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R. DUNCAN LUCE

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PERIODIC EXTENSIVE MEASUREMENT [1,2]

by

R. Duncan Luce

1. Introduction

The classical model for physical measurement includes, as primitives, a non-empty set A, a closed binary (concatenation) operation $\circ$, and an (ordering) relation $\succsim$, and the axioms are (necessary and) sufficient to establish a homomorphism into $\langle \mathbb{R}, +, \geqq \rangle$ [1, 2, 3, 4, 7]. This model accounts for the measurement of many important physical quantities such as mass, length, duration, electrical resistance, etc., but it is unsuitable for many others, among them periodic quantities such as angle and clock time. The purpose of this note is to present axioms in terms of the usual primitives $\langle A, \circ, \succsim \rangle$ that result in a homomorphism into $\langle [O, K], +(\text{mod } K), \geqq \rangle$.

At least two earlier axiomatizations of angles exist. Zassenhaus [8] treated them as abstract entities which are oriented and among which judgments of equality can be made. In addition, he assumed that some pairs of angles, but not all, can be concatenated to form new ones; he imposed this restriction because repeated sums of positive angles ultimately lead to negative ones. This axiomatization differs in several ways from those typical of other extensive quantities. First, an order rather than just an equivalence is usually postulated. Second, unless the intended interpretation actually forces an inherent restriction on concatenation (e.g., the disjointness of events inherent in the additivity of probability), it is usual to let the concatenation operation be closed. Third, orientation is ignored whenever possible; this is true, e.g., of the usual axioms of length and duration measurement. In addition, Zassenhaus' axioms do not adequately highlight the periodic nature of angular measurement.

[1] This work was supported in part by NSF grant GB-6536 to the University of Pennsylvania; it was initiated while I was an Organization of American States Professor at the Pontíficia Universidade Católica do Rio de Janeiro, Brasil.

[2] I wish to thank the referee for suggesting the reformulation in terms of cyclic orders, for pointing out an error in the original proof of Lemma 8, and for other useful suggestions.

The other paper, Lenz [5], has a different set of limitations from our point of view. The main one is that his primitives are far more elaborate than is usual in the theory of measurement; in particular, he postulates a good deal of Hilbert's geometrical structure, defines angles in terms of it, and then adds sufficient axioms to yield the periodic representation. One can view this note as an attempt to isolate the basic abstract structure underlying Lenz' construction. In some respects, our proof is similar to his; the most obvious parallel is the natural embedding of the given structure into an ordinary extensive one. A noticeable difference is the form of the Archimedean axiom which, in the absence of any geometrical structure, must be quite strong, as in other theories of extensive measurement.

2. Axioms and statement of representation theorem

Throughout this note, A is a non-empty set, $\succsim$ a binary relation on A, and $\circ$ a closed operation on a. By $\sim$ we mean both $\succsim$ and $\precsim$ [3], and by $\succ$ we mean both $\succsim$ and not $\precsim$.

DEFINITION 1. *The triple* $\langle A, \succsim, \circ\rangle$ *is a* periodic extensive structure *iff, for all* $a, b, c \in A$,

1. $\langle A, \succsim\rangle$ *is a weak order*, i.e., $\succsim$ *is transitive and connected in the sense that, for all* $a, b \in A$, *either* $a \succsim b$ *or* $b \succsim a$.
2. $\langle A, \sim, \circ\rangle$ *is a weak abelian semigroup* [4].
3. $a \succsim b$ *iff either* (i) $a \circ c \succsim b \circ c \succsim c$, (ii) $c \succ a \circ c \succsim b \circ c$, *or* (iii) $b \circ c \succsim c \succ a \circ c$.
4. *If* $a \succ b$, *then there exists a positive integer* n *such that* $a \succ na$ *and* $nb \succsim b$, *where* $1a = a$ *and* $na = (n-1)a \circ a$.

Axioms 1 and 2 are unexceptional in the theory of extensive measurement and have an obvious interpretation for angles.

Axioms 3 and 4 reflect the fact that when a and b are angles whose sum, $a \circ b$, is less than one cycle, then $a \circ b \succsim a, b$; whereas when $a \circ b$ is greater than one cycle, then $a, b \succ a \circ b$ (see Lemma 2 and Def. 3). Thus, in Axiom 3, if $a \circ c$ and $b \circ c$ are both less than a cycle (case i) or are both greater than a cycle (case ii), the order $a \succsim b$ is unchanged by adding c both to a and to b; but when $a \circ c$ is greater than one cycle and $b \circ c$ is less, the order is reversed (case iii). And Axiom 4 says that if one angle exceeds another, then there is some integer n such that na (= sum of n copies of a) just completes a cycle but nb fails to do so. This is a type of Archimedean Axiom.

[3] Given Axiom 1, it is trivial to show that $\sim$ is an equivalence relation.

[4] That is, $\circ$ is a closed binary operation such that $a \circ (b \circ c) \sim (a \circ b) \circ c$ and $a \circ b \sim b \circ a$ for all $a, b, c \in A$.

THEOREM 1. *$\langle A, \succsim, \circ \rangle$ is a periodic extensive structure with identity e iff for any real $K > 0$ there exists a unique function ϕ from A into* $[0, K)$ *such that*

(i) $a \succsim b$ *iff* $\phi(a) \geqq \phi(b)$,

(ii) $\phi(a \circ b) = \phi(a) + \phi(b) \pmod K$,

(iii) $\phi(e) = 0$.

The most common values of K for angles are 1, 2π, and 360, in which case the unit is called, respectively, the cycle, radian, and degree, and for clock time 12 and 24 are the familiar values.

The axioms of Definition 1 can be reformulated in terms of the following concept due to Rieger [6]; see Fuchs [3], p. 62, and Lenz [5].

DEFINITION 2. *A trinary relation C on a semigroup $\langle A, \circ \rangle$ is a cyclic order iff, for all $a, b, c, d \in A$,*

1. *Exactly one of $C(a, b, c)$ or $C(a, c, b)$ holds.*
2. *$C(a, b, c)$ implies $C(b, c, a)$.*
3. *$C(a, b, c)$ and $C(a, c, d)$ imply $C(a, b, d)$.*
4. *$C(a, b, c)$ implies $C(a \circ d, b \circ d, c \circ d)$ and $C(d \circ a, d \circ b, d \circ c)$.*

If $\langle A, \circ \rangle$ is abelian and has an identity e, C is called Archimedean *iff*

5. *$C(e, a, b)$ implies there exists a positive integer n such that $C(e, a, na)$ and $C(e, nb, b)$.*

If we think of the elements of A as lying in a circle, we can interpret $C(a, b, c)$ to mean that a, b, and c are clockwise in the order a, b, c.

THEOREM 2. *If $\langle A, \succsim, \circ \rangle$ satisfies Axioms $1-3$ of Definition 1 and if C on $A/\sim$ is defined by*

$$C(\mathbf{a}, \mathbf{b}, \mathbf{c}) \textit{ iff } \mathbf{a} \succ \mathbf{b} \succ \mathbf{c},\ \mathbf{b} \succ \mathbf{c} \succ \mathbf{a}, \text{ or } \mathbf{c} \succ \mathbf{a} \succ \mathbf{b},$$

where $\mathbf{a}$ is the equivalence class containing a, then C is a cyclic order. If, in addition, $\langle A, \circ \rangle$ has an identity e and Axiom 4 holds, then C is Archimedean.

Conversely, if C is a cyclic order on an abelian semigroup $\langle A, \circ \rangle$ with identity e and if $\succsim$ on A is defined by

$$a \sim b \textit{ iff } a = b$$
$$a \succ b \textit{ iff } a \neq b \text{ and } C(e, a, b),$$

then $\langle A, \succsim, \circ \rangle$ satisfies Axioms $1-3$ of Definition 1. If C is Archimedean, then $\langle A, \succsim, \circ \rangle$ satisfies Axiom 4.

Combining these two theorems, it is easy to show the following.

COROLLARY. *If $\langle A, \circ \rangle$ is an abelian semigroup with identity e and C is an Archimedean cyclic order, then for any real $K > 0$ there is a unique function ϕ from A into* $[0, K)$ *such that*

(i) $C(a, b, c)$ *iff* $\phi(a) > \phi(b) > \phi(c)$, $\phi(b) > \phi(c) > \phi(a)$, *or* $\phi(c) > \phi(a) > \phi(b)$.

(ii) $\phi(a \circ b) = \phi(a) + \phi(b) \pmod{K}$.

(iii) $\phi(e) = 0$.

The problem of cyclic orders over non-abelian semigroups, which is the natural extension of Rieger's work on cyclic orders over non-abelian groups, appears to be much more complex. As pointed out by a referee, it would be interesting to know under what conditions a cyclically ordered semigroup can be extended to a cyclically ordered group.

3. Preliminary lemmas

In each of the following lemmas we assume that $\langle A, \succsim, \circ \rangle$ is a periodic extensive structure, that $a, b, c \in A$, and that i, j, k, l, m, and n are positive integers.

LEMMA 1. *$a \sim b$ iff $a \circ c \sim b \circ c$; and $a \succ b$ iff either*

(i) $a \circ c \succ b \circ c \succsim c$,

(ii) $c \succ a \circ c \succ b \circ c$, *or*

(iii) $b \circ c \succsim c \succ a \circ c$.

PROOF. Axioms 1 and 3. QED

LEMMA 2. *$a \circ b \succsim a$ iff $a \circ b \succsim b$.*

PROOF. Suppose that, on the contrary, $b \succ a \circ b \succsim a$. From $b \succ a \circ b$, Lemma 1 (with $c = a$) implies three possibilities, but the hypothesis $a \circ b \succsim a$ excludes all but $a \circ b \sim b \circ a \succ a \circ b \circ a \succsim a$. Similarly, Axiom 3 applied to $a \circ b \succsim a$ (with $c = b$) yields $b \succ a \circ b \circ b \succsim a \circ b$. Thus, $a \circ b \circ b \succsim a \circ b \succ a \circ b \circ a$. Using Axiom 2 and Lemma 1 to cancel $a \circ b$, we conclude that $a \succ b$, which is a contradiction. QED

Lemma 2 insures that the following notion is well defined.

DEFINITION 3.

$$r(a, b, = \begin{cases} 1 & \text{if } a, b \succ a \circ b \\ 0 & \text{if } a \circ b \succsim a, b. \end{cases}$$

For angles, $r(a, b)$ is 1 if the sum of the two angles is a complete cycle or more and is 0 if it is less than a complete cycle.

LEMMA 3. *If $a \succsim b$, then $r(a, c) \geqq r(b, c)$.*

PROOF. The only possible violation is $r(a, c) = 0$ and $r(b, c) = 1$, i.e., $a \circ c \succsim c \succ b \circ c$. By Lemma 1, $b \succ a$, contrary to hypothesis. QED

LEMMA 4. $r(a, b)+r(a \circ b, c) = r(b, c)+r(a, b \circ c)$.

PROOF. There are four cases:

(i) $r(a, b) = 0 = r(a \circ b, c)$. By definition $a \circ b \succsim a, b$ and $(a \circ b) \circ c \succsim a \circ b, c$. Using Axiom 2,

$$a \circ (b \circ c) \sim (a \circ b) \circ c \succsim a \circ b \succsim a,$$

and so by Lemma 2, $r(a, b \circ c) = 0$. Suppose $r(b, c) = 1$, then $a \circ b \circ c \succsim c \succ b \circ c$ and so, by Lemma 1, $b \succ a \circ b$, which is contrary to hypothesis. So $r(b, c) = 0$.

(ii) $r(a, b) = 1$, $r(a \circ b, c) = 0$. Thus, $a, b \succ a \circ b$ and $a \circ b \circ c \succsim a \circ b, c$. If $r(b, c) = 0$, then $b \circ c \succsim b, c$. By Lemma 1, $b \succ a \circ b$ implies $b \circ c \succ a \circ b \circ c$ or $a \circ b \circ c \succ c \succ b \circ c$. The latter is impossible, so by Lemma 2, $r(a, b \circ c) = 1$. If $r(b, c) = 1$, then $b, c \succ b \circ c$. So $a \circ b \circ c \succsim c \succ b \circ c$, whence $r(a, b \circ c) = 0$ by Lemma 2.

(iii) $r(a, b) = 0$, $r(a \circ b, c) = 1$. Similar to (ii)

(iv) $r(a, b) = 1 = r(a \circ b, c)$. Thus, $a, b \succ a \circ b$ and $a \circ b, c \succ a \circ b \circ c$. From $a \succ a \circ b \succ a \circ b \circ c$, Lemma 2 implies $r(a, b \circ c) = 1$, and Lemma 1 implies $b \succ b \circ c$, so by Lemma 2, $r(b, c) = 1$. QED

LEMMA 5. *If k and n are positive integers such that $n-k-1$ is a positive integer and $a \succsim b$, then*

$$r[(n-k-1)a, a]+r[(n-k)a, kb] \geqq r(b, kb)+r[(n-k-1)a, (k+1)b].$$

PROOF. Using the commutativity of $\circ$, and so of r, and Lemmas 3 and 4,

$$\begin{aligned} r[(n-k-1)a, a]&+r[(n-k)a, kb] \\ &= r(a, kb)+r[(n-k-1)a, a \circ kb] \\ &= r(kb, a)+r[(n-k-1)a, kb \circ a] \\ &= r[(n-k-1)a, kb]+r[(n-k-1)a \circ kb, a] \\ &\geqq r[(n-k-1)a, kb]+r[(n-k-1)a \circ kb, b] \\ &= r(kb, b)+r[(n-k-1)a, (k+1)b]. \end{aligned}$$

QED

We next introduce a count of the number of complete cycles that is made when the sum na is formed.

DEFINITION 4. $W(1, a) = 0$

$$W(n, a) = \sum_{i=1}^{n-1} r(a, ia), \qquad n \geqq 2.$$

LEMMA 6. $W(m+n, a) = W(m, a)+W(n, a)+r(ma, na)$.

PROOF. For any $m > 0$ and $n = 1$ the assertion is true by definition. By induction and Lemma 4,

$$\begin{aligned} W(m+n, a) &= W(m+n-1, a)+r[(m+n-1)a, a] \\ &= W(m, a)+W(n-1, a)+r[ma, (n-1)a]+r[(m+n-1)a, a] \\ &= W(m, a)+W(n-1, a)+r[(n-1)a, a]+r[ma, na] \\ &= W(m, a)+W(n, a)+r(ma, na). \qquad \text{QED} \end{aligned}$$

Lemma 7. $W(mn, a) = mW(n, a)+W(m, na)$

Proof. For $m = 1$, it is obviously true. By induction on m, and Lemma 6,

$$\begin{aligned} W(mn, a) &= W[(m-1)n, a]+W(n, a)+r[na, (m-1)na] \\ &= (m-1)W(n, a)+W(m-1, na)+W(n, a)+r[na, (m-1)na] \\ &= mW(n, a)+W(m, na). \qquad \text{QED} \end{aligned}$$

Lemma 8. *If $a \succsim b$, then $W(n, a) \geqq W(n, b)$.*

Proof. For $n = 1$, $W(1, a) = 0 = W(1, b)$. For $n = 2$, by Lemma 3,

$$W(2, a) = r(a, a) \geqq r(b, a) = r(a, b) \geqq r(b, b) = W(2, b).$$

For $n \geqq 3$, by Lemmas 3 and 5 (omiting meaningless terms for $n = 3, 4$),

$$\begin{aligned} W(n, a) &= \sum_{i=1}^{n-3} r(a, ia)+r[a, (n-2)a]+r[a, (n-1)a] \\ &\geqq \sum_{i=1}^{n-3} r(a, ia)+r[a, (n-2)a]+r[b, (n-1)a] \\ &\geqq \sum_{i=1}^{n-3} r(a, ia)+r(b, b)+r[(n-2)a, 2b] \\ &= r(b, b)+\sum_{i=1}^{n-4} r(a, ia)+r[a, (n-3)a]+r[(n-2)a, 2b] \\ &\geqq r(b, b)+\sum_{i=1}^{n-4} r(a, ia)+r(b, 2b)+r[(n-3)a, 3b] \\ &\cdots \\ &\geqq \sum_{i=1}^{n-1} r(b, ib) \\ &= W(n, b). \end{aligned}$$

Lemma 9. *If $a \succ b$ and l is a positive integer, then there exists a positive integer k such that $W(k, a) \geqq W(k, b)+l$.*

Proof. By Axiom 4, there exists a positive integer n such that $r[(n-1)a, a] = 1$ and $r[(n-1)b, b] = 0$. By Lemma 8

$$W(n, a) = W(n-1, a)+r[(n-1)a, a] \geqq W(n-1, b)+1 = W(n, b)+1.$$

Observe that if $r(c, jc) = 1$ for all j, then $(j-1)c \succ jc \succ (j+1)c$. Canceling $(j-1)c$ by Lemma 1, $c \succ 2c \succ 3c \succ \cdots$. By Axiom 4, this terminates in a finite number of steps and so we may choose i sufficiently large that $W(i, c) \leqq i-l$. Choosing this i for $c = nb$ and letting $k = in$,

$$\begin{aligned} W(k, a) &= W(in, a) \\ &\geqq iW(n, a) \qquad \text{(Lemma 7)} \\ &\geqq i[W(n, b)+1] \\ &\geqq iW(n, b)+W(i, nb)+l \\ &= W(in, b)+l \qquad \text{(Lemma 7)} \\ &= W(k, b)+l. \end{aligned}$$

4. Imbedding in a non-periodic structure

If a is an angle and m a non-negative integer, then it is natural to interpret (m, a) as the angle consisting of m cycles plus a. For the set of such objects, we have the following natural notions of ordering and concatenation:

DEFINITION 5. *If $\langle A, \succsim, \circ \rangle$ is a periodic extensive structure and N is the set of non-negative integers, define*

$A^* = N \times A$,

$\succsim^*$ *on* A^* *by* $(m, a) \succsim^* (n, b)$ *iff* $m > n$ *or* $m = n$ *and* $a \succsim b$.

$*$ *on* A^* *by* $(m, a) * (n, b) = (m+n+r(a, b), a \circ b)$.

THEOREM 3. *If $\langle A, \succsim, o \rangle$ is a periodic extensive structure, then $\langle A^*, \succsim^*, * \rangle$ is a non-negative extensive structure in the following sense* [7]: *for all $\alpha, \beta, \gamma, \delta \in A^*$,*

1. $\langle A^*, \succsim^* \rangle$ *is weak order;*
2. $\langle A^*, \sim^*, * \rangle$ *is a weak abelian semigroup;*
3. $\alpha \succsim^* \beta$ *iff* $\alpha * \gamma \succsim^* \beta * \gamma$;
4. *if* $\alpha \succ^* \beta$, *then there exists a positive integer n such that* $n\alpha * \gamma \succsim^* n\beta * \delta$;
5. $\alpha * \beta \succsim^* \alpha$.

PROOF. 1. This follows from Axiom 1 of Definition 1.

2. This follows from Axiom 2 of Definition 1 and Lemma 4.

3. Suppose that $(m, a) \succsim^* (n, b)$, and consider the relation between $(m, a) * (p, c) = (m+p+r(a, c), a \circ c)$ and $(n, b) * (p, c) = (n+p+r(b, c), b \circ c)$. If $\succsim^*$ holds, then either $m > n$ or $m = n$ and $a \succ b$. If $m > n$, then the conclusion is $\succ^*$ since either $r(a, c) \geqq r(b, c)$ or $a \circ c \succsim c \succ b \circ c$. If $m = n$ and $a \succsim b$, then by Lemma 3, $r(a, c) \geqq r(b, c)$ and so $\succsim^*$ fol-

lows. To complete the proof we need only show that if $a \succ b$ and $r(a, c) = r(b, c)$, then $a \circ c \succ b \circ c$. If $r(a, c) = r(b, c) = 0$, then $a \circ c \succsim a, c$ and $b \circ c \succsim b, c$. By Lemma 1, $a \succ b$ implies $a \circ c \succ b \circ c \succsim c$. If $r(a, c) = r(b, c) = 1$, then $a, c \succ a \circ c$ and $b, c \succ b \circ c$. By Lemma 1, $a \succ b$ implies $c \succ a \circ c \succ b \circ c$.

4. Let $(m, a) \succ^* (n, b)$, (p, c), and (q, d) be given. Let

$$l = \max[1, q+r(a, c)-p-r(b, c)].$$

If $m > n$, choose k so large that $km > kn+l$, then by Lemma 8,

$$\begin{aligned} k(m, a) * (p, c) &= (km+W(k, a)+p+r(a, c), a \circ c) \\ &> (kn+W(k, b)+q+r(b, d), b \circ d) \\ &= k(n, b) * (q, d). \end{aligned}$$

If $m = n$ and $a \succ b$, then choose k as in Lemma 9 and proceed as above.

5. If $n > 0$, then

$$(m, a) * (n, b) = (m+n+r(a, b), a \circ b) \succ^* (m, a)$$

since $m+n+r(a, b) > m$. If $n = 0$,

$$(m, a) * (0, b) = (m+r(a, b), a \circ b) \succsim^* (m, a)$$

because either $r(a, b) = 1$ or $r(a, b) = 0$ which implies $a \circ b \succsim 1$. QED

5. Proof of Theorem 1.

If $\langle A^*, \succsim^*, *\rangle$ is a non-negative extensive structure, Theorems 2 and 3 of [7] combine to say that there exists an additive homomorphism which is unique up to a similarity transformation. For a periodic structure $\langle A, \succsim, \circ\rangle$ with identity e and a constant $K > 0$, let φ^* be the unique homomorphism of the structure in Theorem 2 for which $\phi^*(1, e) = K$. Set $\phi(a) = \phi^*(0, a)$. Note that $0 \leqq \phi(a) < \phi^*(1, e) = K$. It is obvious that $a \succsim b$ iff $\phi(a) \geqq \phi(b)$. Consider

$$\begin{aligned} \phi(a)+\phi(b) &= \phi^*(0, a)+\phi^*(0, b) \\ &= \phi^*[(0, a) * (0, b)] \\ &= \phi^*[r(a, b), a \circ b]. \end{aligned}$$

But

$$\phi^*(1, a) = \phi^*[(0, a) * (1, e)] = \phi^*(0, a)+\phi^*(1, e) = \phi(a)+K.$$

Thus,

$$\phi(a)+\phi(b) = \phi(a \circ b) \pmod{K}.$$

Since $e \circ e \sim e$ implies $r(e, e) = 0$,

$$2\phi(e) = \phi^*[(0, e) * (0, e)] = \phi^*[r(e, e), e \circ e] = \phi(e)$$

so $\phi(e) = 0$.

The necessity of the axioms is routine to verify. QED

6. Proof of Theorem 2

Suppose $\langle A, \succsim, 0\rangle$ satisfies Axioms 1–3 of Definition 1 and C is defined on $A/\sim$ as in the statement of the theorem. We show the four axioms of Definition 2.

1, 2. Trivial

3. Suppose $C(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ and $C(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d})$. Suppose $\boldsymbol{a} \succ \boldsymbol{b} \succ \boldsymbol{c}$. Since both $\boldsymbol{c} \succ \boldsymbol{d} \succ \boldsymbol{a}$ and $\boldsymbol{d} \succ \boldsymbol{c} \succ \boldsymbol{a}$ lead to the contradiction $\boldsymbol{a} \succ \boldsymbol{a}$, we know $\boldsymbol{a} \succ \boldsymbol{c} \succ \boldsymbol{d}$, and so $\boldsymbol{a} \succ \boldsymbol{b} \succ \boldsymbol{d}$, whence $C(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d})$. The argument is similar in the other six cases.

4. Suppose $C(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. If $\boldsymbol{a} \succ \boldsymbol{b} \succ \boldsymbol{c}$, then by Axiom 3 of Definition 1 either (i) $\boldsymbol{a} \circ \boldsymbol{d} \succ \boldsymbol{b} \circ \boldsymbol{d}$ or (ii) $\boldsymbol{b} \circ \boldsymbol{d} \succ \boldsymbol{d} \succ \boldsymbol{a} \circ \boldsymbol{d}$ and (iii) $\boldsymbol{b} \circ \boldsymbol{d} \succ \boldsymbol{c} \circ \boldsymbol{d}$ or (iv) $\boldsymbol{c} \circ \boldsymbol{d} \succ \boldsymbol{d} \succ \boldsymbol{b} \circ \boldsymbol{d}$. Clearly (i) and (iii) imply $C(\boldsymbol{a} \circ \boldsymbol{d}, \boldsymbol{b} \circ \boldsymbol{d}, \boldsymbol{c} \circ \boldsymbol{d})$ and (ii) and (iv) are impossible since they imply $\boldsymbol{d} \succ \boldsymbol{d}$. If (i) and (iv), then either $\boldsymbol{c} \circ \boldsymbol{d} \succ \boldsymbol{a} \circ \boldsymbol{d} \succ \boldsymbol{d}$, which implies $\boldsymbol{c} \succ \boldsymbol{a}$ (Lemma 1), contrary to assumption, or $\boldsymbol{c} \circ \boldsymbol{d} \succ \boldsymbol{d} \succ \boldsymbol{a} \circ \boldsymbol{d} \succ \boldsymbol{b} \circ \boldsymbol{d}$, whence $C(\boldsymbol{a} \circ \boldsymbol{d}, \boldsymbol{b} \circ \boldsymbol{d}, \boldsymbol{c} \circ \boldsymbol{d})$. The argument is similar for (ii) and (iii) and for the two other cases $\boldsymbol{b} \succ \boldsymbol{c} \succ \boldsymbol{a}$ and $\boldsymbol{c} \succ \boldsymbol{a} \succ \boldsymbol{b}$. The fact that $\langle A, \circ\rangle$ is abelian establishes the other conclusion.

Assume, in addition, Axiom 4 and an identity. For $a \in A$, $a \succsim e$ since if $e \succ a$, Axiom 4 implies there is a positive integer n such that $e \succ ne \sim e$, which is impossible. If $C(\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{b})$, then the only possibility is $\boldsymbol{a} \succ \boldsymbol{b} \succ \boldsymbol{e}$, whence by Axiom 4, there is a positive integer n such that $\boldsymbol{a} \succ \boldsymbol{na} \succ \boldsymbol{e}$ and $\boldsymbol{nb} \succ \boldsymbol{b} \succ \boldsymbol{e}$, whence $C(\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{na})$ and $C(\boldsymbol{e}, \boldsymbol{nb}, \boldsymbol{b})$.

Conversely, suppose $\langle A, \circ\rangle$ is an abelian semigroup with identity e and cyclic order C. Since $\sim$ is $=$, we ignore it. We prove Axioms 1 and 3 of Definition 1 (2 is one of the hypotheses).

1. If $a \succ b$ and $b \succ c$, then $C(e, a, b)$ and $C(e, b, c)$. By Axiom 3 of Definition 2, $C(e, a, c)$, so $a \succ c$. By Axiom 1, either $a \succ b$ or $b \succ a$ for $a \neq b$.

3. If $a \succ b$, $C(e, a, b)$ and so by Axiom 4, $C(e \circ c, a \circ c, b \circ c) = C(c, a \circ c, b \circ c)$. This implies Axiom 3 of Definition 1 if we can show $C(a, b, c)$ implies $a \succ b \succ c$, $b \succ c \succ a$, or $c \succ a \succ b$. Suppose not, then one of the other three possibilities must hold. If $a \succ c \succ b$, then $C(e, a, c)$,

$C(e, a, b)$, and $C(e, c, b)$. By Axiom 2, $C(c, e, a)$ and $C(c, b, e)$, whence by Axiom 3, $C(c, b, a)$. So by Axiom 2, $C(a, c, b)$, which is impossible by Axiom 1. The remaining two cases are similar.

If C is Archimedean, Axiom 4 of Definition 1 follows immediately.

QED

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Professor R. Duncan Luce
Program in the Social Sciences
The Institute for Advanced Study
Princeton, New Jersey 08540
U.S.A.