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# PARTITIONS OF UNITY IN HOMOTOPY THEORY

by

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We prove, roughly, that a map $f$ is a homotopy equivalence if $f$ is locally a homotopy equivalence. We also prove that $p : E \to B$ is a fibration if the restrictions of $p$ to the sets $E_\alpha$ of a suitable covering $(E_\alpha)$ of $E$ are fibrations.

The paper was inspired by talks of Dold (see [5]) and might well be considered a second part to Dold [4]. The essential difference to the work of Dold is that we have to consider numerable coverings of a space $X$ which are closed under finite intersections. We use the fundamental observation of G. Segal ([11], Prop. 4.1) that the „classifying space" of such a covering is homotopy equivalent to X. It seems that this theorem of Segal and the section extension theorem of Dold ([4], 2.7) are the two foundation stones of the theory.

## 1. The main results

A covering $(E_\alpha | \alpha \in A)$ of a space $E$ is called numerable if there exists a locally finite partition of unity $(t_\alpha | \alpha \in A)$ such that the closure of $t_\alpha^{-1}]0, 1]$ is contained in $E_\alpha$. If $\sigma \subset A$ we put

$$A_\sigma = \bigcap_{\alpha \in \sigma} A_\alpha .$$

(From now on we use only non-empty $\sigma$ in this context!) If $B$ is a fixed topological space we have the category Top/$B$ of spaces over $B$ and we have a notion of homotopy and homotopy equivalence over $B$ (see Dold [4], 1).

THEOREM 1. *Let $p : X \to B$ and $q : Y \to B$ be spaces over $B$ and $f : X \to Y$ a map over $B$ (i.e. $qf = p$). Let $U = (X_\alpha | \alpha \in A)$ resp. $V = (Y_\alpha | \alpha \in A)$ be a numerable covering of $X$ resp. $Y$. Assume $f(X_\alpha) \subset Y_\alpha$ and that for every finite $\sigma \subset A$ the map $f_\sigma : X_\sigma \to Y_\sigma$ induced by $f$ is a homotopy equivalence over $B$. Then $f$ is a homotopy equivalence over $B$.*

We call $p : E \to B$ a fibration if it has the covering homotopy property for all spaces (Hurewicz fibration). We call $p : E \to B$ an $h$-fibration if $p$ is homotopy equivalent over $B$ to a fibration. (Then $p$ has the weak covering homotopy property (WCHP) in the sense of Dold [4], 5. See also

[3] for details.) We call $p : E \to B$ shrinkable if $p$ is homotopy equivalent over $B$ to $\mathrm{id} : B \to B$.

THEOREM 2. *Let $p : E \to X$ be a continuous map. Let $U = (E_\alpha | \alpha \in A)$ be a family of subsets of E and let $V = (X_\alpha | \alpha \in A)$ be a numerable covering of X. Assume $p(E_\alpha) \subset X_\alpha$ and that for finite $\sigma \subset A$ the map $p_\sigma : E_\sigma \to X_\sigma$ induced by p is shrinkable. Then p has a section.*

The following theorem answers questions of Dold and D. Puppe (see [5]).

THEOREM 3. *Let $p : E \to B$ be a continuous map. Let $U = (E_\alpha | \alpha \in A)$ be a numerable covering such that for every finite $\sigma \subset A$ the restriction $p_\sigma : E_\sigma \to B$ of p to $E_\sigma$ is a fibration (an h- fibration, shrinkable). Then p is a fibration (an h-fibration, shrinkable).*

The above theorems and their proofs have many corollaries and applications. We mention some of them.

THEOREM 4. *Let $U = (X_\alpha | \in A)$ be a numerable covering of a space X. If all the $X_\sigma$ have the homotopy type of a CW-complex then X has the homotopy type of a CW-complex.*

The hypothesis of Theorem 4 is, for instance, satisfied if all the $X_\sigma$ are either empty or contractible. This in turn is true for spaces which are equi-locally convex (Milnor [9]). Another application of Theorem 4 is the following: If $p : E \to B$ is an $h$-fibration, if $B$ has the homotopy type of a $CW$-complex and if every fibre $p^{-1}(b)$, $b \in B$, has the homotopy type of a $CW$-complex, then $E$ has the homotopy type of a $CW$-complex.

THEOREM 5. *Let $U = (X_\alpha | \alpha \in A)$ be an open covering of X and $V = (Y_\alpha | \alpha \in A)$ an open covering of Y. Let $f : X \to Y$ be a continuous map with $f(X_\alpha) \subset Y_\alpha$.*
(a) *If the $f_\sigma : X_\sigma \to Y_\sigma$ are homotopy equivalences then f induces for every paracompact space Z a bijection*

$$f_* : [Z, X] \to [Z, Y]$$

*of homotopy sets.*

(b) *If the $f_\sigma$ are weak homotopy equivalences then f is a weak homotopy equivalence.*

THEOREM 5(b) is a variant of a result of McCord [8, Theorem 6]. Compare also the special case discussed by Eells and Kuiper [6].

## 2. Homotopy equivalences

In this section we prove Theorems 1, 4 and 5. We begin with the proof of Theorem 1. For simplicity we omit the phrase 'over $B$'. In the following lemmas, for instance, we use cofibrations 'over $B$' and homotopies 'over $B$'.

The covering $U$ of $X$ leads to the classifying space $BX_U$ introduced by G. Segal ([11], p. 108). We recall the basic properties of this space. The map $f$ induces $F : BX_U \to BY_V$, because the construction of $BX_U$ is functorial. We have a commutative diagram

$$\begin{array}{ccc} BX_U & \xrightarrow{F} & BY_V \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle pr} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the vertical maps are homotopy equivalences (Prop. 4.1 of Segal [11]). Note that compactly generated spaces do not enter that proposition. Note also that $BX_U$ is a space 'over $B$' and that pr is a homotopy equivalence 'over $B$'. It is useful to observe that pr is in fact shrinkable – as the proof of Segal shows – and hence in particular an $h$-fibration. The space $BX_U$, being the geometric realisation of a semi-simplicial space, has a functorial filtration by skeletons $BX_U^{(n)}$, $n = 0, 1, 2, \cdots$. We need the following lemma in order to prove that $F$ induces homotopy equivalences

$$F^{(n)} : BX_U^{(n)} \to BY_V^{(n)}.$$

LEMMA 1. *Given a commutative diagram*

$$\begin{array}{ccccc} A_1 & \xleftarrow{f_1} & A_0 & \xrightarrow{f_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_2}\downarrow \\ B_1 & \xleftarrow[g_1]{} & B_0 & \xrightarrow[g_2]{} & B_2 \end{array}$$

*where $f_1$, $g_1$ are cofibrations and $h_0$, $h_1$, $h_2$ are homotopy equivalences. Then $h_0$, $h_1$, $h_2$ induce a homotopy equivalence $h : A \to B$ where $A$ is the push-out of $(f_1, f_2)$ and $B$ the push-out of $(g_1, g_2)$.*

PROOF. The lemma is of course well known, see R. Brown [1], 7.5.7. We sketch a proof because we need the basic ingredient also for other purposes. Using the homotopy theorem for cofibrations (compare [3], 7.42) we can assume without loss of generality that $f_2$ and $g_2$ are cofibrations, too. But then it is clear that Lemma 1 follows from Lemma 2 below. (Compare the detailed proof of a dual lemma in R. Brown and P. R. Heath [2]).

LEMMA 2. *Given a commutative diagram*

$$\begin{array}{ccc} A_0 & \xrightarrow{f} & A_1 \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ B_0 & \xrightarrow[g]{} & B_1 \end{array}$$

*where $f$ and $g$ are cofibrations and $h_0$ and $h_1$ homotopy equivalences. Given a homotopy equivalence $H_0 : B_0 \to A_0$ and a homotopy $\varphi : A_0 \times I \to A_0$ with $\varphi(a, 0) = H_0 h_0(a)$, $\varphi(a, 1) = a$ for $a \in A_0$. Then we can find a homotopy equivalence $H_1 : B_1 \to A_1$ with $fH_0 = H_1 g$ and a homotopy $\psi : A_1 \times I \to A_1$ with $\psi(a, 0) = H_1 h_1(a)$, $\psi(a, 1) = a$ for $a \in A_1$ and*

$$\psi(fa, t) = \begin{cases} f\varphi(a, 2t) & a \varepsilon A_0;\ t \leqq \frac{1}{2} \\ f(a) & a \varepsilon A_0;\ t \geqq \frac{1}{2}. \end{cases}$$

Proof. [3], 2.5.

We can now prove by induction over $n$

LEMMA 3. *The map $F^{(n)} : BX_U^{(n)} \to BY_V^{(n)}$ is a homotopy equivalence.*

PROOF. The space $BX_U^{(0)}$ is the topological sum of the $X_\sigma$, $\sigma \subset A$ finite. Hence $F^{(0)}$ is obviously a homotopy equivalence. We can construct $BX_U^{(n)}$ from $BX_U^{(n-1)}$ via the following push-out diagram

$$\begin{array}{ccc} \coprod_{\tau \in A_n} (X_{q(\tau)} \times \partial\Delta^n) & \xrightarrow{k_n} & BX_U^{(n-1)} \\ \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle J_n} \\ \coprod_{\tau \in A_n} (X_{q(\tau)} \times \Delta^n) & \xrightarrow[K_n]{} & BX_U^{(n)} \end{array}$$

Explanation: $\Delta^n$ is the standard $n$-simplex with boundary $\partial\Delta^n$ and $j_n$ is induced by the inclusion $\partial\Delta^n \subset \Delta^n$. Note that $j_n$ is a cofibration (over $B$!). The topological sum is over $\tau \in A_n$, where

$$A_n = \{(\sigma_0, \cdots, \sigma_n) | \sigma_0 \subsetneqq \cdots \subsetneqq \sigma_n, \sigma_n \subset A \text{ finite}\},$$

and $q(\sigma_0, \cdots, \sigma_n) = \sigma_n$. The map $k_n$ is the attaching map for the $n$-simplices. Lemma 1 gives the inductive step.

As a corollary to the preceding proof we have

LEMMA 4. *The map $J_n : BX_U^{(n-1)} \to BX_U^{(n)}$ is a cofibration.*

We also need

LEMMA 5. *The space $BX_U$ is the topological direct limit of the $BX_U^{(n)}$.*

PROOF. Geometric realisation commutes with direct limits.

In view of Lemma 3 to 5 the following lemma will finish the proof of Theorem 1. Consider a commutative diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{i_1} & X_1 & \xrightarrow{i_2} & X_2 & \longrightarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\ Y_0 & \xrightarrow[I_1]{} & Y_1 & \xrightarrow[I_2]{} & Y_2 & \longrightarrow & \cdots \end{array}$$

where $i_1, i_2, \cdots$ and $I_1, I_2, \cdots$ are cofibrations and $f_0, f_1, \cdots$ are homotopy equivalences. Let $X$ be the topological direct limit of the $i_k$, $Y$ the limit of the $I_k$ and $f : X \to Y$ the map induced by the $f_k$.

LEMMA 6. *The map $f : X \to Y$ is a homotopy equivalence.*

PROOF. (Compare [3], § 10.) Using Lemma 2 we construct inductively homotopy equivalences $F_n : Y_n \to X_n$ with $i_n F_{n-1} = F_n I_n$ and homotopies $\varphi_n : X_n \to X_n$ from $F_n f_n$ to $\mathsf{id}(X_n)$ such that $\varphi_n$ is constant for $t \geqq 1 - 2^{-(n+1)}$ and such that $(i_n \times \mathsf{id})\varphi_n = \varphi_{n-1}$. The $F_n$ and $\varphi_n$ induce $F : Y \to X$ and $\varphi : X \times I \to X$ such that $\varphi(x, 0) = F_n(x)$ and $\varphi(x, 1) = x$ for $x \in X$. Hence $f$ has a homotopy left inverse.

REMARK 1. Lemma 6 shows in particular that $X = \mathsf{lim}\ X_k$ is the homotopy direct limit of the $X_k$ in the sense of Milnor [(10], p. 149), i.e. the projection of the telescope of the $i_n$ onto $X$ is a homotopy equivalence.

REMARK 2. The numerability of the covering $U$ is only used to establish the homotopy equivalence $BX_U \simeq X$. The map $F : BX_U \to BY_V$ is always a homotopy equivalence, if the $f_\sigma$ are homotopy equivalences. There are other cases in which $\mathsf{pr} : BX_U \to X$ is a homotopy equivalence, e.g. if $U$ is closed, finite-dimensional and the inclusions $X_\sigma \subset X_\tau$ are cofibrations.

*Proof of Theorem* 4. We show that $BX_U$ has the homotopy type of a CW-complex. The procedure is the same as in the proof of Theorem 1. If in the diagram

$$A_1 \xleftarrow[f]{} A_0 \xrightarrow[g]{} A_2$$

all spaces have the homotopy type of a *CW*-complex and if $f$ is a cofibration, then the push-out has the homotopy type of a *CW*-complex. This shows inductively that the $BX_U^{(n)}$ have the homotopy type of a *CW*-complex. One finishes the proof using Lemma 5, Lemma 6 and Remark 1.

*Proof of Theorem* 5. Let $U = (X_\alpha | \alpha \in A)$ be any covering of X. Consider $\mathsf{pr} : BX_U \to X$. We claim that for every $\alpha \in A$ the map $\mathsf{pr}_\alpha : \mathsf{pr}^{-1}X_\alpha \to X_\alpha$

is shrinkable. If $U(\alpha)$ is the covering $(X_\alpha \bigcap X_\beta | \beta \in A)$ of $X_\alpha$ we show that its classifying space, $B_\alpha$ say, is canonically homeomorphic to $\mathrm{pr}^{-1}X_\alpha$. The result then follows since $U(\alpha)$ is clearly a numerable covering of $X_\alpha$ because it contains $X_\alpha$. The homeomorphism $B_\alpha \cong \mathrm{pr}^{-1}X_\alpha$ follows along the lines of Gabriel-Zisman [7], Ch. III, 3.2.

Let now $U$ be an open covering of $X$. We show that for a paracompact $Z$ the map $\mathrm{pr}_* : [Z, BX_U] \to [Z, X]$ is bijective. We consider a pull-back diagram

$$\begin{array}{ccc} E & \xrightarrow{q} & BX_U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle pr} \\ Z & \xrightarrow[f]{} & X \end{array}$$

for given $f$. By Corollary 3.2 of Dold [4] we see that $q$ is shrinkable. Let $s : Z \to E$ be a section of $q$. Then gs satisfies $\mathrm{pr} \circ gs = f$ and hence $\mathrm{pr}_*$ is surjective. Injectivity follows similarly; one has to use Prop. 3.1 of Dold [4]. Theorem 5(a) follows.

To prove Theorem 5(b) we show that $F : BX_U \to BY_V$ is a weak homotopy equivalence if the $f_\sigma$ are weak homotopy equivalences. We prove analogues of Lemmas 3 to 6. But this is standard homotopy theory.

## 3. Sections

We prove Theorem 2. We use the notations of the previous section.

We construct a map $s$ such that the following diagram is commutative

$$\begin{array}{ccc} & & E \\ & {\scriptstyle s}\nearrow & \downarrow{\scriptstyle p} \\ BX_U & \xrightarrow[pr]{} & B. \end{array}$$

More precisely we construct inductively maps $s^{(n)} : BX_U^{(n)} \to E$ with $ps^{(n)} = \mathrm{pr}|BX_U^{(n)}$, $J_n s^{(n)} = s^{(n-1)}$, and an additional property to be mentioned soon.

The map

$$s^{(0)} : \coprod_{\sigma \in A_0} X_\sigma \to E$$

is given as follows: $s^{(0)}|X_\sigma \to E$ is a section $X_\sigma \to E_\sigma$ composed with the inclusion $E_\sigma \subset E$. The section exists because $E_\sigma \to B_\sigma$ is shrinkable. The equality $ps^{(0)} = \mathrm{pr}|BX_U^{(0)}$ clearly holds. Suppose $s^{(n-1)}$ is given. We want to extend

$$s^{(n-1)}k_n : \coprod (X_{q(\tau)} \times \partial \Delta^n) \to E$$

over $\coprod (X_{q(\tau)} \times \Delta^n)$. If $\tau = (\sigma_0, \cdots, \sigma_n)$, we impose the additional induction hypothesis that the image of $X_{q(\tau)} \times \partial \Delta^n$ under $s^{(n-1)}k_n$ is contained in $E_{\sigma_0}$. The construction of $s^{(0)}$ agrees with this requirement. With our new hypothesis we have the commutative diagram

$$\begin{array}{ccc} X_{q(\tau)} \times \partial \Delta^n & \xrightarrow{s^{(n-1)}k_n} & E_{\sigma_0} \\ \downarrow{\scriptstyle pr} & & \downarrow \\ X_{q(\tau)} & \xrightarrow[\subset]{} & X_{\sigma_0} \end{array}$$

From Dold [4], Prop. 3.1(b), we see that $s^{(n-1)}k_n$ can be extended over $\coprod X_{q(\tau)} \times \Delta_n$ and hence we can construct $s^{(n)}$ via the push-out diagram entering the proof of Lemma 3. The properties $ps^{(n)} = \mathrm{pr}|BX_U^{(n)}$ and $J_n s^{(n)} = s^{(n-1)}$ are obvious from the construction. We show that $s^{(n)}$ satisfies the additional induction hypothesis. Given $\tau = (\sigma_0, \cdots, \sigma_{n+1})$ we describe

$$k_{n+1} : X_{q(\tau)} \times \partial \Delta^{n+1} \to BX_U^{(n)}.$$

Let $d_i : \Delta^n \to \Delta_i^{n+1}$ be the standard map onto the $i$-th face of $\Delta^{n+1}$ and let $e_i$ be the inverse homeomorphism. Let

$$\partial_i : X_{q(\tau)} \to X_{q(\varepsilon_i \tau)}$$

be the inclusion, where

$$\varepsilon_i \tau = (\sigma_0, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{n+1}).$$

The restriction of $k_{n+1}$ to $X_{q(\tau)} \times \Delta_i^{n+1}$ is $K_n(\partial_i \times e_i)$. By construction of $s^{(n)}$ the image of $s^{(n)}K_n(\partial_i \times e_i)$ is contained in $E_{\sigma_0}$ (for $i > 0$) or $E_{\sigma_1}$ (for $i = 0$). But $E_{\sigma_1} \subset E_{\sigma_0}$, hence $s^{(n)}$ has the desired property. Because of Lemma 5 the maps $s^{(n)}$ combine to give $s : BX_U \to E$.

If $(X_\alpha)$ is numerable then $\mathrm{pr} : BX_U \to X$ has a section $t$ and $st : B \to E$ will then be a section of $p$. This proves Theorem 2.

## 4. Fibrations

If $p : E \to B$ is a map we denote by $W_p$ the subspace

$$W_p = \{(w, e) | w(1) = pe\} \subset B^I \times E,$$

where $B^I$ is the path space with compact open topology. The map

$$\pi_p : E^I \to W_p,$$

defined by $\pi_p(v) = (\mathrm{pv}, \mathrm{v}(1))$, is shrinkable if $p$ is a fibration. Conversely, if $\pi_p$ has a section then $p$ is a fibration.

In general we have a commutative diagram

$$\begin{array}{ccc} W_p & \xrightarrow{k_p} & E \\ & {\scriptstyle j_p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

$k_p(w, e) = e, j_p(w, e) = pe$. The map $k_p$ is a homotopy equivalence and $j_p$ is a fibration. From our definition of $h$-fibrations and Theorem 6.1 of Dold [4] it follows immediately that $p$ is an $h$-fibration if an only if $k_p$ is a homotopy equivalence over $B$.

We have recalled these characterisations of fibrations and h-fibrations because we want to use them in the following proof of Theorem 3.

*Proof of Theorem* 3. To begin with let us assume that the $p_\sigma$ are $h$-fibrations. The $W_{p_\sigma}$ form a numerable covering of $W_p$ and we have $k_p(W_{p_\sigma}) \subset E_\sigma$. Moreover we know that $W_{p_\sigma} \to E_\sigma$ is a homotopy equivalence over $B$ because $p_\sigma$ is an $h$-fibration. We are now in a position to apply Theorem 1, which tells us that $k_p$ is a homotopy equivalence over $B$. Hence $p$ is an $h$-fibration.

Now assume that the $p_\sigma$ are fibrations. We want to show that $\pi_p$ has a section. We use Theorem 2. We have the numerable covering $(W_{p_\alpha} | \alpha \in A)$ of $W_p$ and we have the family of subsets $(E^I_\alpha | \alpha \in A)$. Moreover $E^I_\sigma \to W_p$ is shrinkable because $p_\sigma$ is a fibration. (Note that this is also true if $E_\sigma$ is empty.) Theorem 2 gives the desired section of $\pi_p$.

Finally assume that the $p_\sigma$ are shrinkable, i.e. homotopy equivalences over $B$. Theorem 1 shows that $p$ is a homotopy equivalences over $B$. The proof of Theorem 3 is now finished.

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