COMPOSITIO MATHEMATICA

R. D. ANDERSON

# Proper homotopies in $l_2$-manifolds

*Compositio Mathematica*, tome 23, nº 2 (1971), p. 155-157
<http://www.numdam.org/item?id=CM_1971__23_2_155_0>

COMPOSITIO MATHEMATICA, Vol. 23, Fasc. 2, 1971, pag. 155-157
Wolters-Noordhoff Publishing
Printed in the Netherlands

# PROPER HOMOTOPIES IN $l_2$-MANIFOLDS

by

R. D. Anderson

The purpose of this note is to establish that any two homotopic proper maps into an $l_2$-manifold are properly homotopic. This result answers a question posed to me by David Elworthy. The question arises in the study of degrees and fixed points of maps on such manifolds since proper maps are frequently hypothesized in such study.

All spaces are assumed separable metric. A map $f$ of $X$ into $Y$ is *proper* if for each compact subset $K$ of $Y$, $f^{-1}(K)$ is compact. It is well known that a map is proper iff it is closed and each point inverse is compact. A homotopy $H : X \times I \to Y$ is *proper* if $H$ is a proper map. We say that $K \subset Y$ is a *Z-set* in $Y$ if for each homotopically trivial non-empty open set $U$ in $Y$, $U \setminus K$ is non-empty and homotopically trivial. An *$l_2$-manifold* is a space covered by open sets homeomorphic to Hilbert space $l_2$.

THEOREM: *Let $X$ be a topologically complete space, let $Y$ be an $l_2$-manifold and let $F = \{f_t\}$ be a homotopy of $X \times I$ into $Y$ such that $f_0$ and $f_1$ are proper maps. Then there exists a proper homotopy $G = \{g_t\}$ of $X \times I$ into $Y$ such that $g_0 = f_0$ and $g_1 = f_1$.*

Clearly, the theorem can be proved by showing (*Lemma* 1) the theorem with the added hypothesis that $f_0$ and $f_1$ are onto disjoint $Z$-sets and then by showing (*Lemma* 2) that there exist proper homotopies $H = \{h_t\}$ and $H' = \{h'_t\}$ such that $h_0 = f_0$, $h'_0 = f_1$ and $h_1$ and $h'_1$ are proper maps onto disjoint $Z$-sets in $Y$. I am indebted to Robert Connelly for suggesting this structure of the argument.

The proof of Lemma 1 is almost identical with part of the proof used in [2]. Only one rather minor modification need be made. The essential steps are given below, more details being provided in [2]. In [5], with a modification as noted in Lemma 2.3 and 2.4 of [2], West has shown that (in the notation of our Lemma 1) there exists a homotopy $M = \{m_t\}$ such that (1) $m_0 = f_0$, (2) $m_1 = f_1$, (3) for each $r$, $0 < r < \frac{1}{2}$, $M|(X \times [r, 1-r])$ is a homeomorphism onto a $Z$-set in $Y$ and (4) $\text{Cl}\, M(X \times I)$ is a $Z$-set. It is known, [3], that $Y$ is homeomorphic to $Y \times l_2$ and by Chapman, [4], that there is a homeomorphism $v$ of $Y \times l_2$ onto itself such that the $Z$-set $v(\text{Cl}\, M(Z \times I))$ is contained in $Y \times \{0\}$.

Let $W$ be the image of $X \times I$ under the proper map $\alpha$ defined by the upper semi-continuous decomposition of $X \times I$ consisting of the individual points of $X \times (0, 1)$ and the (compact) inverses of points under the proper maps $m_0$ and $m_1$. Then $W$ is a topologically complete separable metric space. Also $M(X \times I)$ is the continuous 1-1 image of $W$ under the natural map $w$ defined through $X \times I$. For some $p \in Y$, let $u$ be an embedding of $W$ as a closed set in $\{p\} \times l_2$.

Let $W^*$ be the graph of the 1-1 function from $v(M(X \times I))$ onto $u(W)$ defined by $uw^{-1}v^{-1}$. Note that $uw^{-1}v^{-1}|v(M(X \times (\{0\} \cup \{1\})))$ is a homeomorphism. Let $\beta$ be the homeomorphism of $W^*$ onto $u(W)$ included by $\Pi_p : Y \times l_2 \to \{p\} \times l_2$. Finally, as in [2], let $N = \{n_t\}$ be an ambient isotopy of $Y \times l_2$ onto itself preserving each fibre $\{y\} \times l_2$ such that $n_0$ is the identity and $n_1$ carries $v(M(X \times (\{0\} \cup \{1\})))$ homeomorphically onto $\beta^{-1} uw^{-1}v^{-1}(v(M(X \times (\{0\} \cup \{1\}))) \subset W^*$. But then $v^{-1}n_1^{-1}\beta^{-1}u\alpha$ is the desired proper homotopy of $X \times I$ into $Y$ establishing Lemma 1, since $\alpha$ is a proper map and $u$, $\beta^{-1}$, $n_1^{-1}$, and $v^{-1}$ restricted to the appropriate domains are homeomorphisms onto closed sets.

For the proof of Lemma 2, we first observe two elementary properties of proper maps.

(1) The composition of two proper maps is a proper map.

(2) The product of two proper maps is proper, i.e. if $f$ and $g$ are proper maps of A and B into C and D respectively, then the map, $f \times g$, defined on $A \times B$ into $C \times D$ by $(f \times g)(a, b) = (f(a), g(b))$ is a proper map (since any compact subset of $C \times D$ projects onto compact subsets of $C$ and of $D$).

By [3], $Y$ is homeomorphic to $Y \times Q$ where $Q$ is the (compact) Hilbert cube. We henceforth regard $Y$ as $Y \times Q$ with $Q = \prod_{i>0} [0, 1/i]$. Let $\hat{K}$ and $\hat{K}'$ be (obviously proper) isotopies of $Q \times I$ into $Q$ defined by $\hat{K}((q_i), t) = (((3-2t)/3)q_i)$ and $\hat{K}'((q_i), t) = (((3-2t)/3)q_i + 2t/3i)$. By property (2) above, $K = id \times \hat{K}$ and $K' = id \times \hat{K}'$ are proper homotopies of $Y \times (Q \times I)$ into $Y \times Q$. Let $G$ and $G'$ be proper maps of $X \times I$ into $(Y \times Q) \times I$ defined by $G = f_0 \times id$ and $G' = f_1 \times id$. Finally using property (1) above, $H = KG$ and $H' = K'G'$ are the desired proper homotopies (with $Y$ regarded as $Y \times Q$). It is easy to check that $H(X \times \{0\}) = f_0$, $H'(X \times \{0\}) = f_1$, and $H(X \times \{1\}) \cap H'(X \times \{1\}) = \emptyset$. The fact that $H(X \times \{1\})$ and $H'(X \times \{1\})$ are $Z$-sets follows readily from properties of infinite deficiency as in [1] and the fact that $\hat{K}((q_i), 1)$ and $\hat{K}'((q_i), 1)$ have infinite partial deficiency in $Q$ and thus, by [1], can be carried under homeomorphisms of $Q$ onto $Q$ to sets of infinite deficiency in $Q$.

## REFERENCES

R. D. ANDERSON
[1] On Topological Infinite Deficiency, Mich. Math. Journal, 14 (1967) 365–383.

R. D. ANDERSON AND JOHN MCCHAREN
[2] On extending homeomorphisms to Fréchet manifolds, Proceedings of the AMS, 25 (1970) 283–289.

R. D. ANDERSON AND R. M. SCHORI
[3] Factors of Infinite-Dimensional Manifolds, Transactions of the AMS, 142 (1969) 315–330.

T. A. CHAPMAN
[4] Infinite-Deficiency in Fréchet Manifolds, Transactions of the AMS, 148 (1970) 568–573

JAMES E. WEST
[5] Approximating homotopies by isotopies in Fréchet manifolds, Bulletin of the AMS, 75 (1969) 1254–1257.

(Oblatum 23–X–70)

Louisiana State University, Baton Rouge
and
The University of Warwick, Coventry