Compositio Mathematica

Charles Pugh
Michael Shub

Ergodic elements of ergodic actions

Compositio Mathematica, tome 23, nº 1 (1971), p. 115-122
<http://www.numdam.org/item?id=CM_1971__23_1_115_0>

COMPOSITIO MATHEMATICA, Vol. 23, Fasc. 1, 1971, pag. 115–122
Wolters-Noordhoff Publishing
Printed in the Netherlands

ERGODIC ELEMENTS OF ERGODIC ACTIONS

by

Charles Pugh * and Michael Shub **

1. Introduction

A question naturally arising in dynamical systems is: does a flow $\{\varphi_t\}$ have the same nonwandering set as most of the time t maps, φ_t? This and similar questions for recurrency (Poisson stability) we are unable to answer. For ergodicity, however, representation theory of S^1 and Stone's theorem lead to an affirmative answer:

If $\{\varphi_t\}$ is an ergodic flow then each map φ_t, except for a countable set of t-values, is ergodic (See § 2 for definitions).

The proof of this was supplied to us by J. Auslander and Parthasarathy.

In this paper we ask the corresponding question for group actions and answer it completely in the case of R^k-actions. (For other group actions we have partial results and some counter-examples). The analysis is not difficult.

We employ the representation theory of separable locally compact abelian groups, and it seems strange that such simple facts were apparently unknown to the ergodic theorists. Our analysis of R^k-actions, $k \geqq 1$, uses the characterization of cyclic representations not Stone's theorem, as can be found in [2]. We are indebted to R. Palais and H. Levine for conversations leading to the proof of the theorem and to W. Parry for our interest in ergodic theory.

2. Definitions and the main result

Let G be a group and (M, μ) a measure space. An action of G on (M, μ) is a homomorphism $\rho : G \to \text{Auto}\ (M, \mu)$. The elements of Auto (M, μ) are the measurable bijections of M onto itself with measurable inverses. If G has a topology, the homomorphism ρ is required to be continuous in the sense that

$$G \times L^2(M, \mu) \to L^2(M, \mu)$$

$$(g, f) \mapsto f \circ \rho g$$

* Supported by the sloan Foundation and NSF GP8007.

** Nato Postdoctoral fellow.

be continuous. We shall denote the operator on $L^2(M, \mu), f \mapsto f \circ \rho g$, as $\rho_* g$.

The action $\rho : G \to \text{Auto}\,(M, \mu)$ is said to be ergodic iff

1. $\rho(g)$ is measure-preserving for each $g \in G$.
2. If $f \in L^2(M, \mu)$ is invariant (i.e. $\rho_*(g)f = f$ for each $g \in G$) then f is constant.

As is standard, 'equal' means 'equal almost everywhere' and 'constant' means 'equal to a constant-function almost everywhere'. In terms of ρ_* : 1e means that $\rho_* : G \to Un(L^2(M, \mu))$, the unitary operators on $L^2(M, \mu)$.

An *element* g of G is said to be ergodic if ρg is measure preserving and $\rho_* g$ has only constant invariant functions.

We are interested in knowing when an ergodic action has ergodic elements without much restriction on (M, μ) or ρ. We have proved the following

THEOREM 1. *If R^k acts ergodically on (M, μ), $\mu(M) < \infty$, and $L^2(M, \mu)$ is separable, then all the elements of R^k, off a countable of family hyperplanes, are ergodic.*

When we speak of planes, lines, etc., we do not assume that they contain the origin. A more general group is discussed in § 6.

R^k acts on itself naturally by translation $\rho(g) : y \mapsto y+g$. Factoring by Z^k gives the natural action of R^k on T^k, the k-torus. Theorem 1 generalizes the fact that under this natural action every element of R^k, off a countable family of hyperplanes, acts ergodically on T^k. In this case we can identify the hyperplanes as those points $y \in R^k$ whose components $y_1, \cdots, y_k$ are rationally dependent. Unfortunately we can't give such a neat determination of the non-ergodic elements for a general action of R^k.

As some of our work is valid for groups G more general than R^k, we shall only assume

$$G \text{ is } \begin{array}{ll} \text{abelian} & \text{Hausdorff} \\ \text{locally compact} & \text{separable} \end{array}$$

$$\mu(M) < \infty \qquad L^2(M, \mu) \text{ separable}$$

$$\rho : G \to \text{Auto}\,(M, \mu) \text{ is ergodic.}$$

We recall that a unitary representation of G is a homomorphism r sending G into the group of unitary operators on some separable Hilbert space H. Continuity of $(g, h) \mapsto (rg)(h)$ is required. Then $\rho_* : G \to Un(L^2(M, \mu))$ is a unitary representation of G.

3. Cyclic representations

The representation $r : G \to Un(H)$ is cyclic if there is a nonzero vector $v \in H$ such that the only closed subspace of H containing the orbit of v, $\{(rg)(v) : g \in G\}$ is H itself. To any cyclic representation $r : G \to Un(H)$ there corresponds a unique normalized Borel measure β on the character group, $\hat{G} = \mathrm{Hom}(G, S^1)$, such that r is unitarily equivalent to the 'direct integral' representation $m : G \to Un(L^2(\hat{G}, \beta))$ defined by

$$g \mapsto \langle \cdot, g \rangle f(\cdot) \qquad f \in L^2(\hat{G}, \beta)$$

See [2]. Thus for a fixed $g \in G$, we have the unitary operator $L^2(\hat{G}, \beta) \to L^2(\hat{G}, \beta)$ defined by $m(g) : f \mapsto \langle \cdot, g \rangle f(\cdot)$.

Now let us return to the ergodic action $\rho : G \to \mathrm{Auto}(M, \mu)$ and the induced unitary representation $\rho_* : G \to Un(M, \mu)$. The space $L^2(M, \mu)$ may be written as the countable direct sum (of orthogonal closed subspaces)

$$L^2(M, \mu) = C \bigoplus_i H_i$$

where C is the one dimensional space of constant functions and $\rho_* G$ is a cyclic representation on H_i. That is each $\rho_* g$ leaves H_i invariant and (restriction to $H_i) \circ \rho_* : G \to H_i$ is cyclic. For in $C^\perp$ we choose any vector $V \neq 0$ and look at H_i, the smallest closed subspace of $L^2(M, \mu)$ containing its orbit. As ρ_* is a unitary representation, it leaves $C^\perp$ invariant and so $H_i \perp C$. Since $L^2(M, \mu)$ is separable (by assumption) Zorn's lemma lets us proceed inductively to define such an invariant splitting $C \oplus H_1 \oplus H_2 \oplus \cdots = L^2(M, \mu)$.

Let us write ρ_{*i} for (restriction to $H_i) \circ \rho_* : G \to H_i$.

For each i the representation ρ_{*i} is unitarily equivalent to the direct integral representation m_i on $L^2(\hat{G}, \beta_i)$ for some unique normalized Borel measure β_i on $\hat{G}$; $m_i : G \to Un(L^2(\hat{G}, \beta_i))$ defined by

$$(m_i g) : f \mapsto \langle \cdot\, g \rangle f(\cdot)$$

LEMMA 1. *If ε is the identity element of $\hat{G}$ then ε has zero β_i-measure for each i.*

PROOF. If $\beta_i(\varepsilon) > 0$ then let E be the characteristic function of ε. In $L^2(\hat{G}, \beta_i)$, $E \neq 0$ because $\beta_i(\varepsilon) \neq 0$. But $(m_i g)E = \langle \cdot, g \rangle\, E(\cdot)$ and evaluated on any $\chi \in \hat{G}$ this is

$$\langle \chi, g \rangle E(\chi) = \begin{cases} 1 & \text{if } \chi = \varepsilon \\ 0 & \text{if } \chi \neq \varepsilon \end{cases} = E(\chi)$$

Thus $(m_i g)E = E$ for all $g \in G$ and so E would be a nonzero invariant vector for all the operators $m_i g$. But the inverse of the conjugacy

$h_i : H_i \to L^2(\hat{G}, \beta_i)$ carries E on to $h^{-1}_i(E)$, a nonzero invariant vector of $\rho *_i$. Since $h^{-1}_i(E)$ lies in H_i and $H_i \perp C$, $h^{-1}_i(E)$ is nonconstant. Thus $\rho *$ has a nonconstant invariant function in $L^2(M, \mu)$, contradicting the fact that ρ is ergodic. Hence $\beta_i(\varepsilon) = 0$, proving the lemma.

For any $g \in G$ let

$$\ker(g) = \{\chi \in \hat{G} : \langle \chi, g \rangle = 1\}$$

and for any $\chi \in \hat{G}$ let

$$\ker(\chi) = \{g \in G : \langle \chi, g \rangle = 1\} \quad .$$

4. A criterion for non ergodicity

The next lemma distinguishes those elements $g \in G$ which are not ergodic.

LEMMA 2. *If $g_0 \in G$ is not an ergodic element for the action ρ then $\beta_i(ker\, g_0) > 0$ for some i.*

PROOF. That g_0 is not ergodic means $\rho * g_0 : L^2(M, \mu) \to L^2(M, \mu)$ has a nonconstant invariant function, say w. The invariant cyclic decomposition $L^2(M, \mu) = C \oplus_i H_i$ gives $w = c + \sum_i w_i$ and $w_i \neq 0$ for some i. Then w_i is a nonzero invariant vector of $\rho * g_0$ lying in H_i. That is, w_i is a nonzero invariant vector of $\rho *_i g_0$, the restriction of $\rho * g_0$ to H_i. But $\rho *_i$ is cyclic and so is conjugate to the representation $m_i : G \to Un(L^2(\hat{G}, \beta_i))$ defined by

$$m_i g : f \mapsto \langle \cdot, g \rangle f(\cdot)$$

The conjugacy h_i makes

$$\begin{tikzcd} H_i \arrow[r, "\rho *_i g"'] \arrow[d, "h_i"'] & H_i \arrow[d, "h_i"] \\ L^2(\hat{G}, \beta_i) \arrow[r, "m_i g"] & L^2(\hat{G}, \beta_i) \end{tikzcd} \qquad g \in G$$

commute. The conjugate of w_i, $h_i w_i = f$, is a nonzero invariant vector of $m_i g_0$. But since we know the formula defining $m_i g_0$, this says that

$$(1 - \langle \cdot, g_0 \rangle) f(\cdot) = 0$$

in $L^2(\hat{G}, \beta_i)$. Since $f \neq 0$ in $L^2(\hat{G}, \beta_i)$, the factor $1 - \langle \cdot, g_0 \rangle$ vanishes on a set of positive β_i-measure. That is $\{\chi \in \hat{G} : \langle \chi, g_0 \rangle = 1\} = \ker g_0$ has positive β_i-mesaure.

As an immediate corollary we have

PROPOSITION. *The ergodic action ρ has*

(a) *some ergodic elements, if G is not the union of its proper closed subgroups*

(b) *a Baire set of ergodic elements if G is compact and every proper closed subgroup is nowhere dense*

(c) *a contralinear Baire set of ergodic elements if $G = T^k$.*

PROOF. If $g \in G$ is non ergodic then $\beta_i(\ker g) > 0$ for some i and $\ker g$ contains a nonzero element χ, by Lemma 1, so $g \in \ker(\chi)$. This $\ker \chi$ is a proper closed subgroup of G.

(a) From this, (a) is clear.

(b) If G is compact, $\hat{G}$ is countable [3] and so each nonergodic $g \in G$ is contained in some $\ker(\chi)$ where χ ranges over a countable set. The countable union of these nowhere dense subsets, $\ker(\chi)$, includes all non ergodic elements and its complement is the asserted Baire set.

(c) If G is T^k then each $\ker(\chi)$ is a hyperplane for $\chi \in \hat{G} = Z^k$.

By contralinear we mean the complement of the family of rationally dependent sub-tori.

REMARK. For (b), (c) we did not have to use the characterization of a cyclic representation, since in the compact case we may decompose ρ_* into a direct sum of irreducible, 1-dimensional representations.

5. $G = R^k$ and the proof of Theorem 1

Now we shall let $G = R^k$. Since $\hat{R}^k = R^k$, we shall be working with normalized Borel measures β on R^k. For any such β let

$\mathscr{P}_0$ = the set of points $P \in R^k$ with $\beta(P) > 0$.

$\mathscr{P}_n$ = the set of n-planes $P \subset R^k$ with $\beta(P) > 0$ but containing no element of $\mathscr{P}_m$, $m < n$.

This makes $\mathscr{P}_{k+1} = \emptyset$. Let $\mathscr{P} = \mathscr{P}_0 \cup \cdots \cup \mathscr{P}_k$.

LEMMA 3. *$\mathscr{P}$ is at most countable.*

PROOF. If $P, P' \in \mathscr{P}_n$ are not identical then $\beta(P \cap P') = 0$ since $P \cap P'$ is a lower dimensional plane in each. Thus, if $P_1, \cdots, P_m \in \mathscr{P}_n$ are distinct then

$$\beta\Big(\bigcup_1^m P_j\Big) = \sum_1^m \beta(P_j)$$

Since $\beta(R^k) = 1$ there can therefore be no more than m planes $P \in \mathscr{P}_n$

having measure $\geqq 1/m$. Letting $m \to \infty$ we exhaust all of $\mathscr{P}_n$ by finite sets; hence $\mathscr{P}$ is at most countable.

As remarked before, R^k is it own character group where $\langle x, y\rangle = e^{2\pi y(x,y)}$ and (x, y) is the usual dot product in R^k; (We think always of $x \in \hat{G} = R^k, y \in G = R^k$). Then

$$\ker(x) = \{y \in R^k : \langle x, y\rangle = 1\} = \{y \in R^k : (x, y) \in Z\}.$$

LEMMA 4. *If* $\beta(0) = 0$ *and* $B = \{y \in R^k : \beta(\ker y) > 0\}$ *then* B *is contained in countably many hyperplanes.*

PROOF. Since $\beta(\ker y) > 0$, $\ker y$ must contain an element P of $\mathscr{P}$ and since $\beta(0) = 0$ such a $P \neq 0$ may be chosen. All the elements $x \in P$ have y in their kernels; that is $x \in \ker y \Leftrightarrow y \in \ker x$. Hence for each $y \in B$ there exists $P \in \mathscr{P}$ such that $y \in \bigcap_{x \in P-0} \ker x$. Since P ranges over the countable set $\mathscr{P}$, independant of $y \in B$, the lemma is proved.

PROOF OF THEOREM 1. For each normalized Borel measure β_i on R^k as above, the set B_i given in lemma 4 is contained in countably many hyperplanes. For $\beta_i(0) = 0$ is satisfied by Lemma 1. By Lemma 2, if $y \in R^k$ is not ergodic then $y \in B_i$ for some i and the theorem is proved.

6. $G \neq R^k$

Now we give examples of some groups which can act ergodically without having any ergodic elements.

EXAMPLE 1. Give the rationals Q the discrete topology and let $\rho : Q \to \text{Auto}(S^1, d\theta)$ be defined by rotation

$$\rho\left(\frac{p}{q}\right) : e^{2\pi it} \to e^{2\pi i(t+p/q)}$$

The action is ergodic because if f were a nonconstant invariant function for all the operators $\rho_*(p/q)$ it would be an L^2 function which was periodic of every rational period. Such an f cannot exist by taking its Fourier expansion. However, each $\rho_*(p/q)$ has invariant functions $e^{2\pi it} \mapsto e^{2\pi iqmt}$ m being any positive integer.

EXAMPLE 2. Let $m \geqq 2$ and $n \geqq 1$ be integers and let $k = mn$. Let $G = Z_m \oplus Z_k$ and $M = Z_m \oplus Z_m$. Define $\rho : G \to \text{Auto}(M, \mu)$ by

$$\rho(t, t') : (s, s') \mapsto (t+s, t'+s')$$

where $+$ is taken in Z_m and t' is first reduced mod m. The measure μ on $Z_m \oplus Z_m$ equals $1/m^2$ on each point. The action ρ is ergodic because in fact it is transitive. However, for any $g = (t, t') \in G$ there is an invariant function for $\rho_* g$ which is the characteristic function f, of

$T = \{(t, t'), (2t, 2t'), \cdots, (mt, mt')\} \subset M$. The set T is invariant by ρg, so its characteristic function is invariant by $\rho_* g$. Since T has m elements and M has m^2 elements, f is non constnat in $L^2(M, \mu)$.

In example 2, $Z_m \oplus Z_k$ may be replaced by any group G surjecting continuously onto $Z_m \oplus Z_k$:

$$G \to Z_m \oplus Z_k \xrightarrow{\rho} \mathrm{Auto}(Z_m \oplus Z_m, \mu)$$

So if G is of the form $R^k \times T_n \times F$ when F is a finitely generated abelian group then we have answered the question of 'ergodicity $\Rightarrow$ ergodic elements' negatively if F is not cyclic. The case when F is cyclic easily reduces to the study of $G = R^k \times Z$, for which we prove a generalization of theorem 1.

THEOREM 2. *If $G = R^k \times Z$ acts ergodically on (M, μ) then all elements $g \in R^k \times \{1\}$, off a countable union of hyperplanes in $R^k \times \{1\}$, are ergodic.*

Since any Abelian Lie group generated by a compact neighborhood of the identity can be expressed $R^k \times T_n \times F$ where F is a finitely generated discrete group, Theorems 1, 2 and Example 2 answer the question: when does ergodicity of the G action imply the existence of ergodic elements for such a group G? The product decomposition follows from [3] and [4].

PROOF OF THEOREM 2. Instead of the planes $P \in \mathscr{P}_n$ consider translates of closed connected subgroups of $\hat{G} = R^k \times S^1$ having dimension n, positive β measure, and containing no elements of $\mathscr{P}_0 \cup \cdots \cup \mathscr{P}_{n-1}$. By the same argument $\mathscr{P} = \mathscr{P}_0 \cup \cdots \cup \mathscr{P}_{k+1}$ is countable. (It is only necessary to observe that the intersection $P \cap P'$ has dimension $< n$ if $P, P' \in \mathscr{P}_n$ are distinct.) If $g \in R^k \times \{1\}$ is not ergodic then $\beta_i(\ker g) > 0$ for one of the measures β_i arising in § 3. Since $\beta_i(\varepsilon) = 0$, there exist $P \in \mathscr{P}^i, P \neq \varepsilon, P \subset \ker g$, and $\chi \in P, \chi \neq \varepsilon$, can be chosen. We have $\chi = (x, e^{2\pi i \xi})$ with $x \in R^k$ and $0 \leqq \xi < 1$. Then $g \in \ker(\chi)$ and

$$\begin{aligned} R^k \times \{1\} \cap \ker \chi &= \{(y, 1) \in R^k \times \{1\} : \langle \chi, (y, 1)\rangle = 1\} \\ &= \{(y, 1) : e^{2\pi i (x, y) + \xi} = 1\} \\ &= \{(y, 1) : (x, y) + \xi \in Z\} \end{aligned}$$

Now if $x \neq 0$ then it is clear that $R^k \times \{1\} \cap \ker \chi$ is a countable family of hyperplanes. But if $x = 0$ then $\ker(\chi) \cap R^k \times \{1\} = \emptyset$ since $\xi \neq 0$ and $0 \leqq \xi < 1$. [Note that if we had dealt, with say $R^k \times \{2\}$ we would have considered $\{(2y, 2) : (x, y) + 2\xi \in Z\}$. If $x \neq 0$ then this is a countable family of hyperplanes. But if $x = 0$ it is possible for ξ to equal 1/2 and the set to be all of $R^k \times \{2\}$.] See also example 3 below.

Since each non ergodic $g \in R^k \times \{1\}$ belongs to a countable family of hyperplanes $\bigcap_{\chi \in P - \varepsilon} \ker(\chi) \cap R^k \times \{1\}$ and these P are chosen always from the sequence of countable families $\mathscr{P}^1, \mathscr{P}^2, \cdots$, the totality of hyperplanes necessary to contain $\{g \in R^k \times \{1\} : g \text{ is not ergodic}\}$ is at most coutable.

EXAMPLE 3. This example was shown to us by K. Sigmund. Ergodicity of an action $\rho : Z \to \text{Auto}(M, \mu)$ is equivalent to ergodicity of $\rho(\pm 1)$. Here we observe that $\rho(n)$ may fail to be ergodic for all n other than ± 1. This shows that our concentration on the nonergodic element lying in $R^k \times \{1\}$ was necessary.

Let $M = Z_2 \times Z_3 \times \cdots = \prod_{p \in \mathfrak{P}} Z_p$, $\mathfrak{P}$ = the primes. Let μ_p be the normalized measure on Z_p, $\mu_p(S)$ = cardinality $(S)/p$ and let $\mu = \prod_{p \in \mathfrak{P}} \mu_p$. This makes (M, μ) a normalized measure space and $L^2(M, \mu)$ separable. The action $\rho : Z \to \text{Auto}(M, \mu)$ defined by $\rho(n) : (z_p) \mapsto (z_{p+_p} n)$ is ergodic but $\rho(n)$ is not ergodic unless $n = \pm 1$. Ergodicity of ρ follows from some standard ergodic theory [1]. Non ergodicity of $\rho(n)$, $n \neq \pm 1$, is clear since $\rho(n)$ leaves invariant each of the q sets of measure $1/q$, $S^0 = \{(z_p) \in \prod Z_p : z_q = 0\}, \cdots, S^{q-1} = \{(z_p) \in \prod Z_p : z_q = q-1\}$ where q is a prime dividing n.

REFERENCES

BILLINGSLEY

[1] Ergodic Theory and Information. John Wiley and Sons. New York, 1965, p. 17.

G. W. MACKEY

[2] Group representations and Applications. Lectures given at Oxford University, 1966–1967.

L. PONTRIJAGIN

[3] Topological Groups. Princeton University Press.

A. WEIL

[4] L'integration dans les Groupes Topologiques et ses applications .Hermann, Paris (1951).

(Oblatum 22–V–1970)

Institut des Hautes Etudes Scientifiques,
91 - Bures-sur-Yvette.

Prof. C. Pugh
Department of Mathematics
University of California
BERKELEY, Calif. 94720
U.S.A.

Prof. M. Shub
Department of Mathematics
Brandeis University
WALTHAM, Mass. 02154
U.S.A.