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LOCAL CENTRAL LIMIT THEOREM FOR FIRST ENTRANCE OF A RANDOM WALK INTO A HALF SPACE

by

A. J. Stam

1. Introduction, notations

Throughout this paper the following assumptions apply. Let $\bar{X}_k = (X_{k1}, \cdots, X_{kd})$, $k = 1, 2, \cdots$, be independent strictly d-dimensional random vectors with common probability distribution F and characteristic function φ. (The bar distinguishes vectors from scalars and strict d-dimensionality means that the support of F is not contained in a hyperplane of dimension lower than d.) The second moments of the $\bar{X}_i$ will be finite and the first moment vector $\bar{\mu}$ nonzero. We put $\bar{S}_n = \bar{X}_1 + \cdots + \bar{X}_n$, $n = 1, 2, \cdots$,

$$U(A) = \sum_{m=1}^{\infty} F^m(A), \tag{1.1}$$

where the exponent denotes convolution. The distribution function of X_{11} if F_1.

We consider the first entrance of the random walk $\{\bar{S}_n\}$ into the half space $\{\bar{x} : a_1 x_1 + \cdots + a_d x_d \geqq t\}$, where $t > 0$. It is essential that the half line $\bar{x} = c\bar{\mu}$, $c > 0$, intersects the boundary of the half space. For convenience of notation we assume that the x_1-axis of our coordinate system has been chosen in the direction of $\bar{a}$. This implies that we have to assume throughout this paper

$$\mu_1 > 0. \tag{1.2}$$

Now let $N(t) = \min\{n : S_{n1} \geqq t\}$, and let R_t be the joint probability distribution of

$$Z_1(t) - t,\ Z_2(t), \cdots, Z_d(t),$$

where $\bar{Z}(t) = \bar{S}_{N(t)}$. It will be shown in section 3 that R_t for $t \to \infty$ satisfies a local central limit theorem, if either F is nonarithmetic – i.e. $\{\bar{u} : \varphi(\bar{u}) = 1\} = \{0\}$ – or X_{1k} is arithmetic with span 1, $k = 1, \cdots, d$. The approximating probability measure is the product of the well known limiting distribution of $Z_1(t) - t$ and a normal distribution for $Z_2(t)$, $\cdots$, $Z_d(t)$. The corresponding 'marginal' result for $Z_2(t), \cdots, Z_d(r)$ also is derived.

We will need the strict ascending ladder process with respect to the x_1-coordinate, i.e. the random walk $\bar{S}_{n_1}, \bar{S}_{n_2}, \cdots$ in R_d, where $n_1, n_2, \cdots$ are the times at which a strict ascending ladder point occurs in the random walk $S_{11}, S_{21}, S_{31}, \cdots$. We put

$$\bar{Y} = \bar{S}_{n_1}. \tag{1.3}$$

By Wald's identity for expectations we have, since $E\{n_1\} < \infty$ by (1.2),

$$\bar{\nu} \overset{\text{df}}{=} E\{\bar{Y}\} = \bar{\mu}E\{n_1\}. \tag{1.4}$$

By H_1 we denote the probability distribution of Y_1.

Let E denote the covariance matrix of the random variables $X_{1j} - \mu_1^{-1}\mu_j X_{11}, j = 2, \cdots, d$ and ε_{ij} the (i,j)-element of E^{-1}. We put

$$\begin{aligned} &Z(x_1, \cdots, x_d) \\ &\quad = \exp\left[-\tfrac{1}{2}\mu_1 x_1^{-1} \sum_{i=2}^{d} \sum_{j=2}^{d} \varepsilon_{ij}(x_i - \mu_1^{-1}\mu_i x_1)(x_j - \mu_1^{-1}\mu_j x_1)\right], \end{aligned} \tag{1.5}$$

$$L(x_1, \cdots, x_d) = \mu_1^{-1}(2\pi)^{-\rho}(\text{Det } E)^{-\frac{1}{2}} Z(x_1, \cdots, x_d), \tag{1.6}$$

where

$$\rho = \tfrac{1}{2}(d-1). \tag{1.7}$$

If x_1 is kept fixed, $\mu_1^{\rho+1}x_1^{-\rho}L(x_1, x_2, \cdots, x_d)$ considered as a function of $x_2, \cdots, x_d$, is a $(d-1)$-dimensional normal probability density. By C_d we denote the class of continuous functions on R_d with compact support. The indicator function of a set A is written I_A.

Proofs are based on the results obtained in Stam [1].

2. Preliminary lemmas

LEMMA 2.1. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$, then for $g \in C_d$*

$$\lim_{x_1 \to \infty} \left\{ x_1^\rho \int g(\bar{z} - \bar{x}) U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int g(\bar{z})\, d\bar{z} \right\} = 0, \tag{2.1}$$

uniformly in $x_2, \cdots, x_d$.

This is theorem 3.1 of Stam [1], II. We also need theorem 3.2 of the same paper:

LEMMA 2.2. *If there is a Cartesian coordinate system such that the components of $\bar{X}_1$ in this system are arithmetic with span 1 and their joint characteristic function ζ satisfies the condition: $\zeta(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of 2π and $|\zeta(\bar{u})| < 1$ elsewhere and if $E|X_{11}|^\rho < \infty$, then*

$$\lim_{x_1 \to \infty} \{x_1^\rho U(\{\bar{x}\}) - \mu_1^\rho L(\bar{x})\} = 0,$$

uniformly in $x_2, \cdots, x_d$, *if* $\bar{x}$ *is restricted to lattice points of* U.

LEMMA 2.3. *If* F *satisfies the conditions of lemma* 2.1 *and* $g(\bar{x}) = I_{[a,b)}(x_1)g_1(\bar{x})$ *with* $g_1 \in C_d$, *then* (2.1) *holds for* g.

PROOF. We may write $g = h + h_1$ with $h \in C_d$ and $|h_1| \leqq h_2 \in C_d$. Then

$$\left| x_1^\rho \int g(\bar{z}-\bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int g(\bar{z})\, d\bar{z} \right| \leqq$$

$$\left| x_1^\rho \int h(\bar{z}-\bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int h(\bar{z})\, d\bar{z} \right| + \tag{2.2}$$

$$\left| x_1^\rho \int h_2(\bar{z}-\bar{x})U(d\bar{z}) \right| + \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})\, d\bar{z}.$$

Since $L(\bar{x})$ is bounced, we may choose h, h_1 and h_2 so that

$$\mu_1^\rho L(\bar{x}) \int h_2(\bar{z}) dz < \varepsilon/4. \tag{2.3}$$

Then

$$\begin{aligned} \left| x_1^\rho \int h_2(\bar{z}-\bar{x})U(d\bar{z}) \right| &\leqq \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})\, dz \\ &+ \left| x_1^\rho \int h_2(\bar{z}-\bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})\, d\bar{z} \right| \end{aligned} \tag{2.4}$$

and the lemma follows from (2.2), (2.3), (2.4) and lemma 2.1.

LEMMA 2.4. *The random variables* $Y_1, \cdots, Y_d$ *of* (1.3) *have finite second moments. If* $\mu_j = 0$, $j \geqq 2$,

$$\operatorname{cov}(Y_j, Y_k) = E\{n_1\} \operatorname{cov}(X_{1j}, X_{1k}), \qquad j, k = 2, \cdots, d. \tag{2.6}$$

See theorems 1.2, 1.4, 1.5 of Nevels [2].

LEMMA 2.5. *The covariance matrix of the random variables* $Y_j - \nu_1^{-1}\nu_j Y_1$, $j = 2, \cdots, d$, *is* $E\{n_1\} \cdot E$, *where* E *is defined as in section* 1.

PROOF. By (1.4) we have $\nu_1^{-1}\nu_j = \mu_1^{-1}\mu_j$. So

$$Y_j - \nu_1^{-1}\nu_j Y_1 = \sum_{k=1}^{n_1} W_{kj},$$

where $W_{kj} = X_{kj} - \mu_1^{-1}\mu_j X_{k1}$ has expectation zero. The lemma follows from lemma 2.5 by considering the random walk with steps $(X_{k1}, W_{k2}, \cdots, W_{kd})$.

LEMMA 2.6. *If* $E|X_{11}|^\lambda < \infty$, *where* $\lambda > 0$, *then* $E|Y_1|^\lambda < \infty$.

PROOF. See Nevels [2], theorem 1.1.

3. Local limit theorems for R_t

THEOREM 3.1. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$*, we have for $g \in C_d$

$$\lim_{t\to\infty} t^\rho \left| \int g(x_1, x_2-a_2, \cdots, x_d-a_d) R_t(d\bar{x}) - \int g(x_1, x_2-a_2, \cdots, x_d-a_d)\beta(x_1)q_t(x_2, \cdots, x_d)\, d\bar{x} \right| = 0,$$

uniformly in $a_2, \cdots, a_d$. Here

$$\beta(x_1) = 0,\ x_1 \leqq 0,\ \beta(x_1) = \nu_1^{-1}\{1-H_1(x_1)\}, \qquad x_1 > 0, \tag{3.1}$$

and q_t is the $(d-1)$-dimensional normal density with covariance matrix $\mu_1^{-1}tE$ and means $\mu_1^{-1}\mu_j t$, $j = 2, \cdots, d$.

PROOF. First we assume that $X_{11} \geqq 0$ with probability 1. Since $g \in C_d$, it is sufficient to show that

$$\lim_{t\to\infty} \left| t^\rho \int g(x_1, x_2-a_2, \cdots, x_d-a_d)R_t(d\bar{x}) - t^\rho q_t(a_2, \ldots, a_d) \int \beta(x_1)g(\bar{x})\, d\bar{x} \right| = 0, \tag{3.2}$$

uniformly in $a_2, \cdots, a_d$. We have

$$\begin{aligned} & t^\rho \int g(x_1, x_2-a_2, \cdots, x_d-a_d)R_t(d\bar{x}) \\ &= t^\rho \int I_{[t,\infty)}(x_1)g(x_1-t, x_2-a_2, \cdots, x_d-a_d)F(d\bar{x}) \\ &\quad + t^\rho \sum_{m=1}^{\infty} \int\int I_{(-\infty,t)}(x_1)I_{[t,\infty)}(x_1+\xi_1)g(x_1+\xi_1-t, x_2+\xi_2-a_2, \cdots, \\ &\qquad x_d+\xi_d-a_d)F^m(d\bar{x})F(d\bar{\xi}). \end{aligned} \tag{3.3}$$

Here the first term tends to zero for $t \to \infty$, uniformly in $a_2, \cdots, a_d$, since $E|X_{11}|^\rho < \infty$. The second term may be written

$$T_2 = \int \Lambda(\bar{\xi}, t, \bar{a})F(d\bar{\xi}), \tag{3.4}$$

where $\bar{a} = (0, a_2, \cdots, a_d)$ and

$$\Lambda(\bar{\xi}, t, \bar{a}) = t^\rho \int I_{[-\xi_1, 0)}(x_1-t)g(x_1+\xi_1-t, x_2+\xi_2-a_2, \cdots, x_d+\xi_d-a_d)U(d\bar{x}). \tag{3.4a}$$

By lemma 2.3, applied to the function $I_{[-\xi_1, 0)}(x_1)g(\bar{x}+\bar{\xi})$ with $\bar{\xi}$ fixed,

$$(3.5)\quad \Lambda(\bar{\xi}, t, \bar{a}) = \eta(\bar{\xi}, t, \bar{a}) + \mu_1^\rho L(t, a_2, \cdots, a_d) \int I_{[-\xi_1, 0)}(z_1) g(\bar{z}+\bar{\xi})\, d\bar{z},$$

where $\lim_{t\to\infty} \eta(\bar{\xi}, t, \bar{a}) = 0$, uniformly in $a_2, \cdots, a_d$ for fixed $\bar{\xi}$. Equivalently

$$(3.6)\qquad \lim_{t\to\infty} \zeta(\bar{\xi}, t) = 0,$$

for fixed $\bar{\xi}$, where $\zeta(\bar{\xi}, t) = \sup_{\bar{a}} \eta(\bar{\xi}, t, \bar{a})$. We now write

$$(3.7)\qquad \begin{aligned} T_2 &= T_3 + T_4, \\ T_3 &= \int I_{[\frac{1}{2}t, \infty)}(\xi_1) \Lambda(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}), \\ T_4 &= \int I_{[0, \frac{1}{2}t)}(\xi_1) \Lambda(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}). \end{aligned}$$

Since $\int g(\bar{z}-\bar{y}) U(d\bar{z})$ is bounded in $\bar{y}$, we have by (3.4a) and the assumption that $E|X_{11}|^\rho < \infty$,

$$(3.8)\qquad T_3 \leqq c_1 t^\rho \{1 - F_1(\tfrac{1}{2}t)\} \to 0.$$

To T_4 we now apply (3.5) and (3.6) with the Lebesgue dominated convergence theorem. It is noted that L is bounded by a constant and that

$$t^\rho I_{[0, \frac{1}{2}t)}(\xi_1) \leqq 2^\rho (t-\xi_1)^\rho, \qquad 0 \leqq \xi < \tfrac{1}{2}t.$$

So (3.4a) and lemma 2.3 show that $I_{[0,\frac{1}{2}t)}(\xi_1)\Lambda(\bar{\xi}, t, \bar{a})$ and therefore also $I_{[0, \frac{1}{2}t)}(\xi_1)\zeta(\bar{\xi}, t)$ is bounded by a constant. So

$$(3.9)\qquad \lim_{t\to\infty} \left[T_4 - \int I_{[0, \frac{1}{2}t)}(\xi_1) \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) \right] = 0,$$

uniformly in $a_2, \cdots, a_d$, where $\gamma(\bar{\xi}, t, \bar{a})$ is the second term on the right in (3.5). Since $\gamma(\bar{\xi}, t, a)$ is bounded by a constant, (3.9) implies

$$(3.10)\qquad \lim_{t\to\infty} \left[T_4 - \int \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) \right] = 0,$$

uniformly in $a_2, \cdots, a_d$. Now

$$\gamma(\bar{\xi}, t, \bar{a}) = \mu_1^\rho L(t, a_2, \cdots, a_d) \int I_{[0, \xi_1)}(y_1) g(\bar{y})\, d\bar{y},$$

$$\int \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) = \mu_1^\rho L(t, a_2, \cdots, a_d) \int \{1 - F_1(y_1)\} g(\bar{y})\, d\bar{y}.$$

So by (1.5) and (1.6)

$$(3.11)\qquad \int \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) = t^\rho q_t(a_2, \cdots, a_d) \int \beta(y_1) g(\bar{y})\, d\bar{y},$$

and (3.2) follows from (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11).

If $P\{X_{11} < 0\} > 0$, we apply the part of the theorem proved above, to the random walk arising by sampling the $\bar{S}_n$-process at the strict ladder times of the process $\{S_{n1}\}$. It is noted that the first entrance of $\{\bar{S}_n\}$ into the half space $\{x_1 \geqq t\}$ necessarily is a ladder point of $\{S_{n1}\}$. The theorem now follows by lemma 2.5 and (1.4). Lemma 2.6 guarantees that the condition on the absolute moment of order ρ of the x_1-component is satisfied.

THEOREM 3.2. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$, we have for $h \in C_{d-1}$*

$$\lim_{t\to\infty} t^\rho \left| \int h(x_2-a_2, \cdots, x_d-a_d)R_t(d\bar{x}) - \int h(x_2-a_2, \cdots, x_d-a_d)q_t(x_2, \cdots, x_d)\, dx_2 \cdots dx_d \right| = 0,$$

uniformly in $a_2, \cdots, a_d$. Here q_t is the same as in theorem 3.1.

PROOF. Since $h \in C_{d-1}$, it is sufficient to show that, uniformly in $a_2, \cdots, a_d$,

$$\lim_{t\to\infty} t^\rho \left| \int h(x_2-a_2, \cdots, x_d-a_d)R_t(d\bar{x}) - q_t(a_2, \cdots, a_d) \times \int h(x_2, \cdots, x_d)\, dx_2 \cdots dx_d \right| = 0. \tag{3.12}$$

First we assume that $X_{11} \geqq 0$. We then start the proof of (3.2) anew at (3.3), where for $g(x_1, \cdots, x_d)$ we now take $h(x_2, \cdots, x_d)$. We obtain (3.4), (3.5), (3.6), since lemma 2.3 applies to the function $I_{[-\xi_1, 0)}(\xi_1)$ $h(x_2+\xi_2, \cdots, x_d+\xi_d)$ with ξ fixed. To obtain (3.8) and (3.9) we have to take into account the factor $I_{[-\xi_1, 0)}(x_1-t)$ in (3.4a). This means that in the integral in (3.4a) the variable x_1 is restricted to the interval $[t-\xi_1, t)$. We then have in T_3

$$\Lambda(\xi, t, \bar{a}) \leqq t^\rho \int I_{[0,t)}(x_1)|h(x_2+\xi_2-a_2, \cdots, x_d+\xi_d-a_d)|\, U(d\bar{x}). \tag{3.13}$$

By lemma 2.3, for $m \geqq 1$,

$$\int I_{[m, m+1)}(x_1)|h(x_2+\xi_2-a_2, \cdots, x_d+\xi_d-a_d)U(d\bar{x}) \leqq c_2 m^{-\rho},$$

so

$$\Lambda(\xi, t, \bar{a}) \leqq t^\rho\{c_0+c_2 \sum_{m=1}^{[t+1]} m^{-\rho}\}. \tag{3.14}$$

Therefore $T_3 \to 0$, uniformly, since $E|X_{11}|^\rho < \infty$. For $\rho = \frac{1}{2}$ and $\rho = 1$ we have to appeal to the existence of first and second moments. To apply the Lebesgue dominated convergence theorem to T_4 we note that the second term on the right in (3.5) is bounded by $c_3|\xi_1|$ with c_3 a constant. In the same way as (3.14) we obtain

$$I_{[0,\frac{1}{2}t)}(\xi_1)\Lambda(\bar{\xi}, t, \bar{a}) \leqq c_4 t^{\rho} \sum_{[t-\xi_1]}^{[t+1]} m^{-\rho} \leqq c_5|\xi_1|.$$

So $|\zeta(\bar{\xi}, t)| \leqq c_6|\xi_1|$ and (3.9) follows by the existence of first moments. The relation (3.10) also follows and (3.11) is replaced by

$$\int \gamma(\bar{\xi}, t, \bar{a})F(d\bar{\xi}) = t^{\rho} q_t(a_2, \cdots, a_d)\int h(y_2, \cdots, y_d)dy_2 \cdots dy_d.$$

The relation (3.12) now follows from the counterparts of (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11), if $X_{11} \geqq 0$. The proof is concluded in the same way as the proof of theorem 3.1.

THEOREM 3.3. *Let F satisfy the conditions of lemma* 2.2. *For $t > 0$ let $\bar{a}(t)$ be a d-vector such that $0 \leqq a_1(t) \leqq K$ and $t+\bar{a}(t)$ belongs to the F-lattice. Then*

$$\lim_{t\to\infty} t^{\rho}|R_t\{\bar{a}(t)\} - v_1^{-1}H_1(E_t)q_t(a_2(t), \cdots, a_d(t))| = 0,$$

uniformly in $\bar{a}(t)$ for fixed K. Here E_t denotes the open interval $(a_1(t), \infty)$ and q_t the same normal density as in theorem 3.1.

COROLLARY. *If $X_{11}, \cdots, X_{1d}$ are integer valued such that $\varphi(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of 2π and $|\varphi(\bar{u})| < 1$ elsewhere, and if $E|X_{11}|^{\rho} < \infty$, then*

$$\lim_{h\to\infty} h^{\rho}|R_h(\bar{k}) - v_1^{-1}H_1((k_1, \infty))q_h(k_2, \cdots, k_d)| = 0,$$

uniformly in $k_2, \cdots, k_d$, if $h, k_1, \cdots, k_d$ are integers with $h > 0$, $k_1 \geqq 0$.

PROOF. First assume $X_{11} \geqq 0$ with probability 1. We have

$$t^{\rho}R_t\{\bar{a}(t)\} = t^{\rho}P\{\bar{S}_1 = t+\bar{a}(t)\} + T_2,$$

where the first term is dealt with by the existence of EX_{11}^{ρ}

$$T_2 = t^{\rho}\sum_{m=1}^{\infty} P\{S_{m1} < t, \bar{S}_{m+1} = t+\bar{a}(t)\},$$

$$T_2 = t^{\rho}\sum_{m=1}^{\infty}\sum_{\bar{\xi}} P\{X_{m+1} = \bar{\xi}\}P\{S_{m1} < t, \bar{S}_m = t+\bar{a}(t)-\bar{\xi}\},$$

where $\bar{\xi}$ runs through points of the F-lattice. Because of the second factor we may write

$$T_2 = t^{\rho}\sum_{\xi_1 > a_1(t)} F(\{\bar{\xi}\})U(\{t+\bar{a}(t)-\bar{\xi}\}).$$

By lemma 2.2 we have for fixed $\bar{\xi}$

$$t^\rho U\{t+\bar{a}(t)-\bar{\xi}\} = \mu_1^\rho L(t, a_2(t), \cdots, a_d(t))+\eta,$$

where $\eta \to 0$ as $t \to \infty$, uniformly in $\bar{a}(t)$ if $0 \leqq a_1(t) \leqq K$, if $\bar{\xi}$ is kept fixed. The proof now proceeds in the same way as with theorem 3.1. We write $T_2 = T_3+T_4$ where the sum is taken over the sets $\{\xi_1 \geqq \frac{1}{2}t\}$ and $\{a_1(t) < \xi_1 < \frac{1}{2}t\}$, respectively. Handling of T_3 and T_4 requires the same estimations as in the proof of theorem 3.1.

The lattice counterpart of theorem 3.2 is restricted to integer valued $X_{11}, \cdots, X_{1d}$, since under the more general assumptions of theorem 3.3 the lattice description of $Z_2(t), \cdots, Z_d(t)$ is difficult.

THEOREM 3.4. *If $X_{11}, \cdots, X_{1d}$ are integer valued, such that $\varphi(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of 2π and $|\varphi(\bar{u})| < 1$ elsewhere, and if $E|X_{11}|^\rho < \infty$, then*

$$\lim_{h\to\infty} h^\rho |P\{Z_2(h) = k_2, \cdots, Z_d(h) = k_d\}-q_h(k_2, \cdots, k_d)| = 0, \tag{3.15}$$

uniformly in $k_2, \cdots, k_d$. Here $h, k_2, \cdots, k_d$ are integers and q_t is the same normal density as in theorem 3.1.

PROOF. First take $P\{X_{11} \geqq 0\} = 1$. We have

$$\begin{aligned} h^\rho P\{Z_2(h) &= k_2, \cdots, Z_d(h) = k_d\} \\ &= h^\rho P\{X_{11} \geqq h, X_{12} = k_2, \cdots, X_{1d} = k_d\}+T_2, \end{aligned}$$

where the first term tends to zero uniformly in $(k_2, \cdots, k_d)$ as $h \to \infty$ since $E|X_{11}|^\rho < \infty$ and

$$\begin{aligned} T_2 &= h^\rho \sum_{m=1}^{\infty} P\{S_{m1} < h, S_{m+1,1} \geqq h, S_{m+1,r} = k_r, \qquad r = 2, \cdots, d\} \\ &= h^\rho \sum_{m=1}^{\infty} \sum{}' \sum{}'' F^m\{i_1, \cdots, i_d\}F\{j_1, \cdots, j_d\}, \end{aligned}$$

where $\sum'$ and $\sum''$ are subject to the restrictions $i_1 < h$, $i_1+j_1 \geqq h$, $i_r+j_r = k_r$, $r = 2, \cdots, d$. So

$$T_2 = h^\rho \sum_{j_1, \cdots, j_d} F\{j_1, \cdots, j_d\} \sum_{i_1=h-j_1}^{h-1} U\{i_1, k_2-j_2, \cdots, k_d-j_d\}. \tag{3.16}$$

By lemma 2.2 we have for fixed $j_1, \cdots, j_d$ and $h-j_1 \leqq i_1 < h-1$

$$U\{i_1, k_2-j_2, \cdots, k_d-j_d\} = \mu_1^\rho L(h, k_2, \cdots, k_d)+\eta,$$

with $\lim_{h\to\infty} \eta = 0$, uniformly in $k_2, \cdots, k_d$.

The relation (3.15) now follows with (1.5) and (1.6) if passing to the limit in (3.16) under the sum over $j_1, \cdots, j_d$ is justified. This is done by the same methods as in the proof of theorem 3.2.

If $P\{X_{11} < 0\} > 0$ we consider the random walk at the ladder times of the process $\{S_{n1}\}$.

Summary

Let $\bar{X}_1, \bar{X}_2, \cdots$ be independent strictly d-dimensional random vectors, with common distribution, with finite second moments and positive x_1-component of the first-moment vector. Let $\bar{S}_n = \bar{X}_1 + \cdots + \bar{X}_n$, $n = 1, 2, \cdots$, $N(t) = \min \{n: S_{n1} \geqq t\}$ and $\bar{Z}(t) = \bar{S}_{N(t)}$.

If $E|X_{11}|^\rho < \infty$, where $\rho = \frac{1}{2}(d-1)$, the joint distribution of $Z_1(t)-t$, $Z_2(t), \cdots, Z_d(t)$ satisfies a local central limit theorem for $t \to \infty$. The approximating probability measure is the product of the well known limiting distribution for $Z_1(t)-t$ and a normal distribution for $Z_2(t), \cdots, Z_d(t)$. The difference is $o(t^{-\rho})$ as in a local central limit theorem for sums of independent $(d-1)$-vectors.

The theorem is stated and proved for nonarithmetic F and for F restricted to a (rotated) cubic lattice with span 1. A special case of the global version was proved by the author in Zeitschr. für Wahrsch. th. u. verw. Geb. 10 (1968), 81–86.

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