; I like to thank him and several other participants for their contributions when studying and supplementing the oral exposition. Acknowledgement is due to A. Grothendieck and D. Mumford for conversation and correspondence in which they communicated their results (and sketched the proofs), which can be found in sections 2.2, 2.3, 2.4 (but, of course, they cannot be blamed for mistakes or obscurities in the proofs of this paper). We didn't change the informal style of the paper (excercises and motivational remarks) when writing the final version, because it reflects the atmosphere at the conference.

Notation: the set of morphisms from an object X to an object Y in a category $\mathscr{C}$ will be denoted by $\mathscr{C}(X, Y)$, with the exceptions: HOM (–, –) for groups chemes, and Mor (–, –) for schemes.

1. Introduction

These lectures are mainly concerned with lifting problems in algebraic geometry, and to fix the ideas let us make a preliminary definition.

DEFINITION. Let $R \to R_0$ be a surjective ringhomomorphism, and $X_0 \to S_0 = \text{Spec}(R_0)$ a smooth scheme over S_0, respectively a finite flat group scheme over S_0, resp. an abelian scheme over S_0, resp. $\cdots$; we say that X_0 can be lifted to R or to $S = \text{Spec}(R)$ (in the *strong sense*), if there exists a smooth scheme $X \to \text{Spec}(R)$, resp. $\cdots$, such that:

$$\begin{array}{ccc} X\times_S S_0 \cong X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S. \end{array}$$

The scheme X is called a *deformation* of X_0; in case $R_0 = k$ is a field and R is a local artin ring, we call X an infinitesimal deformation of X_0.

In 1959 Grothendieck proved (cf. FGA, page 182–14, coroll. 4) that a smooth curve over a field k can be lifted to any complete local ring having k as residue class field. In the case of higher dimensions the answer is negative in general, as can be seen from an example constructed by Serre, cf. [13], of a smooth variety over a field of characteristic $p \neq 0$ that cannot even be lifted to characteristic zero in the following weak sense (the example given by Serre has dimension 3, but as Mumford remarked, along the same lines one can construct already an example with the same properties of dimension 2):

DEFINITION. Let k be a field of char $(k) = p \neq 0$, and $V_0 \to \mathrm{Spec}\,(k)$ a smooth variety, resp. $\cdots$; we say that V_0 can be lifted to characteristic 0 (in the *weak sense*), if there exists an integral domain R of characteristic zero, a surjective ringhomomorphism $R \to k$, and a smooth scheme $V \to \mathrm{Spec}\,(R)$, resp. $\cdots$, such that $V \otimes_R k \neq V_0$ (both definitions essentially can be found in [13]).

It is easy to see that finite group schemes cannot be lifted to characteristic 0, for example:

EXERCISE. Let p a prime numer, k a field of char$(k) = p$. Construct a finite, *non-commutative* group scheme G_0 over k of rank p^2. Prove that there exists no flat group scheme over any characteristic zero domain which lifts G_0 [Hint: a finite group scheme over an algebraically closed field of characteristic zero is the constant group scheme attached to an abstract group A. However, a group A if order p^2 is commutative. Construct G_0 by finding a finite group scheme of rank p that acts non-trivially on another finite group scheme of rank p; cf. [8], p. 318, example $(-B)$; cf. [14], pp. 6/7, remark],

Some information about the lifting problem is showed in the summary on the next page.

A possible attempt for solving a lifting problem reads as follows:

a) Consider all infinitesimal deformations of V_0 and find a universal family; its parameter space is called 'the local moduli scheme' for V_0; it turns out to be a formal scheme over a noetherian, complete (integral?) ring $\mathcal{O}$ (of characteristic zero?) with residue class field k.

b) Pick an embedding $V_0 \hookrightarrow \boldsymbol{P}_k^n$, and try to lift the divisor class of a hyperplane section $L_0 = V_0 \cdot H_\infty$ to $\mathcal{O}$, or to some quotient $\mathcal{O}/\mathfrak{a}$.

c) Show that $\mathcal{O}/\mathfrak{a}$ admits a characteristic zero integral quotient $\mathcal{O}/\mathfrak{a} \to R$, and apply EGA, III^1 . 5.4.5: 'a formal scheme with a polarization is algebraizable'.

	strong sense		weak sense
smooth algebraic curve *	+ Grothendieck, 1959 FGA, 182–14, cor.11.4	$\Rightarrow$	+
smooth algebraic variety dim $\geqq 2$	–	$\Leftarrow$	– Serre, 1961, [13]
finite group scheme *	–	$\Leftarrow$	–
commutative finite group scheme *	–		+ Mumford and Oort, 1967, [8]
principally polarized abelian variety *	+ Grothendieck and Mumford, between 1960 and 1966	$\Rightarrow$	+
abelian variety	probably –		+ Mumford, 1967 (not yet published, cf. [5])

– means: in general the answer is negative;
+ means: the answer is positive in case R is a complete local ring (strong sense), respectively the answer is positive (weak sense);
* means: the case is discussed in this paper.

This method works for smooth curves over a field, because the obstruction for lifting the scheme as well as the one for lifting any (ample) divisor lies in some second cohomology group over V_0, hence they are zero, and $\mathcal{O} = \mathcal{O}/\mathfrak{a}$ is a characteristic zero integral domain.

In the steps (a) and (b) the liftings can be split into a succession of liftings over surjections of artin rings, and to illustrate (b) we give an excercise, which was already long time ago noticed by Serre and others:

EXERCISE. Given a surjection of local artin rings $R \to R'$, a scheme X over R, an invertible sheaf $L' \in H^1(X', \mathcal{O}^*_{X'})$ over $X' = X \otimes_R R'$. Suppose char $(R/\mathfrak{m}_R) = p \neq 0$. Show the existence of a $q = p^n$ such that $(L')^q$ lifts to X [Hint: suppose $I = \text{Ker}(R \to R')$ has the property $I \cdot \mathfrak{m}_R = 0$, compute the obstruction for lifting L' to R, and prove it is killed under multiplication by p].

The answer in step (a) in case of an abelian variety can be guessed from the work of Kodaira and Spencer, cf. [3], especially II, section 14γ. A complex torus of dimension g is isomorphic to the complex analytic space $\boldsymbol{C}^g/\Gamma$, where $\Gamma \cong \boldsymbol{Z}^{2g}$ is a lattice in $\boldsymbol{C}^g$, i.e. a free additive subgroup of $\boldsymbol{C}^g$ which spans $\boldsymbol{C}^g \cong \boldsymbol{R}^{2g}$ as a real vectorspace. We choose a $\boldsymbol{C}$-base $\{e_1, \cdots, e_g\}$ for $\boldsymbol{C}^g$ such that

$$\Gamma = \boldsymbol{Z} \cdot e_1 + \cdots + \boldsymbol{Z} \cdot e_g + \boldsymbol{Z} \cdot \rho_1 + \cdots + \boldsymbol{Z} \cdot \rho_g;$$

the choice of this $\mathbf{Z}$-base for Γ is unique up to a unimodular transformation, hence it is intuitively plausible that any small change of the $\boldsymbol{\rho}_i$ will yield a variation of the complex structure; in fact the local moduli space of an abelian variety turns out to be smooth on g^2 parameters (cf. theorem 2.2.1), which is the algebraic analogue of the theory of Kodaira and Spencer (cf. [3], II, section 14γ, especially theorem 14.3). These considerations lead to the following

REMARK. The condition char $(R/\mathfrak{m}_R) \neq 0$ in the previous exercise is necessary. In fact, for any abelian variety X_0 over a field of characteristic zero such that dim $(X_0) = g \geqq 2$, there exists an infinitesimal deformation of X over $R = k[\varepsilon]$ which admits no projective embedding over Spec(R). This can be seen as follows (also cf. [9], remark XII.4.2, page 191): an abelian variety (or a complex torus) of dimension g defines a local moduli space Spf$(k[[t_{1,1} \cdots t_{g,g}]])$ (cf. theorem 2.2.1 below; this is canonically the completion of the local ring of $x_0 \in S$, cf. [3], p. 408); a polarization L_0 on X_0 defines an ideal $\mathfrak{a} = \mathfrak{a}(L_0) \subset \mathcal{O} = k[[\cdots t_{i,j} \cdots]]$ generated by $\frac{1}{2}g(g-1)$ elements (cf. theorem 2.3.4 below), and because char $(k) = 0$, the ring $\mathcal{O}/\mathfrak{a}$ is formally smooth (cf. theorem 2.4.1 below, i.e. the classical Riemann equations, cf. [1], p. 69, theorem 2, p. 70, p. 86 last two lines and p. 90, are the correct defining equations infinitesimally); thus L_0 can be lifted to X_t, where X_t is an infinitesimal deformation of X_0 defined by $t : \text{Spec}\,(k[\varepsilon]) \to \text{Spec}\,(\mathcal{O})$, if and only if t is tangent to the locus defined by $\mathfrak{a}(L_0)$, i.e. iff

$$t \in k^{\frac{1}{2}g(g+1)} = (\text{tangent space of } \mathcal{O}/\mathfrak{a}) \subsetneqq k^{g^2} = (\text{tangent space of } \mathcal{O}),$$

where $\neq$ holds because $g \geqq 2$. Because there are countably many polarization types on X_0, we can choose $t \in k^{g^2}$ not tangent to $\mathcal{O}/\mathfrak{a}(L_0)$ for any polarization L_0 on X_0, and we have proved the existence of $X_t \to$ Spec $(k[\varepsilon])$ which does not admit a projective embedding.

In section 2 we reproduce theorems of Grothendieck and Mumford which show that the proposed attempt for solving the lifting problem works for separably polarized abelian varieties. However it seems not to be powerful enough to solve the lifting problem in the case of abelian varieties or in the case of finite commutative group schemes. It was Mumford who proposed studying equal characteristic deformation theory. In fact, this new method works; section 3 illustrates this in case of commutative finite group schemes.

2. Local moduli for (polarized) abelian varieties

2.1. *Pro-representable functors*

This section is mainly taken from [10]. We define the notion of pro-representability only in a restricted sense, referring to [10] and FGA, 195 for more general definitions; also cf. [4].

Fix a field k and a complete, local (noetherian) ring W with maximal ideal $\mathfrak{m}_W$ and residue class field $k = W/\mathfrak{m}_W$ (the interesting cases are: char $(k) = 0$ & $W = k$, or char $(k) = p \neq 0$ & $W = W_\infty(k)$, the ring of infinite Witt-vectors over k).

DEFINITION. We denote by $\mathscr{C}_W$ (or by $\mathscr{C}$) the category of local artinian W-algebras R, together with an isomorphism $k \cong R/\mathfrak{m}_R$ making commutative the diagram

$$\begin{array}{ccc} W & \longrightarrow & R \\ \downarrow & & \downarrow \\ k & \xrightarrow{\sim} & R/\mathfrak{m}_R, \end{array}$$

and $\mathscr{C}_W(R, R')$ consists of all local W-algebra homomorphisms from R to R'.

EXERCISES. 1) Prove that $\mathscr{C}_W$ is a full subcategory of $\mathscr{Alg}_W$.

2) Show that a local W-algebra R is an artinian ring iff R is an artinian W-module. In particular, all $R \in \mathscr{C}_W$ are finitely generated W-modules, and every sub-algebra of R in $\mathscr{C}_W$ is again in $\mathscr{C}_W$.

3) Prove that $\mathscr{C}_W$ has fibered products, and that k is a final object. Hence all finite projective limits exist in $\mathscr{C}_W$.

4) Prove that a morphism $R \to R'$ is epimorphic in $\mathscr{C}_W$ iff it is a surjective map.

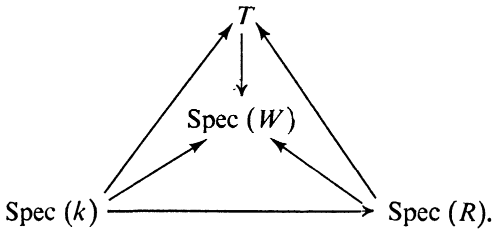
DEFINITION. We denote by $\hat{\mathscr{C}}_W$ the category of all complete, local, noetherian W-algebras $\mathcal{O}$, such that $\mathcal{O}/\mathfrak{m}_\mathcal{O}^n \in \mathscr{C}_W$ for every integer n. The morphisms in $\hat{\mathscr{C}}_W$ are local W-algebra homomorphisms. A functor

$$F : \mathscr{C}_W \to \mathscr{E}ns$$

is called *pro-representable*, if there exists an $\mathcal{O} \in \hat{\mathscr{C}}_W$ and an isomorphism of functors

$$\hat{\mathscr{C}}_W(\mathcal{O}, -) \xrightarrow{\sim} F(-).$$

EXAMPLE. Let $T \to \operatorname{Spec}(W)$ be a noetherian scheme over W, with a k-rational point $t \in T(k)$. For any $R \in \mathscr{C}$ denote by $T(R)$ the set of W-morphisms from Spec (R) to T. We define a functor $F : \mathscr{C} \to \mathscr{E}ns$ as the subfunctor of T given by: $r \in T(R)$ belongs to $F(R)$ iff the following diagram is commutative:



This functor is pro-representable, because

$$\hat{\mathscr{C}}(\hat{\mathcal{O}}_{T,t}, R) \simeq F(R).$$

This shows that from information about $F(R)$, without knowing T, one is able to predict some of the properties of $\mathcal{O}_{T,t}$ that can be read of from its completion, e.g. regularity and dimension.

In general, given a category $\mathscr{C}$, one can define its pro-category Pro ($\mathscr{C}$), cf. FGA, 195; in this way one arrives at a more general definition of pro--representability. Note that in our case $\hat{\mathscr{C}}_W \neq \text{Pro}(\mathscr{C}_W)$. We are going to mention necessary and sufficient conditions for a functor on $\mathscr{C}_W$ to be pro-representable. The following one is clearly necessary.

Definition. Let $\mathscr{C}$ be a category with a final object $\emptyset$ and fibred products. A covariant functor $F : \mathscr{C} \to \mathscr{E}ns$ is called *left exact* iff

(i) $F(\emptyset) = \{pt\}$, and

(ii) F commutes with fibred products, i.e. the natural map

$$F(X \underset{Y}{\times} Z) \simeq F(X) \underset{F(Y)}{\times} F(Z)$$

is bijective.

Definition. A category is called *noetherian* (respectively *artinian*) if for every object $X \in C$, the set of sub-objects of X satisfies the ascending (resp. descending) chain condition. [Note the confusion caused by the contravariant aspect of the functor Spec: a noetherian ring defines a so-called noetherian scheme, and the category of noetherian schemes is an artinian category].

It turns out that representability in the category of noetherian schemes in case of a contravariant functor (many of the functors in algebraic geometry are contravariant) is a delicate affair, cf. [6], whereas (pro-) representability of the same functor restricted to a noetherian sub-category usually is much easier to test because of the following:

'General principle' [1]. A contravariant functor on a noetherian category is (pro-)representable iff it is left exact.

[1] A 'general principle' is a statement which is true under mild extra conditions, depending on the special situation.

For example, if $\mathscr{A}$ is a noetherian, abelian category, its dual category $\mathscr{A}^0$ is artinian, and Gabriel proved that the left exact functors in this case are precisely the pro-representable ones:

$$(\mathrm{Sex}\,(\mathscr{A}^0, \mathscr{A}b))^0 = \mathrm{Pro}\,(\mathscr{A}^0)$$

(cf. [2], in particular the last 5 lines of page 356).

EXERCISE. Suppose $F: \mathscr{C}_W \to \mathscr{E}ns$ is left exact; denote by $k[\varepsilon]$ the ring of dual numbers of k, i.e. $k[\varepsilon] = k[E]/(E^2)$. Show that $F(k[\varepsilon])$ naturally is equiped with the structure of a vectorspace over k (cf. [10], lemma 2.10).

DEFINITION. A functor $F: \mathscr{C} \to \mathscr{E}ns$ is called *formally smooth* if for every surjection $\pi: R \twoheadrightarrow R'$ in $\mathscr{C}$, the map $F\pi: FR \twoheadrightarrow FR'$ is surjective. [Note that in case F is pro-(represented by $\mathcal{O} \in \hat{\mathscr{C}}$, then F is formally smooth iff $\mathcal{O}$ is a formally smooth W-algebra (cf. EGA, 0_{IV}.22.1.4).]

DEFINITION. A surjection $R \to R'$ in $\mathscr{C}_W$ is called *small*, if $I := \mathrm{Ker}\,(R \to R')$ has the property $I \cdot \mathfrak{m}_R = 0$. [Note that a 'small extension' in the terminology of [10] is slightly different from the notion of 'small surjection' defined here.]

EXERCISE. Show that any surjection in $\mathscr{C}$ is the composition of a finite number of small surjections.

THEOREM (2.1.1) (*Schlessinger criterion, cf.* [10], Th. 2.11, *Prop.* 2.5 *also cf. FGA, p.* 195–06, *coroll.*) *A covariant functor* $F: \mathscr{C}_W \to \mathscr{E}ns$ *is pro-representable if and only if F is left exact and*

$$\dim_k (F(k[\varepsilon])) < \infty.$$

It suffices to check left exactness in case $R_1 \to R_2 \leftarrow R_3$, *when the first arrow is a small surjection. If F is pro-represented by* $\mathcal{O} \in \hat{\mathscr{C}}_W$, *is formally smooth, and* $\dim_k(F(k[\varepsilon])) = m$, *then there exists an isomorphism* $\mathcal{O} \cong W[[t_1, \cdots, t_m]]$.

2.2. *Local moduli for abelian varieties*

DEFINITION. A *group object* in a category $\mathscr{S}$ is a contravariant functor $\mathscr{S}^0 \to \mathscr{G}r$ such that after forgetting structure, the functor $\mathscr{S}^0 \to \mathscr{E}ns$ is representable. Equivalently, one has an object $G \in \mathscr{S}$, and a group law on $\mathscr{S}(X, G)$ for each $X \in \mathscr{S}$, such that for every morphism f in $\mathscr{S}$, the map $\mathscr{S}(f, G)$ is a homomorphism of groups.

EXERCISE. Suppose $\mathscr{S}$ has a final object p, and suppose the product $G \Pi G$ exists in $\mathscr{S}$. Then G is a group object in $\mathscr{S}$ iff there exist morphisms

$$\begin{aligned} s &: G\Pi G \to G \text{ (multiplication)}, \\ \iota &: \quad G \ \to G \text{ (inverse)}, \\ \varepsilon &: \quad p \ \ \to G \text{ (neutral element)}, \end{aligned}$$

fitting into some obvious commutative diagrams.

DEFINITION. An A-algebra E is called an A-hyperalgebra, respectively an A-bialgebra, iff Spec (E) is a group scheme over Spec (A), resp. a commutative group scheme over Spec (A). Equivalently: there exist A-algebra homomorphisms $s : E \to E \otimes_A E$, $\iota : E \to E$, and $\varepsilon : E \to A$, such that $\cdots$, cf. CGS, p. I. 1–4.

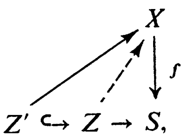
EXERCISE. Let E be an A-hyperalgebra, and put $I_E = \text{Ker}\,(\varepsilon : E \to A)$. Then $E = A \oplus I_E$ (as A-modules). Show that $x \in I_E$ satisfies

$$s(x) \equiv x \otimes 1 + 1 \otimes x \ (\text{mod}\, I_E \otimes I_E).$$

Suppose moreover that $A = k$ is a field of characteristic zero. Show that every local k-hyperalgebra is reduced [*hint*: if $x^n = 0$, $x^{n-1} \neq 0$, compute $(s(x))^n$]. Conclude that every finite group scheme over a field of characteristic zero is reduced. [Any algebraic group scheme in characteristic zero is reduced (Cartier); for references, cf. Inv. Math. 5 (1968), p. 80].

DEFINITION. Let S be a (locally noetherian) scheme and $f : X \to S$ a morphism which is locally of finite presentation; f is said to be *smooth* if for every *affine* scheme Z, for every closed subscheme $Z' \hookrightarrow Z$ defined by a sheaf of nilpotents ideals in $\mathcal{O}_Z$, and for every morphism $Z \to S$ the map

$$\text{Mor}_S(Z, X) \to \text{Mor}_S(Z', X),$$



is surjective.

REMARK. cf. EGA, IV.17.3.1 & 17.1.1; as 'test objects' Z it suffices to consider only schemes of the form Spec (A), where A is a local artin ring cf. EGA, IV.17,5.4 or SGA, 1960, exp. III, th. 3.1 & coroll. 6.8.

DEFINITION. (cf. GIT, chap. 6, def. 6.1). Let $f : X \to S$ be a morphism of schemes. We say X is an *abelian scheme* (abbreviation *AS*) over S, if X is an S-group scheme, and if f is smooth, proper, and has geometrically connected fibres [Note that $s : X\Pi X \to X$ automatically is commutative, cf. GIT, 6.4. coroll. 6.5; in case $S = \text{Spec}\,(k)$, where k is a field we call X an *abelian variety* (*AV*) *over* k].

For the rest of this section we fix an abelian variety X_0 over k, and we define the local moduli functor $M = M_{X_0}$ of X_0 as the covariant functor $M : \mathscr{C}_W \to \mathscr{E}ns$ given by:

$M(R) = \{$isomorphism classes of pairs (X, φ_0), where X is an AS over R, and φ_0 an isomorphism $\varphi_0 : X \otimes_R k \simeq X_0\}$.

THEOREM (2.2.1) (*Grothendieck*). *The functor* $M = M_{X_0}$ *is pro-representable by* $\mathcal{O} = W[[t_{1,1}, \cdots, t_{g,g}]]$, *i.e. there exists an isomorphism*

$$\hat{\mathscr{C}}_W(\mathcal{O}, -) \simeq M(-),$$

where $g = \dim(X_0)$.

The proof of this theorem can be constructed by putting together hints and results from FGA 182–12/13, 195–19, 236–19/20, SGA chap. III, and GIT chap. 6. We gather together some preliminary steps which suffice to make the Schlessinger criterion applicable.

LEMMA (2.2.2) (*cf. SGA, III, lemma* 4.2). *Let* $R \to R'$ *be a surjection in* $\mathscr{C}$, *and* $f : Y \to X$ *a morphism of schemes over R such that* $f' := f \otimes_R R'$ *is an isomorphism. Suppose Y is flat over R. Then f is an isomorphism.*

PROOF. Certainly f induces a homeomorphism on the topological spaces, and the lemma follows from:

Let $R \to R'$ be as before, $I := \text{Ker}(R \to R')$, and $u : M \to P$ a morphism of R-modules. Suppose P is R-flat, and assume $u' := u \otimes_R R' : M/I \cdot M \simeq P/I \cdot P$ is an isomorphism. Then u is an isomorphism.

Put $N = \text{Ker}(u)$, and $Q = P/u(M)$. Applying $- \otimes_R R'$ to the exact sequence

$$0 \to N \to M \to P \to Q \to 0$$

we obtain $Q/I.Q = 0$, so $Q = IQ = \cdots I^n Q = 0$ for some n, as I is nilpotent. Thus

$$0 = \text{Tor}_1^R(P, R/I) \to N/IN \to M/IM \to P/IP \to 0,$$

because P is R-flat. Hence $N/IN = 0$, hence $N = 0$ as before, and the lemma is proved.

LEMMA (2.2.3). *Let* $R \to R'$ *be a* small *surjection of local artin rings with kernel I,* $k = R/\mathfrak{m}_R$, $S = \text{Spec}(R)$, $S' = \text{Spec}(R')$, $S_0 = \text{Spec}(k)$. *Let* $Z \to S$ *be a flat morphism of finite type,* $Z' = Z \times_S S'$, $Z_0 = Z \times_S S_0$, *and denote by* Θ_{Z_0} *the sheaf of germs of k-derivations from* $\mathcal{O}_{Z_0}$ *into itself. We write* $\text{Aut}_S(Z, S')$ *for the set of S-automorphisms of Z which induce the identity on Z'. There is a canonical isomorphism*

$$\Gamma(Z_0, \Theta_{Z_0}) \otimes_k I \simeq \text{Aut}_S(Z, S').$$

Sketch of the PROOF: The elements $D \in \Gamma(Z_0, \Theta_{Z_0}) \otimes_k I$ and $f \in \mathrm{Aut}_S(Z, S')$ correspond to each other in the following way:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I\mathcal{O}_Z & \longrightarrow & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_{Z'} & \longrightarrow & 0 \\ & & & & {\scriptstyle f = id + D}\downarrow & & \| & & \\ 0 & \longrightarrow & I\mathcal{O}_Z & \longrightarrow & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_{Z'} & \longrightarrow & 0 \end{array}$$

(with the obvious identifications made).

LEMMA (2.2.4) (*cf. SGA, exp. III, th.* 4.1). *Let R be a noetherian ring, $I \subset \sqrt{0}$ a nilpotent ideal, and $R' = R/I$; let $X' \to S' = \mathrm{Spec}\,(R')$ be a smooth scheme. For every $x \in X'$ there exists an open neighbourhood $x \in U' \subset X'$ and a smooth morphism $U \to S = \mathrm{Spec}\,(R)$ such that $U \otimes_R R' \cong U'$; if moreover $x \in U' \cap V'$, $V \to S$ smooth and $V' = V \otimes_R R'$, then for every affine neighbourhood $x \in W' \subset U' \cap V'$ there exists an isomorphism making commutative the diagram*

$$\begin{array}{ccc} U|W' & \xrightarrow{\sim} & V|W' \\ & \nwarrow \quad \nearrow & \\ & W' & \end{array}$$

(*'locally X' can be lifted to R, and the lift is unique locally up to a* (*non-canonical*) *isomorphism'*).

This follows directly from the following statements:

EGA IV^4.17.11.4: a morphism $Y \to S$ is smooth in $y \in Y$ iff there exists an open neighbourhood $y \in U \subset Y$, an integer g and an étale S-morphism $U \to S[t_1, \cdots, t_g]$;

EGA IV^4.18.1.2: $I \subset \sqrt{0} \subset A \to A' = A/I$; the functor $Y \mapsto Y \otimes_A A'$ is an equivalence of the categories of schemes étale over Spec (A), respectively étale over Spec (A').

NOTATION. Let $R \to R'$ be a surjection of local artin rings, $S = \mathrm{Spec}\,(R)$, and $X' \to S' = \mathrm{Spec}\,(R')$ a smooth scheme over S'. We write:

$\mathbf{L} = \mathbf{L}(X'; R \to R') = \{$isomorphism classes of pairs (X, φ') such that $X \to S$ is smooth, and $\varphi' : X \times_S S' \to X'$ an isomorphism$\}$.

PROPOSITION (2.2.5) (*cf. FGA*, 182–12/13, *SGA, III, th.* 6.3, *prop.* 5.1). *Take the same notations as above, and suppose $I = \mathrm{Ker}\,(R \to R')$, $I \cdot \mathfrak{m}_R = 0$.*

i) *There exists an element*

$$\mathfrak{o} = \mathfrak{o}(X'; R \to R') \in H^2(X_0, \Theta_{X_0}) \underset{k}{\otimes} I$$

such that $\mathfrak{o} = 0$ iff $\mathbf{L} = \mathbf{L}(X'; R \to R') \neq \emptyset$.

ii) *If* $\mathfrak{o} = 0$, *then any* $(X, \varphi') \in \mathbf{L}$ *yields a bijection*

$$\iota_X : H^1(X_0, \Theta_{X_0}) \underset{k}{\otimes} I \xrightarrow{\sim} \mathbf{L}.$$

iii) *For any* S'*-isomorphism* $\psi' : Y' \xrightarrow{\sim} X'$ *of smooth* S'*-schemes, one has a diagram*

$$H^2(Y_0, \Theta_{Y_0}) \underset{k}{\otimes} I \xrightarrow[d\psi_0]{} H^2(Y_0, \psi_0^* \Theta_{X_0}) \underset{k}{\otimes} I \xleftarrow[\psi_0^*]{} H^2(X_0, \Theta_{X_0}) \underset{k}{\otimes} I$$

and

$$d\psi_0(\mathfrak{o}(Y')) = \psi_0^*(\mathfrak{o}(X')).$$

Proof. i) By (2.2.4) we can choose an open, affine covering $\{U'_\alpha\}$ of X' such that every $U'_\alpha \to S'$ can be lifted to a smooth, affine scheme $Z_\alpha \to S$. We write $U'_{\alpha\beta} = U'_\alpha \cap U'_\beta$, and similarly for more indices; because X' is separated over S', we conclude $U'_{\alpha\beta}$ is affine (cf. EGA, I. 5.5.6), hence there exists a morphism

$$\xi_{\beta\alpha} : Z_\alpha | U'_{\alpha\beta} \to Z_\beta | U'_{\alpha\beta}$$

which reduces to $\varphi_\beta^{-1} \cdot \varphi_\alpha$ over S', where the isomorphisms $\varphi_\alpha : Z_\alpha \times_S S' \xrightarrow{\sim} U'$ are given by the lifting. Note that $\xi_{\alpha\beta}$ is an isomorphism by (2.2.2). We define

$$\xi^\gamma_{\beta\alpha} = \xi_{\beta\alpha} | U'_{\alpha\beta\gamma} : Z_\alpha | U'_{\alpha\beta\gamma} \xrightarrow{\sim} Z_\beta | U'_{\alpha\beta\gamma},$$

and we write

$$c_{\alpha\beta\gamma} = (\xi^\beta_{\gamma\alpha})^{-1} \cdot \xi^\alpha_{\gamma\beta} \cdot \xi^\gamma_{\beta\alpha} \in \mathrm{Aut}_S(Z_\alpha | U'_{\alpha\beta\gamma}).$$

Clearly $c_{\alpha\beta\gamma} \times_S S'$ is the identity on $U'_{\alpha\beta\gamma}$, hence by (2.2.3) $c_{\alpha\beta\gamma}$ defines an element of $\Gamma((U_{\alpha\beta\gamma})_0, \Theta_{X_0}) \otimes_k I$, also denoted by $c_{\alpha\beta\gamma}$. We claim $c = \{c_{\alpha\beta\gamma}\}$ is a 2-cocycle, and hence it defines a cohomology class

$$[c] \in H^2(\{(U_{\alpha\beta})_0\}, \Theta_{X_0}) \otimes I = H^2(X_0, \Theta_{X_0}) \underset{k}{\otimes} I.$$

To prove this we compute the coboundary of c. For every pair (α, β) the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Aut}_S(Z_\alpha | U'_{\alpha\beta}, S') & & \\
\uparrow \cong & \overset{\sim}{\searrow} \text{ via } \varphi_\alpha & \\
 & & H^0((U_{\alpha\beta})_0, \Theta_{X_0}) \otimes_k I, \\
 & \overset{\sim}{\nearrow} \text{ via } \varphi_\beta & \\
\mathrm{Aut}_S(Z_\beta | U'_{\alpha\beta}, S') & &
\end{array}$$

where the vertical map sends an automorphism a of $Z_\beta | U'_{\alpha\beta}$ to $(\xi_{\beta\alpha})^{-1} \cdot a \cdot \xi_{\beta\alpha}$. Note that the group $\mathrm{Aut}_S(Z_\alpha, S')$ is commutative for every α;

$$(\partial c)_{\alpha\beta\gamma\delta} = c_{\beta\gamma\delta} \cdot c_{\alpha\gamma\delta}^{-1} \cdot c_{\alpha\beta\delta} \cdot c_{\alpha\beta\gamma}^{-1}$$

is represented on $Z_\alpha | U'_{\alpha\beta\gamma\delta}$ by the automorphism

$$[\xi_{\beta\alpha}^{-1}((\xi_{\delta\beta}^{\gamma})^{-1} \cdot \xi_{\delta\gamma}^{\beta} \cdot \xi_{\gamma\beta}^{\delta})\xi_{\beta\alpha}] \cdot [(\xi_{\gamma\alpha}^{\delta})^{-1} \cdot (\xi_{\delta\gamma}^{\alpha})^{-1} \cdot \xi_{\delta\alpha}^{\gamma}] \\ \cdot [(\xi_{\delta\alpha}^{\beta})^{-1} \cdot \xi_{\delta\beta}^{\alpha} \cdot \xi_{\beta\alpha}^{\delta}] \cdot [(\xi_{\beta\alpha}^{\gamma})^{-1} \cdot (\xi_{\gamma\beta}^{\alpha})^{-1} \cdot \xi_{\gamma\alpha}^{\beta}].$$

The 4 square brackets commute mutually; call them B_1, B_2, B_3 and B_4. If we write them up in the order $B_2 \cdot B_3 \cdot B_1 \cdot B_4$ the factors cancel, thus proving $\partial c = 0$.

It is not difficult to check that the class $[c]$ does not depend on the choices of U_α and $\xi_{\beta\alpha}$.

It is equally clear that c is a coboundary iff the isomorphisms $\xi_{\beta\alpha}$ can be changed so as to define a schematic glueing-together of the pieces $Z_\alpha | U'_{\alpha\beta}$.

ii) Suppose given $\boldsymbol{L} \neq \emptyset$, take $(X, \varphi') \in \boldsymbol{L}$, and consider

$$d_{\alpha\beta} \in \Gamma((U'_{\alpha\beta})_0), \Theta_{X_0}) \underset{k}{\otimes} I \cong \mathrm{Aut}_S(X_{\alpha\beta}, S'), \; X_\alpha = X | U_\alpha,$$

a cocycle on a covering $\{U_\alpha\}$. As the cocycle condition is fulfilled the system $(X_\alpha, d_{\alpha\beta})$ defines a scheme X^d together with an isomorphism $X^d \otimes_R R' \cong X'$. Moreover X^d is isomorphic to X (inducing the identity on X') iff d is a coboundary, and another set of representatives for the class $[d]$ yields the same element of $\boldsymbol{L}$. Thus the map $d \mapsto X^d$ is well-defined, and clearly it is injective. To see it is a bijection, we define its inverse: suppose $(Y, \psi') \in \boldsymbol{L}$. Choose

$$\varphi_\alpha : Y_\alpha | U' \cong X, \text{ and write } d_{\alpha\beta} = \varphi_\beta \cdot (\varphi_\alpha | X_{\alpha\beta})^{-1}.$$

Clearly we have thus defined an element $[d] \in H^1(X_0, \Theta_{X_0}) \otimes_k I$, and we have given the inverse of the map

$$\iota_X : H^1(X_0, \Theta_{X_0}) \underset{k}{\otimes} I \cong \boldsymbol{L}, \; d \mapsto X^d,$$

which is therefore bijective.

iii) We define $d\psi_0 : \Theta_{Y_0} \to \psi_0^* \Theta_{X_0}$ by $\psi_0 = \psi' \otimes_{R'} k$, and $a \mapsto \psi_0 \cdot a \cdot \psi_0^{-1}$. The map ψ_0^* is defined as follows: on an open set $U_0 \subset Y_0$, by definition $\Gamma(U_0, \psi_0^* \Theta_{X_0}) \cong \Gamma(\psi_0(U_0), \Theta_{X_0})$, because ψ_0 is an isomorphism. Suppose $\mathfrak{o}(X')$ is given by the cocycle $\{c_{\alpha\beta\gamma}^{(X')}\}$ as in the first part of the proof; we arrive at a cocycle $\{c_{\alpha\beta\gamma}^{(Y')}\}$ on Y_0 simply by 'conjugating' all isomorphisms with ψ'. This cocycle defines the element $\mathfrak{o}(Y')$, which proves the proposition.

Proposition (2.2.6). *Let $\pi : R \to R'$ be a small surjection in $\mathscr{C}$, X_0 an AV over k, M the moduli functor defined by X_0, and $(X', \varphi_0) \in MR'$.*

Forgetting the group structure on the abelian varieties involved yields a bijection:

$$L = L(X' : R \to R') \underset{\kappa}{\overset{\sim}{\leftarrow}} \overset{-1}{\pi}(X', \varphi_0) \subset MR \xrightarrow{\pi} MR'$$

(*we write* π *instead of* $M\pi$).

Proof. Suppose given $(Y, \psi_0) \in MR$ such that $\pi(Y, \psi_0) = (X', \varphi_0)$; we choose an isomorphism $\psi' : Y \otimes R' \to X'$ such that $\varphi_0 \cdot (\psi' \otimes k) = \psi_0$, and we define

$$\kappa(Y, \psi_0) = (Y, \psi') \in L.$$

This map κ is well defined: in fact, suppose $\mu' : Y \otimes R' \xrightarrow{\sim} X'$ has the property $\varphi_0 \cdot (\mu' \otimes k) = \psi_0$; then

$$a := (Y \otimes R' \xrightarrow{\psi'} X') \cdot (X' \xrightarrow{(\mu')^{-1}} Y \otimes R')$$

is an automorphism of the abelian scheme X', in particular $a(0) = 0$, such that $a \otimes k$ is the identity; hence by GIT, p. 116, coroll. 6.2 we conclude $\psi' = \mu'$, thus the map κ is well defined.

Injectivity of κ: suppose $\kappa(Y, \psi_0) = \kappa(Z, \mu_0)$, i.e. there exists a morphism of S-schemes $b : Y \to Z$, $S = \text{Spec}(R)$, such that

$$\mu_0 \cdot (b \otimes k) = \psi_0.$$

Denote by f_y, respectively ε_Y the structural morphism, resp. the zero section of Y; consider $h := b - b \cdot \varepsilon_Y \cdot f_Y$; clearly

$$h \cdot \varepsilon_Y = b \cdot \varepsilon_Y - b \cdot \varepsilon_Y = 0,$$

and because ψ_0 and μ_0 are isomorphisms of abelian varieties:

$$\mu_0 \cdot (h \otimes k) = \psi_0;$$

hence by GIT, p. 117, coroll. 6.4 we conclude b is a homomorphism (and hence an automorphism) of abelian schemes over S, thus (Y, ψ_0) and (Z, μ_0) define the same element of MR.

Surjectivity of κ : apply GIT, p. 124, prop. 6.15. q.e.d.

Proof of theorem (2.2.1), by checking the conditions appearing in the Schlessinger condition. Obviously Mk = one element.

Consider a commutative square (not necessarily cartesian) in $\mathscr{C}$:

$$(*) \qquad \begin{array}{ccc} Q & \xrightarrow{\chi} & R \\ \rho \downarrow & & \downarrow \pi \\ T & \xrightarrow[\mu]{} & R', \end{array}$$

such that π and ρ are small surjections. Write $I = \text{Ker}(\pi)$, $J = \text{Ker}(\rho)$;

clearly $\chi(J) \subset I$, thus we arrive at a commutative diagram with exact columns:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ J & \longrightarrow & I \\ \downarrow & & \downarrow \\ Q & \longrightarrow & R \\ \downarrow & & \downarrow \\ T & \longrightarrow & R' \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

Applying M to (*) we obtain a new commutative diagram, which defines a map $\omega : MQ \to MT \times_{MR'} MR$. In this situation we have a natural map

$$\mathrm{id} \otimes \chi : H^2(X_0, \Theta_{X_0}) \underset{k}{\otimes} I \to H^2(X_0, \Theta_{X_0}) \underset{k}{\otimes} I;$$

let $(Y, \psi_0) \in MT$, and $(X', \varphi_0) = \mu(Y, \psi_0)$(we write μ instead of $M\mu$, etc.). We claim

$$(\mathrm{id} \otimes \chi)(\mathfrak{o}(Y)) = \mathfrak{o}(X').$$

In fact, let Z be a lifting of an open subset of Y of Q; then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Aut}_Q(Z, T) & \longrightarrow & \Gamma(X_0, \Theta_{X_0}) \otimes_k J \\ \downarrow & & \downarrow \\ \mathrm{Aut}_R(Z \otimes R', T) & \longrightarrow & \Gamma(X_0, \Theta_{X_0}) \otimes_k I, \end{array}$$

and the statement follows immediately, using the definition of the obstruction elements given in the proof of (2.2.5). Now suppose the diagram (*) to be cartesian, and assume π is a small surjection; then automatically $J \xrightarrow{\sim} I$ is an isomorphism, and ρ is a small surjection. If $MT \times_{MR'} MR = \emptyset$, certainly ω is bijective; thus suppose we can choose $((Y, \psi_0), (X, \varphi_0)) \in MT \times_{MR'} MR$. Then by (2.2.5), and by what is just said,

$$(\mathrm{id} \otimes \chi)(\mathfrak{o}(Y; Q \to T)) = \mathfrak{o}(\pi(X, \varphi_0); R \to R') = 0,$$

thus, again by (2.2.5), there exists (Z, ψ_0) lifting Y to Q. Thus we arrive at a diagram:

$$\begin{array}{ccc}
H^1(X_0, \Theta_{X_0}) \otimes J & \xrightarrow[\sim]{\mathrm{id}\otimes\chi} & H^1(X_0, \Theta_{X_0}) \otimes I \\
\sim \downarrow \iota_Z & & \sim \downarrow \iota_{\chi(Z)} \\
L(Y; Q \to T) & \longrightarrow & L(X'; R \to R') \\
\cap \downarrow \kappa^{-1} & & \cap \downarrow \kappa^{-1} \\
MQ & \xrightarrow{\chi} & MR,
\end{array}$$

with $(X', \varphi_0) = \pi(X, \varphi_0)$, and with the middle horizontal arrow being defined by base change; direct verification shows the upper square to be commutative, and commutativity of the lower square follows from the construction, cf. (2.2.6). Hence by (2.2.5ii) and (2.2.6) we conclude that χ induces a bijection

$$\overset{-1}{\rho}(Y, \psi_0) \xrightarrow{\sim} \overset{-1}{\pi}(X', \varphi_0),$$

and thus left exactness of the functor M is proved.

Next we check the finite dimensionality of $M(k[\varepsilon])$. Because $k[\varepsilon] \to k$ has a section, obstruction for lifting X_0 is zero, and by (2.2.5ii) we conclude

$$M(k[\varepsilon]) \cong H^1(X_0, \Theta_{X_0}) \otimes k \cdot \varepsilon \cong H^1(X_0, \Theta_{X_0}),$$

hence finite dimensionality; note that X_0 is a group scheme, hence $\Theta_{X_0} \cong$ $\cong \mathcal{O}_{X_0} \otimes_k \mathfrak{t}$, thus

$$M(k[\varepsilon]) \cong H^1(X_0, \mathcal{O}_{X_0}) \otimes \mathfrak{t} \cong \mathfrak{t}^t \otimes \mathfrak{t}$$

(canonical isomorphisms), where $\mathfrak{t}$ is the tangent space at the zero element of X_0, and $H^1(X_0, \mathcal{O}_{X_0}) = \mathfrak{t}^t$ is the tangent space of the zero element of the dual abelian variety X_0^t; conclusion:

$$\dim_k (M(k[\varepsilon])) = g^2.$$

The last point to be proven is the smoothness. Suppose $R \to R'$ is a small surjection in $\mathscr{C}$ with kernel I, and $(X', \varphi_0) \in MR'$. We know that $\mathfrak{o}(X'; R \to R')$ is 'invariant under automorphisms" (cf. 2.2.5.iii); consider the inverse map $i : X' \to X'$. As X_0 is an abelian variety,

$$H^2(X_0, \mathcal{O}_{X_0}) = H^1(X_0, \mathcal{O}_{X_0}) \wedge H^1(X_0, \mathcal{O}_{X_0})$$

(cf. GA, VII.31, prop. 16), and the effect of i on each of the first three factors of the right hand side of

$$H^2(X_0, \Theta_{X_0}) \otimes I = (\mathfrak{t}^t \wedge \mathfrak{t}^t) \otimes \mathfrak{t} \otimes I$$

is to change the sign, hence

$$\mathfrak{o}(X'; R \to R') = -\mathfrak{o}(X'; R \to R');$$

if char $(k) \neq 2$, this proves the obstruction vanishes, thus proving that $MR \to MR'$ is surjective. In order to deal with the case of characteristic two also, we present another proof.

Write $P' = X' \times_{S'} X'$; this is an AS over S', and its lifting is obstructed by $\mathfrak{o}(P') \in H^2(P_0, \Theta_{P_0}) \otimes I$. The two projections of P_0 on its two factors give two injections:

$$i_1, i_2 : H^2(X_0, \Theta_{X_0}) \rightrightarrows H^2(P_0, \Theta_{P_0}).$$

Denote $i_1 \otimes id_I$ by i_1 and similarly for the index 2; one checks

$$\mathfrak{o}(P') = i_1(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X'))$$

(take an affine covering of both factors, construct the obstruction elements of both factors, lift the products of one open set of the first and one open set of the second covering, and take the transition function as for the factors; the cocycle thus obtained defines $\mathfrak{o}(P')$, because the cohomology class does not depend on all these choices, and the formula is proved). Define $a' \in \mathrm{Aut}_{S'}(P')$ by

$$a'(x, y) = (x+y, y),$$

and apply (2.2.5iii):

$$\mathfrak{o}(P') = (a_0^*)^{-1}(da_0)(\mathfrak{o}(P')) = 2i_1(\mathfrak{o}(X')) + i_2(\mathfrak{o}(X')),$$

thus $i_1(\mathfrak{o}(X')) = 0$, or $\mathfrak{o}(X') = 0$, which proves that the functor M is formally smooth, and the proof of theorem (2.2.1) is concluded.

Let V_0 be a smooth variety over k. We define $N = N_{V_0} : \mathscr{C}_W \to \mathscr{E}ns$, the local moduli functor given by V_0:

$$NR = \{\cong \text{ classes of } (V, \varphi_0) | V \to \mathrm{Spec}\,(R) \text{ is smooth, and } \varphi_0 : V \otimes_R k \xrightarrow{\sim} V_0\}.$$

Suppose $H^0(V_0, \Theta_{V_0}) = 0$; any $a \in \mathrm{Aut}_R(V)$, $(V, \varphi_0) \in NR$, such that $a \otimes_R k \to id_{V_0}$ is the identity itself, $a = id_V$, and hence:

Proposition. (2.2.7). *Let $\pi : R \to R'$ be a small surjection in $\mathscr{C}$, V_0 a smooth variety over k such that $H^0(V_0, \Theta_{V_0}) = 0$, N the moduli functor given by V_0, and $(V', \varphi_0) \in NR'$. The natural map*

$$L = L(V', R \to R') \xrightarrow{\sim} \overset{-1}{\pi}(V', \varphi_0) \subset NR \xrightarrow{\pi} NR'$$

is bijective.

Thus the arguments given above prove the following

Theorem (2.2.8) (*cf. FGA*, 195–18, *prop.* 4.1). *Let V_0 be a smooth variety over k, with $H^0(V_0, \Theta_{V_0}) = 0$, and $\dim_k H^1(V_0, \Theta_{V_0}) = m < \infty$. The local moduli functor $N = N_{V_0}$ is pro-representable. If $H^2(V_0, \Theta_{V_0}) = 0$ the functor is formally smooth, and*

$$\mathscr{C}_W(W[[t_1, \cdots, t_m]], -) \simeq N(-).$$

For the complex analytic analogue, cf. M. Kuranishi – New proof for the existence of locally complete families of complex structures. Proc. Conf. Complex Analysis (Minneapolis, 1964), 142–154. Springer Verlag, Berlin, 1965.

2.3. *Local moduli for polarized abelian varieties.*

Let S be a prescheme and $\pi : X \to S$ be a morphism of finite type such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_S$; suppose π admits a section. For any prescheme Y we write

$$\mathrm{Pic}\,(Y) = H^1(Y, \mathcal{O}_Y^*),$$

and we define $\quad \mathrm{Pic}_{X/S} : \mathscr{S}ch_S^0 \to \mathscr{G}r$

by $\quad \mathrm{Pic}_{X/S}\,(T) = \mathrm{Pic}\,(X \times_S T)/\mathrm{Pic}\,(T).$

If this functor is representable, the resulting S-prescheme is called the Picard scheme of X over S, denoted by $\mathbf{Pic}_{X/S}$. In case X is an abelian scheme over S, we define

$$X^t := \mathbf{Pic}_{X/S}^{\tau} = \mathbf{Pic}_{X/S}^0.$$

Various properties, such as the last equality and such as the smoothness of $X^t \to S$ will be used (cf. [12], th. 5, or LAV, p. 100, coroll. 3; cf. GIT, p. 117, prop. 6.7). For any $L \in \mathrm{Pic}(X)$ we define

$$\Lambda L : X \to X^t \text{ by } \Lambda L = (s - p_1 - p_2 : X \times_S X \to X)^*(L)$$

(sometimes denoted by φ_L; cf. LAV, p. 75, GIT, 6.2, MAV, p. 60).

Lemma (2.3.1) (*cf. MAV, p.* 74). *For any* $R \in \mathscr{C}_W$, and any abelian *scheme* X *over* Spec $(R) = S$ *there is an exact sequence*:

(1) $$0 \to X^t(R) \to \mathrm{Pic}\,(X) \xrightarrow{\Lambda} \mathrm{HOM}_S\,(X, X^t),$$

and for any surjection $R \twoheadrightarrow R'$ *in* $\mathscr{C}$, we *write* $X' = X \otimes_R R'$, *and the following diagram is commutative and exact*:

(2) $$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ 0 & \longrightarrow & X^t(R) & \longrightarrow & \mathrm{Pic}\,(X) & \longrightarrow & \mathrm{HOM}_S(X, X^t) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (X')^t(R') & \longrightarrow & \mathrm{Pic}\,(X') & \longrightarrow & \mathrm{HOM}_S\,(X', (X')^t) \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

PROOF. The map $X^t(R) \to \mathrm{Pic}\,(X)$ is given by

$$X^t(R) \subset \mathrm{Mor}_S(S, \mathbf{Pic}_{X/S}) \rightrightarrows \mathrm{Pic}\,(X).$$

If (2) is commutative, has exact columns and exact bottom row, exactness of (1) follows. So all we need to prove is exactness of (1) in case $R = k$, and commutativity plus 'vertical' exactness of (2) in all cases.

In case $R = k$, a field, exactness of the sequence (1) is known (cf. LAV, p. 90, th. 2). Commutativity of the diagram (2) is obvious from the definitions. Because X^t is smooth over S, surjectivity of $X^t(R) \to X'^t(R')$ follows. The last fact to be proved follows from GIT, p. 116, coroll. 6.2:

$$\begin{array}{ccc} \mathrm{HOM}_S(X, X^t) & \hookrightarrow & \\ \downarrow & & \mathrm{HOM}\,(X_0, X_0^t), \\ \mathrm{HOM}_{S'}(X', X'^t) & \nearrow & \end{array}$$

which concludes the proof of the lemma.

REMARK. If we work over $k = \boldsymbol{C}$, the field of complex numbers, an analogous sequence can be derived as follows. From the exact sequence

$$0 \to \boldsymbol{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

we derive

$$\begin{array}{ccccccc} H^1(X, \boldsymbol{Z}) & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^*) & \searrow & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X^t(\boldsymbol{C}) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X, \boldsymbol{Z}). \end{array}$$

An algebraic structure on the complex torus X exists iff there exists a class in

$$H^2(X, \boldsymbol{Z}) \to H^2(X, \boldsymbol{R}) = H^2_{dR}(X)$$

which comes from an ample sheaf. This notion of polarization can be defined over any field via Λ as pointed out above.

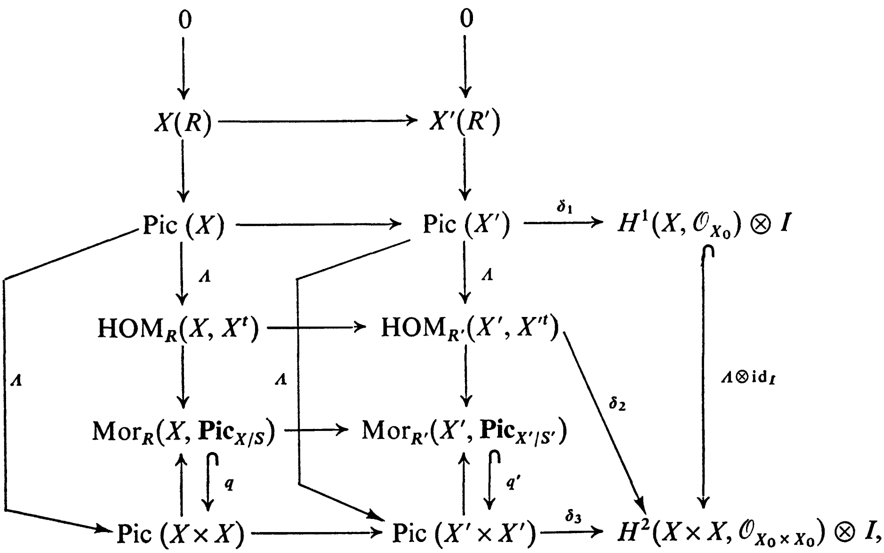
DEFINITION. Let R be a local artin ring, and $\pi : X \to S = \mathrm{Spec}\,(R)$ an abelian scheme. A homomorphism $\lambda : X \to X^t$ is called a *quasi-polarization* if there exists a divisor class $L \in \mathrm{Pic}\,(X)$ such that $\Lambda(L) = \lambda$. If moreover L is relatively ample for π, we say λ is a *polarization* on X.

REMARK. This definition coincides with GIT, page 120, definition 6.2, in case R is a local artin ring because of the following:

LEMMA (2.3.2). *Let* $R \twoheadrightarrow R'$ *be a small surjection,* $L' \in \mathrm{Pic}\,(X')$, *and sup-*

pose there exsists a homomorphism $\lambda : X \to X^t$ *which lifts* $\Lambda L' = \lambda'$ *to* R; *then there exists* $L \in \mathrm{Pic}\,(X)$ *such that* $\Lambda L = \lambda$.

PROOF. First we show λ' can be lifted to R if and only if L' can be lifted to R. In case char $(R/\mathfrak{m}_R) \neq 2$ this is easy, using GIT, page 121, prop. 6.10 (because the obstructions for lifting λ' and L' are elements of vector spaces over k). In the general case one considers the following diagram:



where

$$\Lambda : H^1(X, \mathcal{O}_{X_0}) \to H^2(X \times X, \mathcal{O}_{X_0 \times X_0})$$

is defined by $\Lambda = s^* - p_1^* - p_2^*$, which is known to be an *injective* homomorphism (cf. [11], p. IX. 8, lemma 2.4), and where q is defined as the kernel map of

$$(i_1 : X \to X \times X)^* : \mathrm{Pic}\,(X \times X) \to \mathrm{Pic}\,(X)$$

(and analogously for q'). As

$$X \xleftarrow{s - p_1 - p_2} X \times X \xleftarrow{i_1} X$$

is the zero map, $\Lambda : \mathrm{Pic}\,(X) \to \mathrm{Pic}\,(X \times X)$ factors through q, and because of functoriality of the Λ-operation on the exact sequences involving Pic, we obtain

$$\begin{aligned}(\Lambda \otimes \mathrm{id}_I)\delta_1(L') &= \delta_2(\Lambda(L')) = \\ &= \delta_2(\lambda \otimes_R R') = \delta_3(q(\lambda) \otimes_R R') = 0,\end{aligned}$$

and by injectivity of $\Lambda \otimes id_I$ this implies $\delta_1(L') = 0$. Thus L' can be lifted to R, and by (2.3.1) this lift can be chosen such that $\Lambda L = \lambda$. q.e.d.

DEFINITION. Let X_0 be an abelian variety over k, and $\Lambda L_0 = \lambda_0 \in \mathrm{HOM}(X_0, X_0^t)$ a quasi-polarization. We define a functor

$$P_{X_0, \lambda_0} = P : \mathscr{C}_W \to \mathscr{E}ns$$

by

$$PR = \{\cong \text{ classes of } (X, \lambda, \varphi_0) | (X, \lambda) \text{ is a quasi-polarized abelian scheme over } R, \text{ and } \varphi_0 : (X, \lambda) \otimes_R k \xrightarrow{\sim} (X_0, \lambda_0)\}.$$

THEOREM (2.3.3) (*Mumford*). *Fix the notations* $k, W, X_0, \lambda_0, g = \dim(X_0), \mathcal{O} = W[[t_{11}, \cdots, t_{gg}]]$ *as above, and* $d := \dim_k H^2(X_0, \mathcal{O}_{X_0})$; *the functor P is a subfunctor of M, and there exist d elements* $a_1, \cdots, a_d \in \mathcal{O}$, $\mathfrak{a} := (a_1, \cdots, a_d) \cdot \mathcal{O}$, *so that $\mathcal{O}/\mathfrak{a}$ pro-represents P*:

$$\begin{array}{ccc} \mathscr{C}_W(\mathcal{O}, -) & \xrightarrow{\sim} & M(-) \\ \uparrow & & \uparrow \\ \mathscr{C}_W(\mathcal{O}/\mathfrak{a}, -) & \xrightarrow{\sim} & P(-) \end{array}$$

(*note that* $H^2(X_0, \mathcal{O}_{X_0}) = H^1(X_0, \mathcal{O}_{X_0}) \wedge H^1(X_0, \mathcal{O}_{X_0})$, *so* $d = \frac{1}{2}g(g-1)$).

PROOF. From the 'rigidity lemma' GIT, p. 116, Corollary 6.2 we conclude that P is a subfunctor of M (this is the advantage of working with $\Lambda L = \lambda$, rather than with L). Clearly $P(k) = \{pt\}$. Consider a cartesian square

$$\begin{array}{ccc} Q & \xrightarrow{\chi} & R \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ T & \xrightarrow{\tau} & R' \end{array} \qquad J = \mathrm{Ker}(\rho) \cong I = \mathrm{Ker}\,\pi,$$

with π (and hence ρ) a small surjection. Thus we obtain

$$\begin{array}{ccc} PQ & \longrightarrow & PT \times_{PR'} MR \\ \downarrow & & \downarrow \\ MQ & \xrightarrow{\sim} & MT \times_{MR'} MR \end{array}$$

and we show the top line to be surjective (injectivity being obvious): let $((Y, \mu, \psi_0), (X, \lambda, \varphi_0))$ be an element of the right hand upper corner, and let $(Z, \alpha_0) \in MQ$ correspond with $((Y, \mu_0), (X, \varphi_0))$. There exists $K \in \mathrm{Pic}(Y)$ with $\Lambda K = \mu$; as $\mu \otimes_Q R' = \lambda \otimes_R R'$ by the previous lemma we can choose $L \in \mathrm{Pic}(X)$ such that $L \otimes_R R' = K \otimes_Q R'$. We consider the following commutative diagram of sheaves on X_0:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes \mathcal{O}_{X_0} & \xrightarrow{i} & \mathcal{O}_X^* & \xrightarrow{\pi} & \mathcal{O}_{X'}^* & \longrightarrow & 1 \qquad i(a \otimes f) = 1 + af \\ & & \uparrow \cong & & \uparrow \chi & & \uparrow \tau & & \\ 0 & \longrightarrow & J \otimes \mathcal{O}_{X_0} & \longrightarrow & \mathcal{O}_X^* & \xrightarrow{\rho} & \mathcal{O}_Y^* & \longrightarrow & 1; \end{array}$$

thus we obtain a commutative exact diagram

$$\begin{array}{ccccc} \mathrm{Pic}\,(X) & \longrightarrow & \mathrm{Pic}\,(X') & \longrightarrow & H^2(X, \mathcal{O}_{X_0}) \otimes I \\ \uparrow \chi & & \uparrow \tau & & \uparrow \cong \\ \mathrm{Pic}\,(Z) & \xrightarrow{\tau} & \mathrm{Pic}\,(Y) & \xrightarrow{\delta} & H^2(X, \mathcal{O}_{X_0}) \otimes J; \end{array}$$

as $\delta(L \otimes_R R') = 0$, we conclude $\delta(K) = 0$; so there exists a divisor class $D \in \mathrm{Pic}\,(Z)$ lifting K to Q, and because $P \subset M$ we conclude $(Z, \Lambda D, \alpha_0)$ maps onto the pair $((Y, \mu, \psi_0), (X, \lambda, \varphi_0))$. Thus P is pro-representable.

Exercise. Let $P \subset M$ be two functors from $\mathscr{C}_W$ into $\mathscr{E}ns$; suppose both are pro-representable, say $\mathscr{C}_W(\mathcal{O}, -) \xrightarrow{\sim} M(-)$, then there exists an ideal $\mathfrak{a} \subset \mathcal{O}$, such that $P \subset M$ is pro-represented by $\mathcal{O}/\mathfrak{a} \twoheadleftarrow \mathcal{O}$.

So we are given $\mathfrak{a} \subset \mathcal{O}$ such that $\mathcal{O}/\mathfrak{a}$ pro-represents P. By the Artin--Rees lemma (cf. [7], p. 7, th. 3.7) we can choose an integer $n > 0$ so that

$$\mathfrak{m}^n \cap \mathfrak{a} = \mathfrak{m}(\mathfrak{m}^{n-1} \cap \mathfrak{a})$$

(here $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}$). Note the exact sequence

$$0 \to \mathfrak{m}\mathfrak{a} + \mathfrak{m}^n \to \mathfrak{a} + \mathfrak{m}^n \to \mathfrak{a}/\mathfrak{m}\mathfrak{a} \to 0.$$

We are going to find a set of generators for $\mathfrak{a}$ by finding their initial terms in $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$. Let $R := \mathcal{O}/(\mathfrak{m}\mathfrak{a} + \mathfrak{m}^n)$, $R' := \mathcal{O}/\mathfrak{a} + \mathfrak{m}^n$, and

$$0 \to I = \mathrm{Ker}\,(\rho) \to R \xrightarrow{\rho} R' \to 0; \qquad R, R' \in \mathscr{C}_W.$$

As $\mathcal{O}/\mathfrak{a}$ pro-represents P, the surjection $\mathcal{O}/\mathfrak{a} \twoheadrightarrow R'$ transforms the universal object over $\mathcal{O}/\mathfrak{a}$ into $(X', \Lambda L' = \lambda', \varphi_0) \in PR'$. Thus by the exact sequence

$$\mathrm{Pic}\,(X) \to \mathrm{Pic}\,(X') \xrightarrow{\delta} H^2(X, \mathcal{O}_{X_0}) \otimes I,$$

and by choosing a k-base $\{\xi_1, \cdots, \xi_d\}$ for $H^2(X, \mathcal{O}_{X_0}) \cong k^d$, we yield

$$\delta L' = \sum_{i=1}^{d} \xi_i \otimes \bar{a}_i, \qquad \bar{a}_i \in I \cong \mathfrak{a}/\mathfrak{m}\mathfrak{a};$$

then we choose $a_i \in \mathfrak{a}$ so that $\bar{a}_i = a_i \bmod \mathfrak{m}\mathfrak{a}$, and we write $\mathfrak{b} = (a_1, \cdots, a_d)\mathcal{O} \subset \mathfrak{a}$. Consider $R'' = \mathcal{O}/\mathfrak{b} + \mathfrak{m}\mathfrak{a} + \mathfrak{m}^n$, and the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & J & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & 0. \end{array}$$

Thus we obtain

$$\begin{array}{ccccc} H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^2(X, \mathcal{O}_{X'}^*) & \longrightarrow & H^2(X, \mathcal{O}_{X_0}) \otimes I \\ \downarrow & & \| & & \downarrow f \\ H^1(X, \mathcal{O}_{X''}^*) & \longrightarrow & H^2(X, \mathcal{O}_{X'}^*) & \longrightarrow & H^2(X, \mathcal{O}_{X_0}) \otimes J, \end{array}$$

and by commutativity we obtain

$$f(\sum \xi_i \otimes \bar{a}_i) = \sum \xi_i \otimes (a_i \bmod (\mathfrak{b}+\mathfrak{m}\mathfrak{a}+\mathfrak{m}^n)) = 0$$

as $a_i \in \mathfrak{b}$. Thus $(X', \lambda', \varphi_0)$ can be lifted to R'', and the result defines a $\mathscr{C}_W$-morphism

$$\begin{array}{ccc} (X'', \lambda'', \varphi_0) : \mathcal{O}/\mathfrak{a} & \longrightarrow & R'' \\ \uparrow & & \uparrow \\ \mathcal{O} & \longrightarrow & R \end{array}$$

because of universality of $\mathcal{O}/\mathfrak{a}$. Thus $\mathfrak{a} \subset \mathfrak{b}+\mathfrak{m}\mathfrak{a}+\mathfrak{m}^n$, and one easily concludes $\mathfrak{a} = \mathfrak{b}$, which proves the theorem.

In general it seems difficult to give an explicit description of $\mathfrak{a}$. In particular it does not seem easy to determine whether $p \in \sqrt{\mathfrak{a}}$ or not in case char $(k) = p$, and W a characteristic zero domain. However in special cases one can achieve the result and thus prove the existence of a lifting to characteristic zero.

2.4. *Abelian varieties having a separable polarization*

THEOREM (2.4.1) (*Grothendieck*). *Let (X_0, λ_0) be a polarized abelian variety. Its moduli functor P is formally smooth if and only if the polarization λ_0 is separable.*

REMARK. It can be proved (cf. MAV, p. 234, Coroll. 1) that any polarization $\Lambda L_0 = \lambda_0$ can be obtained from an isogeny $\varphi : X \to Y$ and a *principal* polarization $\Lambda K_0 = \mu_0 : Y \xrightarrow{\sim} Y^t$ (a polarization μ is called principal if it is an isomorphism), $\lambda_0 = \varphi^t \mu_0 \varphi = \Lambda(\varphi^* K_0)$. Clearly a polarization λ_0 is smooth if and only if Ker (λ_0) is an étale group scheme over k; it can be proved that λ_0 is smooth if and only if the degree of λ_0, which is the rank of Ker (λ_0), is prime to char (k).

We sketch the proof of the implication in the theorem we need. Suppose λ_0 to be separable. The polarization λ_0 yields a map $f: H^1(X, \Theta_{X_0}) \to H^2(X, \mathcal{O}_{X_0})$ ('cup product with the fundamental class of L_0'), which can be defined by the commutativity of the following diagram:

$$\begin{array}{ccc} H^2(X, \mathcal{O}_{X_0}) \cong H^1(X, \mathcal{O}_{X_0}) \wedge H^1(X, \mathcal{O}_{X_0}) & = & \mathfrak{t}^t \wedge \mathfrak{t}^t \\ \uparrow f & & \uparrow \mathrm{id} \wedge d\lambda_0 \\ H^1(X, \Theta_{X_0}) \cong H^1(X, \mathcal{O}_{X_0}) \otimes \mathfrak{t} & & \cong \mathfrak{t}^t \otimes \mathfrak{t}, \end{array}$$

where $\mathfrak{t}$ is the tangent space at $0 \in X_0$, and $\mathfrak{t}^t$ the tangent space at $0 \in X_0^t$. Let $R \to R'$ be a small surjection, $(X', \lambda', \varphi_0) \in PR'$, and choose some $(X, \varphi_0) \in MR$ which lifts $(X', \varphi_0) \in MR'$. Let $(Y, \psi_0) \in MR$, be some lift of (X', φ_0), and write

$$\operatorname{Pic}(Y) \to \operatorname{Pic}(X) \overset{\delta_X}{\to} H^2(X, \mathcal{O}_{X_0}) \otimes I.$$

Direct verification shows that

$$\delta_Y(L') = \delta_X(L') + (f \otimes \mathrm{id}_I)(\imath_X^{-1}(Y, \psi')),$$

where

$$\imath_X : H^1(X, \Theta_{X_0}) \otimes I \simeq L(X'; R \to R')$$

is the homomorphism defined in (2.2.5), and (Y, ψ') the pair corresponding to (Y, ψ_0) via (2.2.6). As λ_0 is separable, f is surjective, so given L', we can choose Y so that $\delta_Y(L') = 0$, which proves smoothness of the functor P.

Corollary (2.4.2). *Any abelian variety which admits a separable polarization (in particular any AV which admits a principal polarization) can be lifted to characteristic zero.*

Proof. Choose a separable polarization $\lambda_0 : X_0 \to X_0^t$, and take $W = W_\infty(k)$. As the functor P_{X_0, λ_0} is formally smooth, (X_0, λ_0) can be lifted to a formal abelian scheme with a polarization $(\mathfrak{X}, \lambda)$ over W, i.e. there exists a system (X_i, λ_i) of polarized abelian schemes over $W_i = W/(p^{i+1})$, so that

$$(X_{i+1}, \lambda_{i+1}) \otimes_W W_i \cong (X_i, \lambda_i), \qquad i \geqq 0.$$

By EGA, III^1.5.4.5 the formal scheme $\mathfrak{X}$ is algebraizable, i.e. there exists an abelian scheme (with a polarization) $X \to \operatorname{Spec}(W)$ such that $X \otimes_W W_i \cong X_i$, which proves the liftability for abelian varieties admitting a separable polarization to characteristic zero.

Remark. Let $P = P_{X_0, \lambda_0}$ be pro-represented by $\mathcal{O}/\mathfrak{a}$; clearly the pair (X_0, λ_0) can be lifted to characteristic zero iff $p^n \notin \mathfrak{a}$ for all n, i.e. $p \notin \sqrt{\mathfrak{a}}$, where $p = \operatorname{char}(k)$.

REMARK. (Grothendieck and Mumford) Theorem (2.4.1) can be made more precise in the following sense: let $P \cong \mathscr{C}_W(\mathscr{O}/\mathfrak{a}, -)$ be the local moduli functor defined by the separably polarized AV (X_0, λ_0); the isomorphism $\mathscr{O} \cong W[[t_{11}, \cdots, t_{ij}, \cdots, t_{gg}]]$ can be choosen in such a way that

$$\mathfrak{a} = (\cdots, t_{ij} - t_{ji}, \cdots) \cdot \mathscr{O}.$$

3. Commutative finite group schemes can be lifted to characteristic zero [2]

In case the local moduli scheme defined by a smooth projective variety V_0 (or by a polarized abelian variety, or by a finite group scheme) is not formally smooth over W, a solution of the lifting problem does not follow directly from the unequal characteristic case obstruction calculus. However, an entirely different method might yield the answer, and in fact it does in case of abelian schemes, and of commutative finite group schemes. In this section we outline this idea in the last case.

Let k be a field of characteristic p, and let α_p be defined by

$$\alpha_p = \operatorname{Ker}(F : \boldsymbol{G}_a \to \boldsymbol{G}_a)$$

i.e. $\alpha_p = \operatorname{Spec}(E)$, where

$$E = k[\tau],\ \tau^p = 0,$$
$$s\tau = \tau \otimes 1 + 1 \otimes \tau,$$
$$\iota\tau = -\tau,\ \varepsilon\tau = 0.$$

Although this group scheme cannot be lifted in the strong sense (cf. [8], p. 317/18, example $(-A)$), we try to convince the reader in three different ways it can be lifted to characteristic zero in the weak sense.

FIRST METHOD. Suppose first $p = 2$, and try to classify all R-bialgebra's $E = R \cdot 1 \otimes I_E$, where $I_E = \operatorname{Ker}(\varepsilon : E \to R)$ is free of rank *one* (so Spec (E) is a group scheme of rank $p = 2$ over Spec (R), where R is any (commutative, $1 \in R$) ring. Let $I_E = R \cdot \tau$, $s\tau = \tau \otimes 1 + 1 \otimes \tau - c\tau \otimes \tau$, $\tau^2 = a\tau$, $\iota\tau = \gamma\tau$. We show $ac = p = 2$ if and only if E is an R-bialgebra with these definitions. In fact

$$(s\tau)^2 = s(\tau^2) \text{ implies } (ac-2)(ac-1) = 0$$

and

$$\varepsilon(\tau) = (1 \otimes \iota)s(\tau) \text{ implies } \gamma(ac-1) = 1,$$

so $ac = 2$. This proves that α_2 can be lifted to characteristic zero, e.g. choose $R = \boldsymbol{Z}_p[A, C]/(AC-p)$, $p = 2$. In fact in the same way we can

[2] This chapter is of an expository nature, and the reader should not expect more than easy examples.

prove α_p can be lifted to characteristic zero, using the following classification due to Tate:

THEOREM (*cf.* [14]). *Let p be a prime number. There exists a polynomial* $D_p \in \mathbf{Z}_p[X, Y]$, *such that for any complete, local ring R, of residue characteristic p, any group scheme G of rank p over R can be given in the form*

$$G = \operatorname{Spec}(A), \quad A = R[\tau]/(\tau^p - a\tau),$$
$$s\tau = \tau_1 + \tau_2 + c \cdot D_p(\tau \otimes 1, 1 \otimes \tau), \quad ac = p \cdot 1 \in R,$$

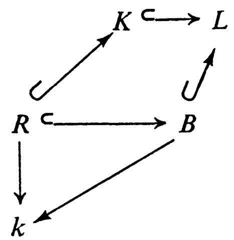
and conversely these formulas determine a group scheme of rank p.

In this method it becomes clear 'how much ramification' we have to impose on R in order to get a lifting of α_p. However, this method does not seem the appropriate one to be generalized (classification of all group schemes of rank p^2 already seems a difficult question).

SECOND METHOD. Let k be an algebraically closed field of characteristic p, such that there exists an elliptic curve C_0, whose Hasse invariant is zero. The group $C_0(k)$ has no points of order p, and hence

$$\alpha_p \cong \operatorname{Ker}(F : C_0 \to C_0^{(p)})$$

is the only subgroup scheme of C_0 of rank p. By the methods of chapter 2 we know that C_0 is liftable (use 2.3.3, or 2.4.2), so choose a characteristic zero integral complete local ring R with residue class field $R/\mathfrak{m}_R = k$, and choose an abelian scheme of dimension 1 $C \to \operatorname{Spec}(k)$, such that $C \otimes_R k \cong C_0$.



Let $K = Q(R)$, the field of fractions of R; as char $(K) = 0$, we know that C has points of order p rational over the algebraic closure of K, so we can choose a finite extension $K \subset L$, and $t \in C(L)$ such that $t \neq 0$ and $p \cdot t = 0$. Let B be the integral closure of R in L, let N_L be constant group over L defined by the abstract group $\mathbf{Z}/p\mathbf{Z} \cong \{0, t, 2t, \cdots, (p-1)t\} \subset C(L)$, and let M be the B-group scheme defined as the kernel of $[p] : C_B \to C_B$ (it is B-flat, cf. GIT, p. 122, lemma 6.12). Then

$$\begin{array}{ccc} & & N_L \\ & & \big\downarrow {\scriptstyle i_L} \\ M & \longleftarrow & M_L \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}(B) & \longleftarrow & \operatorname{Spec}(L) \end{array}$$

and we use:

LEMMA (*Tate*). *Let B be a complete discrete valuation ring, $L = Q(B)$ its field of fractions, and M, respectively $M_L = M \otimes_B L$, N_L finite group schemes as indicated above flat over B, resp. flat over L; then there exists a B-flat subgroup scheme $i : N \hookrightarrow M$ such that $N \otimes_B L \cong N_L$ and $i \otimes_B L = i_L$.*

Once arrived at this point, we are done, because $N_0 = N \otimes_B k \hookrightarrow M \otimes_B k \hookrightarrow C_0$, and the rank of N_0 is p, so $N_0 \cong \alpha_p$.

In this method again ramification has to show up, indeed it does in the extension $K \hookrightarrow L$. Also this method does not seem very fruitful for generalizations, hence we try another one:

THIRD METHOD. Let char $(k) = p$, $T = \operatorname{Spec}(k[t]) = A^1_k$, and consider the additive linear group over T,

$$\boldsymbol{G}_{a,T} = \operatorname{Spec}(k[t][X]), \qquad sX = X \otimes 1 + 1 \otimes X, \text{ etc.}$$

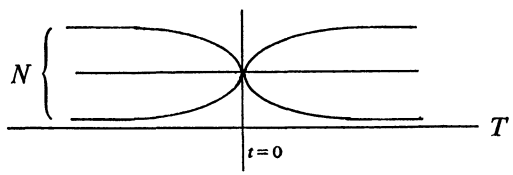
We define the T-homomorphism

$$\rho : \boldsymbol{G}_{a,T} \to \boldsymbol{G}_{a,T}$$

by $\rho^*(X) = X^p - tX$, and clearly

$$0 \to N \to \boldsymbol{G}_{a,T} \xrightarrow{\rho} \boldsymbol{G}_{a,T} \to 0,$$

$N = \operatorname{Ker}(\rho)$ is a finite flat T-group scheme of rank p (it is defined by $N = \operatorname{Spec}(k[t,x]/(x^p - tx))$, $sx = x \otimes 1 + 1 \otimes x$, etc.).



Clearly $N_0 = (k[t] \to k, t \mapsto 0)^*(N)$, the fibre over the point $t = 0$, is $N_0 \cong \alpha_p$, and for any point $\xi \in \bar{k}$, $\xi \neq 0$, $N_\xi = (k[t] \to \bar{k}, t \mapsto \xi)^*(N)$ is the constant group scheme given by $\boldsymbol{Z}/p\boldsymbol{Z}$ over $\bar{k}$ (N_ξ is separable because $X^p - \xi X$ is a separable polynomial). Thus we have constructed a deformation of α_p into a group scheme which clearly can be lifted. Hence we can conclude α_p can be lifted to characteristic zero in the following way: let $K = k(t)$, $\eta \in T(K)$ given by the inclusion $k[t] \to K$, $W = W_\infty(K)$ the ring of infinite Witt vectors over K, and $R := k[t] \times_K W$ those Witt vectors of which the first coordinate is in $k[t]$

$$\begin{array}{ccccc} & & R & \longrightarrow & W = W_\infty(K) \\ & & \downarrow & & \downarrow \rho \\ k & \xleftarrow{\;0\;} & k[t] & \xhookrightarrow{\;\eta\;} & K \end{array}$$

As $X^p - tX \in K[X]$ is separable, the group scheme $N_K = N \otimes_{k[t]} K$ is étale over K, hence there exists a lift of it to W. It is clear that this lift can be defined by equations whose reductions by ρ have coefficients in $k[t]$, thus there exist a finite flat group scheme $M \to \operatorname{Spec}(R)$ such that $M \otimes_k k[t] \otimes_{k[t]} k = \alpha_p$ and as R is an integral domain of characteristic zero, we have proved liftibility of α_p. In this method again the information about the ramification of the 'smallest ring R' which could be used seems to go beyond control, but this last method at least turned out to be generalizable to arbitrary rank. The main conclusion of this last method is:

General principle. Let $X_0 \to \operatorname{Spec}(k)$ be given (where X_0 is a finite group scheme, or a polarized smooth variety or $\cdots$), suppose there exist an *integral* scheme $T \in \mathrm{Sch}_k$, with $t : \operatorname{Spec}(k) \to T$, and $X \to T$ (where X is a finite flat T group scheme, or a polarized smooth variety over T, or $\cdots$) such that:

(i) $X \times_T (t : \operatorname{Spec}(\mathrm{k}) \to T) \cong X_0$, (i.e. X is an *equal* characteristic deformation of X_0) and

(ii) $X \times_T (\{\eta\} \to T, \text{ generic point}) = X_\eta$ is liftable to characteristic zero.

Then X_0 is liftable to characteristic zero.

The proof of this principle in the case of finite group schemes was outlined above (and cf. [8], lemma 2.1), in other cases the proof can be given using existence of the (global or local) moduli scheme. This new method of reducing lifting questions to equal characteristic deformation indeed solves the problems we encountered in chapter 2: Mumford can show (cf. [5]), that any polarized abelian variety X_0 in characteristic $p > 0$ admits an equal characteristic deformation such that the generic fibre X_η is an 'ordinary' abelian variety (an abelian variety Y over a field K of characteristic $p > 0$ is called ordinary, if $Y(\overline{K})$ has exactly $p^{\dim Y} - 1$ points of order p; that is the maximum); comparing the lifting theory of abelian varieties and their associated p-divisible groups Serre and Tate proved that ordinary abelian varieties are liftable (together with the polarization).

Thus modulo many details which still have to be published, Mumford concludes all abelian varieties can be lifted to characteristic zero.

The good choice for the integral scheme T plus $X \to T$ is the difficult point, and we illustrate the difficulties by trying to do a good choice in the case of finite commutative group schemes.

FIRST TRY. Fix k, char $(k) = p$, fix an integer $n > 0$, and let

$$\Psi : \mathcal{S}ch_k^0 \to \mathcal{E}ns$$

be defined by

$$\Psi(T) = \{ \cong \text{ classes of } N \to T, \text{ finite flat commutative group scheme of rank } n\}.$$

If Ψ were representable by an integral scheme we would be done, because the generic fibre is an étale group scheme. But, in general Ψ is not representable: let $k = \bar{k}$, $n = p$, and consider $N = \text{Ker}\,(\rho) \to T$ as in the third method. Then $N_0 \cong \alpha_p$, and $N_t \cong \mathbf{Z}/p\mathbf{Z}$ for all $0 \neq t \in k$; this *jump phenomenon* certainly prevents Ψ from being representable (because, suppose $\Psi \cong \text{Mor}_k(-, S)$, then $N \in \Psi(T)$ defines a morphism $f_N = f$: $T \to S$, and $f(0) \neq f(t)$ for all $t \neq 0$ and $f(t_1) = f(t_2)$ for all $t_1 \neq 0 \neq t_2$, which is impossible).

This certainly is not a difficult obstacle, so:

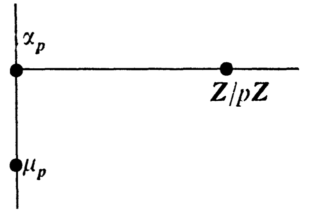
SECOND TRY: Data as above, and

$$\Phi : \mathcal{S}ch_k^0 \to \mathcal{E}ns$$

is defined by: fix $E = R.1 \oplus R\tau_2 \oplus \cdots \oplus R\tau_n$, $\varepsilon\tau_i = 0$, for each R, and

$$\Phi(\text{Spec}\,(R)) = \{\text{all } R\text{-bialgebra structures on } E\}$$

($E \in Alg_R$ is called an R-bialgebra, if and only if, Spec $(E) \to$ Spec (R) is a commutative group scheme). Claim: this functor is representable by a noetherian affine scheme $T \to$ Spec (k); namely write out all possible R--linear maps $m : E \otimes_R E \to E$, $s : E \to E \otimes_R E$, $\iota : E \to E$; these are given by a finite number of coefficients, and imposing the conditions for the R-bialgebra structure provides us with a number of equations among these coefficients, which proves representability of Φ. However, the result $\Phi \cong \text{Mor}_k(-, T)$ in general is not an integral scheme. For example take $p = 2 = n$; we have seen in the first method that



the resulting T is given by: $T = \text{Spec}\,(k[A, C]/(AC-2))$, $s\tau = \tau \otimes 1+1 \otimes \tau - c\tau \otimes \tau$, $m(\tau \otimes \tau) = A\tau$, and $\iota\tau = \tau$. This suggests we should delete all but one of the components of T. This surgery we can do directly on the functor level, and as we are mainly interested in local group schemes, and as every local group scheme over a perfect field is given by a truncated polynomial ring (Dieudonné and Cartier (cf. SGAD, Exp. VII_B, 5.4)), we define:

THIRD TRY. Let k be a *perfect* field, char $(k) = p$, and fix integers $\nu_1, \cdots, \nu_m$, $\nu_i \geqq 1$. Define

$$\Theta_\nu = \Theta : \mathscr{S}ch_k^0 \to \mathscr{E}ns$$

by: $E = R[X_1, \cdots, X_m]/(X_1^{p \exp \nu_1}, \cdots, X_m^{p \exp \nu_m}) = R[X]/(X^{p \exp \nu})$ we write $X^{p \exp m} = X^{p^m}$, $\varepsilon X_i = 0$,

$$\Theta(\text{Spec}\,(R)) = \{\text{all } R\text{-bialgebra structures on } R[X]/(X^{p \exp \nu})\}.$$

Certainly this functor is representable $\Theta \cong \text{Mor}_k\,(-, T)$, and nice fact: T is 'contractible' in the following sense: The multiplicative semigroup A_k^1 acts on T such that the zero point of each orbit equals $0 \in T$, where 0 corresponds to $s\tau_i = \tau_i \otimes 1+1 \otimes \tau_i$, $\tau_i = X_i \bmod (X^{p \exp \nu})$; this can be seen as follows: if $s \in \Theta(R)$,

$$s\tau_i = \sum a_{i, \alpha, \beta} \tau^\alpha \otimes \tau^\beta$$

(we use multi-index notation

$$X^\alpha = X_1^{\alpha_1} \times \cdots \times X_m^{\alpha_m}, \qquad \alpha = (\alpha_1, \cdots, \alpha_m)),$$

and $\lambda \in k$, then define

$$s_\lambda \tau_i = \sum \lambda^{|\alpha|+|\beta|-1} a_{i, \alpha, \beta} \tau^\alpha \otimes \tau^\beta.$$

Clearly $s_\lambda \in \Theta(R)$ (compute $s(\tau_i/\lambda)$ for $\lambda \neq 0$, or write out coassociativity of s), and for every s we obtain $s_0 = 0 \in \Theta(k)$. Thus in order to prove T to be irreducible, it suffices to show $0 \in T(k) = \Theta(k)$ is a nonsingular point. In fact this is true in case $m = 1$ (local group schemes with tangent dimension one) as we might guess from the example of the second try (we have deleted the horizontal axis), but T is not smooth at 0 if $m > 1$ and not all ν_i equal:

EXAMPLE. $m = 2$, $\nu_1 = 1$, $\nu_2 = 2$, $R = k[\delta]$, $\delta^{p+1} = 0$, $\rho : R \twoheadrightarrow R' = k[\varepsilon]$, $\delta \mapsto \varepsilon$, $\varepsilon^2 = 0$, and $s \in \Theta(R')$ is given by

$$s'\tau_1 = \tau_1 \otimes 1+1 \otimes \tau_1+\varepsilon\tau_2 \otimes \tau_2$$
$$s'\tau_2 = \tau_2 \otimes 1+1 \otimes \tau_2;$$

check this is a coassociative comultiplication; the important point to note is, that $s : E' \to E' \otimes E'$, $E' = R'[\tau_1, \tau_2]$ is a ring homomorphism, because

$$(s'\tau_1)^p = (\cdots)^p = \varepsilon^p\tau_2^p \otimes \tau_2^p = 0, \qquad \text{because } p \geqq 2.$$

However there does not exist any R-algebra homomorphism $s : E \to E \otimes E$ such that $s \otimes R' = s'$, because

$$(s\tau_1)^p = (\tau_1 \otimes 1 + 1 \otimes \tau_1 + \delta\tau_2 \otimes \tau_2 + \delta^2(\cdots))^p \neq 0$$

because $\delta^p \neq 0$ (all these calculations are so easy because we still are in the 'equal characteristic case'). Thus $\Theta(\rho) : \Theta R \to \Theta R'$ is not surjective, and T is not smooth at $s' \otimes_{R'} k = 0 \in T(k)$.

LAST TRY. Fix data as before, and remark that for any *reduced* $R \in Alg_k$,

$$s \in \Theta(R), \qquad s\tau_i = \sum a_{i\alpha\beta}\tau^\alpha \otimes \tau^\beta, \qquad 1 \leqq i \leqq m,$$

$(\tau^\alpha)^{p \exp v_i} \neq 0$ & $(\tau^\beta)^{p \exp v_i} \neq 0$ imply $a_{i\alpha\beta} = 0$. Thus we can proceed the surgery, we can produce the scheme T_{red} by changing the functor as follows:

$$\textstyle\sum_v = \sum : \mathscr{S}ch_k^0 \to \mathscr{E}ns, \qquad v = (v_1, v_2, \cdots, v_m)$$

is defined by: for any $R \in Alg_k$ (not necesserily reduced)

$$\textstyle\sum(\text{Spec}\,(R)) = \{s \in \Theta(\text{Spec}\,(R)) \text{ such thay } s\tau_i \text{ satisfies condition}(Pv)_i\},$$

where

$$s\tau_i = \sum a_{i\alpha\beta}\tau^\alpha \otimes \tau^\beta$$

is said to satisfy condition (Pv_i) if $\alpha_j \cdot p^{v_i} < p^{v_j}$ and $\beta_j \cdot p^{v_i} < p^{v_j}$ imply $a_{i\alpha\beta} = 0$ (this is the same condition as explained abcve to be true automatically in case R is reduced).

Using the lifting theory of Lazard of formal groups, we were able to prove:

THEOREM (cf. [8], *theorem* 3.1). $\sum : \mathscr{S}ch_k^0 \to \mathscr{E}ns$ *is represented by an affine space* $T = A_k^n$, *in particular* T *is irreducible.*

From this we conclude easily liftability of finite commutative group schemes to characteristic zero (for details, cf. [8]). In fact, we only need to prove the case N_0 is local. Then $N_0 = \text{Spec}\,(E_0)$, and $E_0 \cong k[\tau_1, \cdots, \tau_m]$, $\tau_i^{p \exp v_i} = 0$; the resulting T has a point which corresponds to (a rigidifacation of) the finite group scheme

$$\mu_{p^{v_1}} \times \cdots \times \mu_{p^{v_m}}.$$

From this one concludes that the generic point of T corresponds to a group scheme whose dual is étale, hence which certainly can be lifted to characteristic zero. This solves the lifting problem for finite commutative group schemes, and we conclude by thanking the nordic countries for their hospitality and friendship crystalized during the nordic summer school in algebraic geometry 1970.

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