COMPOSITIO MATHEMATICA

I. N. STEWART

Infinite-dimensional Lie algebras in the spirit of infinite group theory

<http://www.numdam.org/item?id=CM_1970__22_3_313_0>

INFINITE-DIMENSIONAL LIE ALGEBRAS
IN THE SPIRIT OF INFINITE GROUP THEORY

by

I. N. Stewart

Both notations and the numbering of sections will be carried over from the previous paper. For convenience, however, a separate list of references will be given.

3. Lie algebras, all of whose subalgebras are n-step subideals

A theorem of J. E. Roseblade [12] states that if G is a group such that every subgroup K of G is subnormal in at most n steps, i.e. there exists a series of subgroups

$$K = K_0 \lhd K_1 \lhd \cdots \lhd K_n = G,$$

then G is nilpotent of class $\leqq f(n)$ for some function $f : \mathbf{Z} \to \mathbf{Z}$.

This chapter is devoted to a proof of the analogous result for Lie algebras over fields of arbitrary characteristic.

3.1 *Subnormality and completions*

It might be thought that we could prove the theorem for Lie algebras over $\mathbf{Q}$ by a combination of Roseblade's result and the Mal'cev correspondence, as follows:

Suppose L is a Lie algebra over $\mathbf{Q}$, such that every subalgebra $K \leqq L$ satisfies $K \lhd^n L$. By a theorem of Hartley [5] p. 259 (cor. to theorem 3) $L \in \mathrm{L}\mathfrak{N}$. We may therefore form the corresponding group $\mathscr{G}(L)$. Clearly every *complete* subgroup H of G satisfies $H \lhd^n G$. If we could show that every subgroup of G is boundedly subnormal in its completion, we could use Roseblade's theorem to deduce the nilpotence (of bounded class) of G, hence of L.

This approach fails, however – we shall show that a locally nilpotent torsion-free group need not be subnormal in its completion, let alone boundedly so.

Let $T_n(\mathbf{Q})$ denote the group of $(n+1) \times (n+1)$ unitriangular matrices over $\mathbf{Q}$, $U_n(\mathbf{Q})$ the Lie algebra of all $(n+1) \times (n+1)$ zero-triangular matrices over $\mathbf{Q}$. Similarly define $T_n(\mathbf{Z})$, $U_n(\mathbf{Z})$.

If H is a subnormal subgroup of G let $d(H, G)$ be the least integer d for which (in an obvious notation) $H \lhd^d G$. d is the *defect* of H in G.

LEMMA 3.1.1
$$d(T_n(Z), T_n(Q)) = n.$$

PROOF:

Let $T = T_n(Q)$, $S = T_n(Z)$, $d = d(S, T)$.. Then $d \leq n$ since T is nilpotent of class n. We show that $S \lhd^{n-1} T$ is *false*. Suppose, if possible, that $S \lhd^{n-1} T$. Then for all $s \in S$, $t \in T$ we would have

$$(t, {}_{n-1}s) \in S$$

(where $(a, {}_m b)$ denotes $(\cdots (a, b), b), \cdots, b)$.) Taking logarithms,

$$\log(t, {}_{n-1}s) \in \log(S).$$

By the Campbell-Hausdorff formula, remembering that T is nilpotent of class n, this means that

$$[\log(t), {}_{n-1}\log(s)] \in \log(S).$$

We choose $s \in S$ in such a way as to prevent this happening.
Consider the matrix

$$X = \begin{bmatrix} 0 & x & 0 & \cdots & 0 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

Then

$$\exp(X) = \begin{bmatrix} 1 & x & x^2/2! & x^3/3! & \cdots & x^n/n! \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

So if we put

$$s = \exp \begin{bmatrix} 0 & n! & 0 & \cdots & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

then $s \in S$.

Let

$$t = \exp \begin{bmatrix} 0 & \lambda & 0 & \cdots & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

where for the moment λ is an arbitrary element of Q. An easy induction shows that

$$[\log(t), {}_{n-1}\log(s)] = \begin{bmatrix} 0 & \cdots & 0 & \alpha \\ & & & \\ & & & \\ & & & \end{bmatrix} = A \text{ (say)},$$

where $\alpha = \lambda \cdot (n!)^{n-1}$

Now

$$\exp(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 & \alpha \\ & & & & \\ & & & & \\ & & & & 1 \end{bmatrix}$$

and we can choose $\lambda \in Q$ so that $\alpha \notin Z$. Thus $\exp(A) \notin S$, so $A \notin \log (S)$, a contradiction. This shows $d \geqq n$, so that $d = n$ as claimed.

COROLLARY 1

There is no bound to the defect of a nilpotent torsion-free group in its completion.

PROOF:

$T_n(Q)$ is easily seen to be the completion of $T_n(Z)$.

COROLLARY 2

A locally nilpotent torsion-free group need not be subnormal in its completion.

PROOF:

Let
$$V = \mathop{\mathrm{Dr}}_{n=1}^{\infty} T_n(\mathbf{Z}).$$

Then
$$\overline{V} = \mathop{\mathrm{Dr}}_{n=1}^{\infty} T_n(\mathbf{Q}).$$

If V were subnormal in $\overline{V}$ then $V \lhd^m \overline{V}$ for some $m \in \mathbf{Z}$, so that $T_{m+1}(\mathbf{Z}) \lhd^m T_{m+1}(\mathbf{Q})$ contrary to lemma 3.1.1.

3.2 *Analogue of a theorem of P. Hall*

We prove the theorem we want directly for Lie algebras, using methods based on those of Roseblade. Throughout the chapter all Lie algebras will be over a fixed but arbitrary field $\mathfrak{k}$ (of arbitrary characteristic). We introduce 3 new classes of Lie algebras:

$$L \in \mathfrak{D} \Leftrightarrow (H \leq L \Rightarrow H \operatorname{si} L)$$

$$L \in \mathfrak{D}_n \Leftrightarrow (H \leq L \Rightarrow H \lhd^n L)$$

$$L \in \mathfrak{F} \Leftrightarrow (H < L \Rightarrow I_L(H) > H).$$

(The last condition is known as the *idealiser condition*).

Throughout this chapter $\mu_i(m, n, \cdots)$ will denote a positive-integer valued function depending only on those arguments explicitly shown.

Our first aim is to show that if $H \lhd L$, $H \in \mathfrak{N}_c$, and $L/H^2 \in \mathfrak{N}_d$, then $L \in \mathfrak{N}_{\mu_1(c,d)}$ for some function μ_1. For the purposes of this chapter it is immaterial what the exact form of μ_1 is; but it is of independent interest to obtain a good bound. The group-theoretic version, with $\mu_1(c, d) = \binom{c+1}{2}d - \binom{c}{2}$, is due to P. Hall [4]; the result for Lie algebras with this bound is proved by Chong-Yun Chao [2] (stated only for finite-dimensional algebras). In [14] A. G. R. Stewart improves Hall's bound in the group-theoretic case to $cd + (c-1)(d-1)$ and shows this is best possible. We add a fourth voice to the canon by showing that similar results hold for Lie algebras (using essentially the same arguments). A few preliminary lemmas are needed to set up the machinery.

LEMMA 3.2.1

If L is a Lie algebra and $A, B, C \leq L$ then

$$[[A, B], C] \leq [[B, C], A] + [[C, A], B].$$

PROOF:
From the Jacobi identity.

LEMMA 3.2.2
If L is a Lie algebra and A, B, C ≤ L then

$$[[A, B], {}_nC] \leqq \sum_{\substack{i+j=n \\ i, j \geqq 0}} [[A, {}_iC], [B, {}_jC]].$$

PROOF:
Use induction on n. If $n = 1$ lemma 3.2.1 gives the result. Suppose the lemma holds for n. Then

$$[[A, B], {}_{n+1}C] = [[[A, B], {}_nC], C]$$
$$\leqq \sum_{i+j=n} [[[A, {}_iC], [B, {}_jC]], C] \text{ by hypothesis}$$
$$\leqq \sum_{i+j=n} [[A, {}_{i+1}C], [B, {}_jC]] + [[A, {}_iC], [B, {}_{j+1}C]]$$

by lemma 3.2.1

$$= \sum_{i+j=n+1} [[A, {}_iC], [B, {}_jC]]$$

and the induction step goes through.

THEOREM 3.2.3
Let L be a Lie algebra, $H \lhd L$, such that $H \in \mathfrak{N}_c$ and $L/H^2 \in \mathfrak{N}_d$. Then $L \in \mathfrak{N}_{\mu_1(c,d)}$ where

$$\mu_1(c, d) = cd + (c-1)(d-1).$$

Further, this bound is best possible.

PROOF:
Induction on c. If $c = 1$ the result is obvious. If $c > 1$, then for any r with $1 \leqq r \leqq c$ we have $M_r = H/H^{r+1} \lhd N_r = L/H^{r+1}$. $M_r \in \mathfrak{N}_r$ and $N_r/M_r^2 \in \mathfrak{N}_d$ so inductively we may assume

$$L^{2rd-r-d+2} \leqq H^{r+1} \quad 1 \leqq r \leqq c-1.$$

Now $\quad L^{2rd-c-d+2} \leqq [H^2, {}_{2cd-2d-c+1}L]$

$$\leqq \sum_i [[H, {}_iL], [H, {}_{2cd-2d-c+1-i}L]]$$

summed over the interval $0 \leqq i \leqq 2cd-2d-c+1$ (by lemma 3.2.2). Each such i belongs to an interval

$$2(j-1)d-d-(j-1)+1 \leqq i < 2jd-d-j+1 \quad (1 \leqq j \leqq c).$$

Condider an arbitrary j. By induction if $j \neq 1$, and since $H \lhd L$ if $j = 1$, we have

$$[[H, {}_iL], [H, {}_{2cd-2d-c+1-i}L]]$$
$$\leq [H^j, L^{2d(c-j)-d-(c-j)+2+2dj-d-j-i} \cap H]$$

(also using the fact that $[H, {}_tL] \leq L^{t+1}$)

$$\leq [H^j, L^{2d(c-j)-d-(c-j)+2} \cap H] \text{ since } 2dj-d-j \geq i$$
$$\leq [H^j, H^{c-j+1} \cap H] \text{ by induction if } c-j \neq 0, \text{ and}$$

obviously if $c-j = 0$

$$\leq H^{c+1}$$
$$= 0.$$

Thus $L^{2cd-c-d+2} = 0$ and the induction hypothesis carries over. The result follows.

Next we show that this value of μ_1 is best possible, in the sense that for all $c, d > 0$ there exist Lie algebras L, H satisfying the hypotheses of the theorem, such that L is nilpotent of class *precisely* $cd+(c-1)(d-1)$.

Now in [14] A. G. R. Stewart constructs a nilpotent *torsion-free* group G having a normal subgroup N with N nilpotent of class c, G/N' nilpotent of class d, and G nilpotent of class precisely $cd+(c-1)(d-1)$. Let $\bar{G}$ be the completion of G, $\bar{N}$ the completion of N. Put $L = \mathcal{L}(\bar{G})$, $H = \mathcal{L}(\bar{N})$. Using the results of chapter 2 it is easily seen that these have the required properties.

3.3 The class $\mathfrak{X}_n$

Write $L \in \mathfrak{X}_n \Leftrightarrow \langle H^L \rangle^n \leq H$ for all $H \leq L$.

LEMMA 3.3.1
$\mathfrak{X}_n = \mathrm{QS}\mathfrak{X}_n$.

PROOF:
Trivial.

LEMMA 3.3.2
$\mathfrak{D}_n \cap \mathfrak{A}^2 \leq \mathfrak{X}_n$.

PROOF:
Let $H \leq L \in \mathfrak{D}_n \cap \mathfrak{A}^2$, so that $L^{(2)} = 0$. We show by induction on m that

$$\langle H^L \rangle^m \leq H^m + \sum_{i=1}^{\infty} [[H, {}_iL], {}_{m-1}H].$$

$m = 1$:

$$\langle H^L \rangle = H + \sum_{i=1}^{\infty} [H, {}_iL] \text{ obviously. } (*)$$

$m = 2$:

$$\langle H^L \rangle^2 = [H + \Sigma[H, {}_iL], H + \Sigma[H, {}_jL]] \text{ from } (*)$$

$$\leqq [H, H] + \Sigma[[H, {}_iL], H]$$

since $L^2 \in \mathfrak{A}$.

$m > 2$:

$$\langle H^L \rangle^m \leqq [H^{m-1} + \Sigma[[H, {}_iL], {}_{m-2}H], H + \Sigma[H, {}_jL]]$$

$$\leqq H^m + \Sigma[[H, {}_iL], {}_{m-1}H]$$

since $L^2 \in \mathfrak{A}$.

Now if $L \in \mathfrak{D}_n$ it is clear that $[L, {}_nH] \leqq H$, and consequently $\langle H^L \rangle^n \leqq H^n + H = H$, which shows that $L \in \mathfrak{X}_n$ as claimed.

LEMMA 3.3.3
If K is a minimal ideal of $L \in \text{L}\mathfrak{N}$ then $K \leqq \zeta_1(L)$.

PROOF:
See Hartley [5] lemma 10 p. 269.

LEMMA 3.3.4
If $K \lhd L \in \text{L}\mathfrak{N}$ and $K \in \mathfrak{F}_h$, then $K \leqq \zeta_h(L)$.

PROOF:
Induction on h. If $h = 0$ the result is clear. Let $0 = K_0 < K_1 < \cdots < K_\alpha = K$ be a series of ideals $K_i \lhd L$ ($i = 0, \cdots, \alpha$) such that the series cannot be refined (this exists since K is finite-dimensional). Then K_{i+1}/K_i is a minimal ideal of L/K_i. By our induction hypothesis $K_{\alpha-1} \leqq \zeta_{h-1}(L)$, and $K_\alpha + \zeta_{h-1}(L)/\zeta_{h-1}(L)$ is a minimal ideal of $L/\zeta_{h-1}(L)$, so by lemma 3.3.3 it is contained in $\zeta_1(L/\zeta_{h-1}(L))$ which implies $K \leqq \zeta_h(L)$. The result follows.

LEMMA 3.3.5
If $H \leqq L \in \mathfrak{N}_r \cap \mathfrak{G}_s$ then $H \in \mathfrak{F}_{\mu_2(r,s)}$ where

$$\mu_2(r, s) = s + s^2 + \cdots + s^r.$$

PROOF:
It is sufficient to show $L \in \mathfrak{F}_{\mu_2(r,s)}$. Now L is spanned (qua vector space) by commutators of the form $[g_1, \cdots, g_i]$ ($i \leqq r$) where the g_j are chosen from the given set of s generators. This gives the result.

Next we need an unpublished theorem of B. Hartley:

THEOREM 3.3.6 (Hartley)
$\mathfrak{J} \leqq \text{L}\mathfrak{N}$.

PROOF:
Let $L \in \mathfrak{J}$, and let M be maximal with respect to $M \leqq L$, $M \in \text{L}\mathfrak{N}$

(such an M exists by a Zorn's lemma argument). Let $u \in I = I_L(M)$. Then $K = M + \langle u \rangle \leqq L$. $L \in \mathfrak{F}$ so $K \in \mathfrak{F}$, from which it is easy to deduce that K has an ascending series $(U_\alpha)_{\alpha \leqq \sigma}$ with $U_1 = \langle u \rangle$. Then

$$U_\alpha = (M \cap U_\alpha) + (\langle u \rangle \cap U_\alpha)$$
$$= (M \cap U_\alpha) + \langle u \rangle,$$

so

(*) $U_{\alpha+1} = (M \cap U_{\alpha+1}) + U_\alpha$.

We show by transfinite induction on α that $U_\alpha \in \mathrm{L}\mathfrak{N}$. $U_1 = \langle u \rangle \in \mathfrak{A} \leqq \mathrm{L}\mathfrak{N}$. $M \cap U_{\alpha+1} \lhd U_{\alpha+1}$ (since $M \lhd K$) and $M \cap U_{\alpha+1} \in \mathrm{L}\mathfrak{N}$; also $U_\alpha \lhd U_{\alpha+1}$ and $U_\alpha \in \mathrm{L}\mathfrak{N}$. By Hartley [5] lemma 7 p. 265 and (*) $U_{\alpha+1} \in \mathrm{L}\mathfrak{N}$. At limit ordinals the induction step is clear. Hence $U_\sigma = K \in \mathrm{L}\mathfrak{N}$. By maximality of M we have $K = M$, so $I = M$. But $L \in \mathfrak{F}$ so $M = L$. Therefore $L \in \mathrm{L}\mathfrak{N}$ which finishes the proof.

LEMMA 3.3.7
$\mathfrak{D}_n \leqq \mathfrak{D} \leqq \mathrm{L}\mathfrak{N}$.

PROOF:
Clearly $\mathfrak{D}_n \leqq \mathfrak{D} \leqq \mathfrak{F}$. Now use theorem 3.3.6.

LEMMA 3.3.8
If $x \in L \in \mathfrak{X}_n$, then $\langle x^L \rangle \in \mathfrak{N}_n$.

PROOF:
$\langle x^L \rangle^n \leqq \langle x \rangle$ since $L \in \mathfrak{X}_n$. If $\langle x^L \rangle^n = 0$ we are home. If not, then $\langle x \rangle = \langle x^L \rangle^n$ ch $\langle x^L \rangle \lhd L$, so $\langle x \rangle \lhd L$. Thus $x \in C_L(x) \lhd L$, so $\langle x^L \rangle \leqq C_L(x)$ and $\langle x^L \rangle^{n+1} = 0$ as claimed.

LEMMA 3.3.9
$\mathfrak{A}^2 \cap \mathfrak{D}_n \leqq \mathfrak{N}_{\mu_3(n)}$.

PROOF:
Let $L \in \mathfrak{A}^2 \cap \mathfrak{D}_n$. $L^n = \langle [x_1, \cdots, x_n]^L : x_i \in L \rangle$. Let $X = \langle x_1, \cdots, x_n \rangle$. By lemma 3.3.2 $L \in \mathfrak{X}_n$, so if $x \in L$, then $\langle x^L \rangle \in \mathfrak{N}_n$ by lemma 3.3.8. Let $T = \langle X^L \rangle = \langle x_1^L \rangle + \cdots + \langle x_n^L \rangle$, a sum of n $\mathfrak{N}_n$-ideals of L. By Hartley [5] lemma 1 (iii) p. 261 $T \in \mathfrak{N}_{n^2}$. Thus $X \in \mathfrak{N}_{n^2} \cap \mathfrak{G}_n$, so by lemma 3.3.5 every subalgebra of X has dimension $\leqq r = \mu_2(n^2, n)$. $L \in \mathfrak{X}_n$ so $T^n \leqq X$. $Y = \langle [x_1, \cdots, x_n]^L \rangle \leqq T^n \leqq X$ so dim $(Y) \leqq r$. By lemma 3.3.7 $\mathfrak{D}_n \leqq \mathrm{L}\mathfrak{N}$, and $Y \lhd L$; consequently lemma 3.3.4 applies and $Y \leqq \zeta_r(L)$. Thus $L^n \leqq \zeta_r(L)$, and $L \in \mathfrak{N}_{n+r}$.

We may therefore take $\mu_3(n) = n + \mu_2(n^2, n)$.

LEMMA 3.3.10
$\mathfrak{A}^d \cap \mathfrak{D}_n \leqq \mathfrak{N}_{\mu_4(n,d)}$.

PROOF:

Induction on d. If $d = 1$ we may take $\mu_4(n, 1) = 1$. If $d = 2$, then by lemma 3.3.9 we may take $\mu_4(n, 2) = \mu_3(n)$. If $d > 2$, let $M = L^{(d-2)}$. Then $M \in \mathfrak{A}^2 \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu_3(n)}$ by lemma 3.3.9, an $L/M^2 \in \mathfrak{A}^{d-1} \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu_4(n,d-1)}$ by induction. By theorem 3.2.3

$$L \in \mathfrak{N}_{\mu_4(n,d)}$$

where

$$\mu_4(n, d) = ab + (a-1)(b-1),$$
$$a = \mu_3(n), \ b = \mu_4(n, d-1).$$

LEMMA 3.3.11

If $0 \neq A \vartriangleleft L \in \mathfrak{N}$ then $A \cap \zeta_1(L) \neq 0$.

PROOF:

See Schenkman [13] lemma 8.

Define

$$\alpha(L) = \{x \in L : \langle x^L \rangle \in \mathfrak{A}\}$$
$$\alpha_n(L) = \{x \in L : \langle x^L \rangle \in \mathfrak{A} \cap \mathfrak{F}_n\}.$$

LEMMA 3.3.12

If $L = \langle \alpha_n(L) \rangle$ then $L \in \mathfrak{N}_n$.

PROOF:

L is generated by abelian ideals, so by lemma 1 (iii) of Hartley [5] p. 261 $L \in \mathfrak{L}\mathfrak{N}$. Let the abelian ideals which generate L and are of dimension $\leq n$ be $\{A_\lambda : \lambda \in \Lambda\}$. By lemma 3.3.4 $A_\lambda \leq \zeta_n(L)$ so $L = \zeta_n(L)$ as required.

LEMMA 3.3.13

If $H = \langle \alpha(H) \rangle$ and $H \in \mathfrak{X}_n$ then $H \in \mathfrak{A}^{[\mu_5(n)]}$.

PROOF:

It is easily seen that $H^n = \langle [x_1, \cdots, x_n]^H : x_i \in \alpha(H) \rangle$. Let $X = \langle x_1, \cdots, x_n \rangle$. $\langle X^H \rangle = T = \langle x_1^H \rangle + \cdots + \langle x_n^H \rangle \in \mathfrak{N}_n$ by Hartley [14] lemma 1 (iii) p. 261. Since $H \in \mathfrak{X}_n$ $T^n \leq X \in \mathfrak{G}_n \cap \mathfrak{N}_n$. Therefore if $Y = \langle [x_1, \cdots, x_n]^H \rangle$ then $Y \leq T^n \leq X$ so by lemma 3.3.5 $Y \in \mathfrak{F}_{\mu_2(n,n)}$. $Y \leq \langle x_1^H \rangle \in \mathfrak{A}$ so $Y \in \mathfrak{A} \cap \mathfrak{F}_{\mu_2(n,n)}$. Therefore $H^n \leq \langle \alpha_{\mu_2(n,n)}(H) \rangle = D$, say, and $D = \langle \alpha_{\mu_2(n,n)}(D) \rangle$. Thus $H/D \in \mathfrak{A}^{n-1}$, and by lemma 3.3.12 $D \in \mathfrak{N}_{\mu_2(n,n)} \leq \mathfrak{A}^{[\mu_2(n,n)]}$. Therefore $H \in \mathfrak{A}^{[\mu_5(n)]}$ where

$$\mu_5(n) = n - 1 + \mu_2(n, n).$$

LEMMA 3.3.14

$\mathfrak{X}_n \leq \mathfrak{A}^{[\mu_6(n)]}$.

PROOF:

Let $H \leqq L \in \mathfrak{X}_n$. Then $H \geqq \langle H^L \rangle^n \lhd L$. $\langle H^L \rangle / \langle H^L \rangle^n \in \mathfrak{N}_{n-1}$, so by Hartley [5] lemma 1 (ii) p. 261 $H/\langle H^L \rangle^n \lhd^{n-1} \langle H^L \rangle / \langle H^L \rangle^n$, so $H \lhd^{n-1} \langle H^L \rangle \lhd L$. Thus $H \lhd^n L$ and $L \in \mathfrak{D}_n$. Hence $\mathfrak{X}_n \leqq \mathfrak{D}_n \leqq \text{L}\mathfrak{N}$ by lemma 3.3.7.

By lemma 3.3.8 $x \in L \Rightarrow \langle x^L \rangle \in \mathfrak{N}_n$. So if we define

$$L_1 = \Sigma\{A : A \lhd L, A \in \mathfrak{A}\}$$

then $L_1 > 0$ (since e.g. $0 \neq \zeta_1(\langle x^L \rangle) \leqq L_1$). Similarly let

$$L_{i+1}/L_i = \Sigma\{A : A \lhd L/L_i, A \in \mathfrak{A}\}.$$

Then

$$0 < L_1 \leqq L_2 \leqq \cdots.$$

Let $y \in L$. Then $Y = \langle y^L \rangle \lhd L$ and $Y \in \mathfrak{N}_n$. An easy induction shows $\zeta_i(Y) \leqq L_i$ so $y \in L_n$. Therefore $L_n = L$. By lemma 3.3.1 $L_{i+1}/L_i \in \mathfrak{X}_n$, and clearly we have $L_{i+1}/L_i = \langle \alpha(L_{i+1}/L_i) \rangle$, so by lemma 3.3.13

$$L_{i+1}/L_i \in \mathfrak{A}^{\mu_5(n)}.$$

Thus $L \in \mathfrak{A}^{\mu_6(n)}$ where $\mu_6(n) = n\mu_5(n)$.

We have now set up most of the machinery needed to prove the main result by induction; this is done in the next section.

3.4 The Induction Step

LEMMA 3.4.1
$\mathfrak{D}_n = \text{QS}\mathfrak{D}_n$.

PROOF:
Trivial.

LEMMA 3.4.2
$\mathfrak{D}_1 = \mathfrak{N}_1 = \mathfrak{A}$.

PROOF:

Let $x, y \in L \in \mathfrak{D}_1$. Then $\langle x \rangle, \langle y \rangle \lhd L$. If x and y are linearly independent then $[x, y] \in \langle x \rangle \cap \langle y \rangle = 0$. If x and y are linearly dependent then $[x, y] = 0$ anyway. Thus $L \in \mathfrak{A} = \mathfrak{N}_1$.

We now define the *ideal closure series* of a subalgebra of a Lie algebra. Let L be a Lie algebra, $K \leqq L$. Define $K_0 = L$, $K_{i+1} = \langle K^{K_i} \rangle$. The series

$$K_0 \geqq K_1 \geqq \cdots \geqq K_n \geqq \cdots$$

is the ideal closure series of K in L.

LEMMA 3.4.3
1) *If $K = L_n \lhd L_{n-1} \lhd \cdots \lhd L_0 = L$ then $L_i \geqq K_i$ for $i = 0, \cdots, n$.*
2) *$K \lhd^n L$ if and only if $K_n = K$.*

PROOF:

1) By induction. For $i = 0$ we have equality. Now $K_{i+1} = \langle K^{K_i} \rangle \leqq \langle K^{L_i} \rangle \leqq L_{i+1}$ so the induction step goes through.

2) Clearly $K_{i+1} \lhd K_i$, so that if $K_n = K$ then

$$K = K_n \lhd K_{n-1} \lhd \cdots \lhd K_0 = L.$$

On the other hand, if $K \lhd^n L$ then

$$K = L_n \lhd L_{n-1} \lhd \cdots \lhd L_0 = L,$$

and by part (1) $K \leqq K_n \leqq L_n = K$.

LEMMA 3.4.4
Let $H \leqq L \in \mathfrak{D}_n$, H_i the i-th term of the ideal closure series of H in L. Then $H_i/H_{i+1} \in \mathfrak{D}_{n-i}$.

PROOF:

$$H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_{i+1} \lhd H_i \lhd \cdots \lhd H_0 = L.$$

Suppose $H_{i+1} \leqq K \leqq H_i$. If $j \leqq i$ then $K_j \leqq H_j$ by lemma 3.4.3.1, so $K_i \leqq H_i$. But $H \leqq H_{i+1} \leqq K$ so an easy induction on j shows that $H_j \leqq K_j$. Thus $H_i = K_i$. But $L \in \mathfrak{D}_n$ so $K \lhd^n L$, and K has ideal closure series

$$K = K_n \lhd K_{n-1} \lhd \cdots \lhd K_i \lhd \cdots \lhd K_0 = L.$$

Therefore

$$K = K_n \lhd K_{n-1} \lhd \cdots \lhd K_i = H_i, \text{ and } K \lhd^{n-i} H_i.$$

Thus $K/H_{i+1} \lhd^{n-1} H_i/H_{i+1}$ and the lemma is proved.

It is this result that provides the basis for an induction proof of our main result in this chapter, which follows:

THEOREM 3.4.5
$\mathfrak{D}_n \leqq \mathfrak{N}_{\mu(n)}$.

PROOF:

As promised, by induction on n.

If $n = 1$ then by lemma 3.4.2 we may take $\mu(1) = 1$. If $n > 1$ let $L \in \mathfrak{D}_n$, $H \leqq L$. By lemma 3.4.4, if $i \geqq 1$ $H_i/H_{i+1} \in \mathfrak{D}_{n-i} \leqq \mathfrak{D}_{n-1} \leqq \mathfrak{N}_{\mu(n-1)}$ by inductive hypothesis. Let $m = \mu(n-1)$. Then certainly $H_i/H_{i+1} \in \mathfrak{A}^m$, and so $H_1^{(m(n-1))} \leqq H$ for all $H \leqq L$. Let $Q = H_1/H_1^{(m(n-1))} \in \mathfrak{D}_n \cap \mathfrak{A}^{m(n-1)}$. By lemma 3.3.10 $Q \in \mathfrak{N}_c$, where $c = \mu_4(n, m(n-1))$. Thus $Q^{c+1} = 0$ so $H_1^{c+1} \leqq H_1^{(m(n-1))} \leqq H$, so that

$L \in \mathfrak{X}_{c+1}$. By lemma 3.3.14 $L \in \mathfrak{A}^d$ where $d = \mu_6(c+1)$. Finally therefore $L \in \mathfrak{A}^d \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu(n)}$ by lemma 3.3.10, where

$$\mu(n) = \mu_4(n, \mu_6(1 + \mu_4(n, (n-1) \cdot \mu(n-1)))).$$

The theorem is proved.

REMARK

The value of $\mu(n)$ so obtained becomes astronomical even for small n, and is by no means best possible. However, without modifying the argument it is hard to improve it significantly.

It is not hard to see that this is equivalent to the following theorem, which is stated purely in finite-dimensional terms:

THEOREM 3.4.6.

There exists a function $\delta(n)$ of the integer n, taking positive integer values, and tending to infinity with n, such that any finite-dimensional nilpotent Lie algebra of class n has a subalgebra which is not a $\delta(n)$-step subideal.

This result holds both for characteristic zero and the modular case.

Using the Mal'cev correspondence we can now prove:

THEOREM 3.4.7

Let G be a complete torsion-free R-group (in the sense of lemma 2.1.2) such that if H is a complete subgroup of G then $H \lhd^n G$. Then G is nilpotent of class $\leq \mu(n)$.

PROOF:

Let $x \in G$, $X = \{x^\lambda : \lambda \in \mathbf{Q}\}$. Since G is a complete R-group $X \cong \mathbf{Q}$ (under addition) so X is abelian and complete. Therefore $\langle x \rangle \lhd X \lhd^n G$, so $\langle x \rangle$ is subnormal in G and G is a Baer group (Baer calls then *nilgroups*) so is locally nilpotent (Baer [1] § 3 Zusatsz 2). G is also complete and torsion-free so we may form the Lie algebra $\mathscr{L}(G)$ over $\mathbf{Q}$. If $K \leq \mathscr{L}(G)$ then $\mathscr{G}(K)$ is a complete subgroup of G (theorem 2.4.2) so $\mathscr{G}(K) \lhd^n G$. By lemma 2.4.5 $K \lhd^n \mathscr{L}(G)$. By theorem 3.4.5 $\mathscr{L}(G) \in \mathfrak{D}_n \leq \mathfrak{N}_{\mu(n)}$. By theorem 2.5.4 G is nilpotent of class $\leq \mu(n)$.

We may also recover Roseblade's original result for the case of torsion-free goups. Suppose G is a torsion-free group, every subgroup of which is subnormal of defect $\leq n$. Then G is a Baer group so is locally nilpotent. Let $\bar{G}$ be the completion of G (Note: we must again avoid Mal'cev and appeal either to Kargapolov or Hall in order to maintain algebraic purity). Then every complete subgroup of $\bar{G}$ is the completion of its intersection with G (Kuroš [8] p. 257) which is $\lhd^n G$. By lemma 2.4.4 we deduce that every complete subgroup of $\bar{G}$ is $\lhd^n \bar{G}$. $\bar{G}$ is a complete R-group, so theorem 3.4.6 applies.

We have not been able to decide whether or not $\mathfrak{D} = \mathfrak{N}$. The corresponding result for groups is now known to be false (Heineken and Mohamed [6]) but their counterexample is a p-group; so we cannot use the Mal'cev correspondence to produce a counterexample for the Lie algebra case.

4. Chain Conditions in special classes of Lie algebras

We now investigate the effect of imposing chain conditions (both maximal and minimal) on more specialised classes of Lie algebras, with particular regard to locally nilpotent Lie algebras. Application of the Mal'cev correspondence then produces some information on chain conditions for complete subgroups of complete locally nilpotent torsion-free groups. Additional notation will be as in [15].

4.1 *Minimal Conditions*

[15] Lemma 2.7 immediately implies

PROPOSITION 4.1.1
$\mathfrak{L}\mathfrak{N} \cap \text{Min}-\lhd^2 = \mathfrak{N} \cap \mathfrak{F}$.

If we relax the condition to Min$-\lhd$ lemma 2.6 of [15] shows that $\mathfrak{L}\mathfrak{N} \cap \text{Min}-\lhd \leqq \mathfrak{E}\mathfrak{A} \cap \mathfrak{Z}$. But in contrast to proposition 4.1.1 we have

PROPOSITION 4.1.2
$\mathfrak{L}\mathfrak{N} \cap \text{Min}-\lhd \nleqq \mathfrak{N} \cup \mathfrak{F}$.

PROOF:
Let $\mathfrak{k}$ be any field. Let A be an abelian Lie algebra of countable dimension over $\mathfrak{k}$, with basis $(x_n)_{0 < n \in \mathbf{Z}}$. There is a derivation σ of A defined by

$$x_i \sigma = x_{i-1} \qquad (i > 1)$$
$$x_1 \sigma = 0.$$

Let L be the split extension (Jacobson [7] p. 18) $A \oplus \langle \sigma \rangle$. Clearly $L \in \mathfrak{L}\mathfrak{N}\backslash(\mathfrak{N} \cup \mathfrak{F})$. Let $A_i = \langle x_1, \cdots, x_i \rangle$. We show that the only ideals of L are 0, A_i $(i > 0)$, A, or L. For let $I \lhd L$, and suppose $I \nleqq A$. Then there exists $\lambda \neq 0$, $\lambda \in \mathfrak{k}$, and $x \in A$, such that $\lambda\sigma + x \in I$. Then $x_i = [\lambda^{-1} x_{i+1}, \lambda\sigma + x] \in I$ so $A \leqq I$. Thus $x \in I$, so $\sigma \in I$, and $I = L$.

Otherwise suppose $0 \neq I \leqq A$. For some $n \in \mathbf{Z}$ we have

$$x = \lambda_n x_n + \lambda_{n-1} x_{n-1} + \cdots + \lambda_1 x_1 \in I$$

where $0 \neq \lambda_n$, $\lambda_i \in \mathfrak{k}$ $(i = 1, \cdots, n)$. Then $[\lambda_n^{-1} x, {}_{n-1}\mathbf{Q}] = x_1 \in I$. Suppose inductively that $A_m \leqq I$ for some $m < n$. Then $[\lambda_n^{-1} x, {}_{n-m-1}\sigma]$

$\in I$, and this equals $x_{m+1}+y$ for some $y \in A_m$. Thus $x_{m+1} \in I$ and $A_{m+1} \leq I$. From this we deduce that either $I = A_n$ for some n or $I = A$.

Thus the set of ideals of L is well-ordered by inclusion, so $L \in \text{Min} - \vartriangleleft$.

For Lie algebras satisfying Min−si we may define a *soluble radical* (which has slightly stronger properties when the underlying field has characteristic zero).

THEOREM 4.1.3

Let L be a Lie algebra over a field of characteristic zero, satisfying Min-si. Then L has a unique maximal soluble ideal $\sigma(L)$. $\sigma(L) \in \mathfrak{F}$ and contains every soluble subideal of L.

PROOF:

Let $F = \delta(L)$ be the $\mathfrak{F}$-residual of L (i.e. the unique minimal ideal F with $L/F \in \mathfrak{F}$), $\beta(L)$ the Baer radical (see [15]). Let $\dim (L/F) = f$, $\dim (\beta(F)) = b$. Both f and b are finite. Define $B_1 = \beta(L)$, $B_{i+1}/B_i = \beta(L/B_i)$. By [15] lemmas 2.4 and 2.7 $B_i \in \text{E}\mathfrak{A} \cap \mathfrak{F}$. $B_i \cap F \vartriangleleft F$ and as in [15] theorem 3.1 F has no proper ideals of finite codimension, so by the usual centraliser argument $B_i \cap F$ is central in F, so $B_i \cap F \leq \beta(F)$. $\dim (B_i) = \dim(B_i \cap F) + \dim(B_i + F/F) \leq b + f$. Consequently $B_{i+1} = B_i$ for some i. Let $\sigma(L) = B_i$. Then $\sigma(L) \vartriangleleft L$, $\sigma(L) \in \text{E}\mathfrak{A} \cap \mathfrak{F}$. $L/\sigma(L)$ contains no abelian subideals, and hence no soluble subideals, other than 0. Thus $\sigma(L)$ contains every soluble subideal of L as claimed.

For the characteristic $p \neq 0$ case we prove rather less:

THEOREM 4.1.4

Let L be a Lie algebra over a field of characteristic > 0, and suppose $L \in \text{Min-si}$. Then L has a unique maximal soluble ideal $\sigma(L)$, and $\sigma(L) \in \mathfrak{F}$.

PROOF:

Let $F = \delta(L)$ as in the previous theorem. Suppose $S \vartriangleleft L$, $S \in \text{E}\mathfrak{A}$. Then $S \in \text{E}\mathfrak{A} \cap \text{Min-si} \leq \mathfrak{F}$, so $F \cap S \in \mathfrak{F}$. The usual argument shows $F \cap S \leq \zeta_1(F) \in \mathfrak{F} \cap \mathfrak{A}$. Let $\dim(\zeta_1(F)) = z$, $\dim (L/F) = f$. Then $\dim(S) = \dim(F \cap S) + \dim(S + F/F) \leq z + f$. Clearly the sum of two soluble ideals of L is a soluble ideal; the above shows that the sum of *all* the soluble ideals of L is in fact the sum of a finite number of them, so satisfies the required conclusions for $\sigma(L)$.

Suppose now that $\mathfrak{W}$ denotes the class of Lie algebras L such that every non-trivial homomorphic image of L has a non-trivial abelian subideal; and let $\mathfrak{B}$ denote the class of all Lie algebras L such that every non-trivial homomorphic image of L has a non-trivial abelian ideal. Then immediately we have

THEOREM 4.1.5

1) *For fields of characteristic zero*

$$\mathfrak{W} \cap \text{Min-si} = E\mathfrak{A} \cap \mathfrak{F}.$$

2) *For arbitrary fields*

$$\mathfrak{V} \cap \text{Min-si} = E\mathfrak{A} \cap \mathfrak{F}.$$

PROOF:

If L satisfies the hypotheses then we must have $L = \sigma(L) \in E\mathfrak{A} \cap \mathfrak{F}$ as required. The converse is clear.

Digression

It is not hard to find alternative characterisations of the classes $\mathfrak{V}$, $\mathfrak{W}$. $\mathfrak{V}$ is clearly the class of all Lie algebras possessing an ascending $\mathfrak{A}$-series of ideals. These are the Lie analogues of the *SI**-groups of Kuroš [8] p. 183. $\mathfrak{W}$ is the Lie analogue of Baer's subsoluble groups (see [1]), which Phillips and Combrink [9] show to be the same as *SJ**-groups (same reference for notation). A simple adaptation of their argument shows that $\mathfrak{W}$ consists precisely of all Lie algebras possessing an ascending $\mathfrak{A}$-series of subideals. We omit the details.

A useful corollary of theorem 4.1.5 follows from

LEMMA 4.1.6

A minimal ideal of a locally soluble Lie algebra is abelian.

PROOF:

Let N be a minimal ideal of $L \in \text{LE}\mathfrak{A}$ and suppose $N \notin \mathfrak{A}$. Then there exist $a, b \in N$ such that $[a, b] = c \neq 0$. By minimality $N = \langle c^L \rangle$ so there exist $x_1, \cdots, x_n \in L$ such that $a, b \in \langle c, x_1, \cdots, x_n \rangle = H$, say. $L \in \text{LE}\mathfrak{A}$ so $H \in E\mathfrak{A}$. Now $C = \langle c^H \rangle \lhd H$, and $a, b \in C$, so $c = [a, b] \in C^2$ ch $C \lhd H$, so $c \in C^2 \lhd H$, and $C = C^2$. But $C \leq H \in E\mathfrak{A}$, a contradiction. Thus $N \in \mathfrak{A}$.

COROLLARY

$$\text{ELE}\mathfrak{A} \cap \text{Min-si} = E\mathfrak{A} \cap \mathfrak{F}.$$

PROOF:

It is sufficient to show $\text{LE}\mathfrak{A} \cap \text{Min-si} \leq E\mathfrak{A} \cap \mathfrak{F}$. By lemma 4.1.6 $\text{LE}\mathfrak{A} \cap \text{Min-si} \leq \mathfrak{V}$ (since $\text{LE}\mathfrak{A}$ is Q-closed). Theorem 4.1.5 finishes the job.

4.2. *Maximal Conditions*

Exactly as in [15] we may define maximal conditions for subideals, namely Max-si, Max-$\lhd^n$, and Max-$\lhd$. We do not expect any results like theorem 2.1 of [15] and confine our attention mainly to Max-$\lhd$.

LEMMA 4.2.1

$$E\mathfrak{A} \cap \text{Max-}\lhd \leq \mathfrak{G}.$$

PROOF:

We show by induction on d that $\mathfrak{A}^d \cap$ Max-$\lhd \leqq \mathfrak{G}$. If $d = 1$ then $L \in \mathfrak{A} \cap$ Max-$\lhd \leqq \mathfrak{F} \leqq \mathfrak{G}$. Suppose $L \in \mathfrak{A}^d \cap$ Max-$\lhd$, and let $A = L^{(c-1)}$. $L/A \in \mathfrak{A}^{d-1}$ and $L/A \in$ Max-$\lhd$, so $L/A \in \mathfrak{G}$ by induction. $A \in \mathfrak{A}$. There exists $H \in \mathfrak{G}$ such that $L = A + H$ (Let H be generated by coset representatives of A in L corresponding to generators of L/A.) By Max-$\lhd$ there exist $a_1, \cdots, a_n \in A$ such that $A = \langle a_1^L \rangle + \cdots + \langle a_n^L \rangle$. But if $a \in A$, $h \in H$, then $[a_i, a+h] = [a_i, h]$ so $A + H = \langle a_1^H \rangle + \cdots + \langle a_n^H \rangle = \langle a_1, \cdots, a_n, H \rangle \in \mathfrak{G}$.

REMARK

It is not true that $E\mathfrak{A} \cap$ Max-$\lhd \leqq \mathfrak{F}$. The example discussed in Hartley [5] section 7 p. 269 shows this – indeed it shows that even $E\mathfrak{A} \cap$ Max-$\lhd \cap$ Min-$\lhd$ is not contained in $\mathfrak{F}$. This contrasts with a well known theorem of P. Hall which states that a soluble group satisfying maximal and minimal conditions for normal subgroups is necessarily finite.

It is easy to show that $E\mathfrak{A} \cap$ Max-$\lhd^2 = E\mathfrak{A} \cap \mathfrak{F}$.

LEMMA 4.2.2

Let $H \lhd L \in LE\mathfrak{A} \cap$ Max-$\lhd$. Then $H = 0$ or $H^2 < H$.

PROOF:

Let $P = \bigcap H^{(\alpha)}$. Then P ch $H \lhd L$ so $P \lhd L$. Suppose if possible $P \neq 0$. Then there exists K maximal with respect to $K \lhd L$, $K < P$. P/K is a minimal ideal of $L/K \in LE\mathfrak{A}$, so by lemma 4.1.6 $P/K \in \mathfrak{A}$, so that $P^2 < P$ contradicting the definition of P. Thus $P = 0$ (so $H^2 < H$) or $H = 0$.

LEMMA 4.2.3

If $H \leqq L \in \mathfrak{N}$ and $L = H + L^2$, then $H = L$.

PROOF:

We show by induction on n that $H + L^n = L$. If $n = 2$ this is our hypothesis. Now $H + L^n = H + (H + L^2)^n = H + H^n + L^{n+1} = H + L^{n+1}$, so $L = H + L^{n+1}$ as required. For large enough n $L^n = 0$ so $L = H$.

LEMMA 4.2.4

Let L be any Lie algebra with $P \lhd L$, $H \leqq L$, such that $L = H + P^2$. Then $L = H + P^n$ for any integer n.

PROOF:

We show $P = (H \cap P) + P^n$. Now $P = (H \cap P) + P^2$. Modulo P^n we are in the situation of lemma 4.2.3, so $P \equiv (H \cap P) \pmod{P^n}$, which provides the result.

Let $\mathfrak{Y}$ be any class of Lie algebras, L any Lie algebra. Define

$$\lambda(L, \mathfrak{Y}) = \bigcap \{N : N \vartriangleleft L, L/N \in \mathfrak{Y}\}.$$

LEMMA 4.2.5

If $L \in$ LE$\mathfrak{A} \cap$ Max-$\vartriangleleft$ *and* $L_k = \lambda(L, \mathfrak{N}^k)$, *then* $L/L_k \in$ E$\mathfrak{A}$.

PROOF:

Induction on k. If $k = 0$ the result is trivial. If $k \geqq 0$ assume $L/L_k \in$ E$\mathfrak{A}$. Then $L/L_k^2 \in$ E$\mathfrak{A} \cap$ Max-$\vartriangleleft \leqq \mathfrak{G}$ (by lemma 4.2.1). Thus there exists $H \leqq L$, $H \in \mathfrak{G}$, such that $L = H + L_k^2$ (coset representatives again). Since $L \in$ LE$\mathfrak{A}$ $H \in \mathfrak{A}^d$ for some d. Let $Q \vartriangleleft L$ with $L/Q \in \mathfrak{N}^{k+1}$. Then there exists $P \vartriangleleft L$ with $Q \leqq P$, $P/Q \in \mathfrak{N}$, $L/P \in \mathfrak{N}^k$. By definition $L_k \leqq P$ so $L_k^2 \leqq P^2$ and $L = H + P^2$. By lemma 4.2.4 $L = H + P^n$ for any n, so $L = H + Q$ ($P/Q \in \mathfrak{N}$). $L/Q \cong H/(H \cap Q) \in \mathfrak{A}^d$. L_{k+1} is the intersection of all such Q, so by standard methods L/L_{k+1} is isomorphic to a subalgebra of the direct sum of all the possible L/Q, all of which lie in $\mathfrak{A}^d$. Therefore $L/L_{k+1} \in \mathfrak{A}^d$ as claimed.

LEMMA 4.2.6

If $L \in$ L$(\mathfrak{N}^k) \cap$ Max-$\vartriangleleft$, *then* $L/L_k \in \mathfrak{N}^k$. *Thus* L_k *is the unique minimal ideal* I *of* L *with* $L/I \in \mathfrak{N}^k$.

PROOF:

By lemma 4.2.5 (since $\mathfrak{N}^k \leqq$ E$\mathfrak{A}$) $L/L_k \in$ E$\mathfrak{A}$. But $L/L_k \in$ Max-$\vartriangleleft$ so by lemma 4.2.1 $L/L_k \in \mathfrak{G}$. The usual argument shows that there exists $X \leqq L$, $X \in \mathfrak{G}$, $L = L_k + X$. Then $L/L_k \cong X/(L_k \cap X)$. $X \in \mathfrak{N}^k$ since $L \in$ L$(\mathfrak{N}^k)$ so $L/L_k \in \mathfrak{N}^k$.

THEOREM 4.2.7

L$(\mathfrak{N}^k) \cap$ Max-$\vartriangleleft \leqq \mathfrak{G} \cap \mathfrak{N}^k$.

PROOF:

Clearly all we need show is that if $L \in$ L$(\mathfrak{N}^k) \cap$ Max-$\vartriangleleft$ then $L \in \mathfrak{G}$. Define L_k as above. Suppose if possible that $L_k \neq 0$. Then $L_k \vartriangleleft L$, so by lemma 4.2.2 $L_k^2 < L_k$. By definition and lemma 4.2.6, $L_{k+1} \leqq L_k^2$, so that $L_{k+1} < L_k$. But $L/L_{k+1} \in$ E$\mathfrak{A} \cap$ Max-$\vartriangleleft$ (lemma 4.2.5) $\leqq \mathfrak{G}$ (lemma 4.2.1). The usual argument now shows $L/L_{k+1} \in \mathfrak{N}^k$, so that $L_k \leqq L_{k+1}$, a contradiction. Thus $L_k = 0$, and $L \cong L/L_k \in$ E$\mathfrak{A} \cap$ Max-$\vartriangleleft$ (lemma 4.2.5) $\leqq \mathfrak{G}$ (lemma 4.2.1).

COROLLARY

L$\mathfrak{N} \cap$ Max-$\vartriangleleft = \mathfrak{F} \cap \mathfrak{N}$.

PROOF:

Put $k = 1$ and note that $\mathfrak{G} \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$.
Compare this with Proposition 4.1.2.

4.3 *Mal'cev Revisited*

In order to apply the results of chapter 2 to obtain corresponding theorems for locally nilpotent torsion-free groups, we must find what property of the complete locally nilpotent torsion-free group G corresponds to the condition $\mathscr{L}(G) \in \mathfrak{F}$.

LEMMA 4.3.1

Let G be a complete locally nilpotent torsion-free group. Then $\mathscr{L}(G) \in \mathfrak{F}$ if and only if G is nilpotent and of finite rank (in the sense of the Mal'cev special rank, see Kuroš [8] p. 158).

PROOF:

If $\mathscr{L}(G) \in \mathfrak{F}$ then $\mathscr{L}(G) \in \mathfrak{F} \cap \mathfrak{N}$ so has a series

$$0 = L_0 \lhd L_1 \lhd \cdots \lhd L_n = \mathscr{L}(G)$$

such that $\dim(L_{i+1}/L_i) = 1$ $(i = 0, \cdots, n-1)$. Thus G has a series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with $G_i = \mathscr{G}(L_i)$. By lemma 2.4.2.5 $G_{i+1}/G_i \cong \mathscr{G}(L_{i+1}/L_i) \cong \mathbf{Q}$ (additive group). $\mathbf{Q}$ is known to be of rank 1, and it is also well-known that extensions of groups of finite rank by groups of finite rank are themselves of finite rank. Thus G is of finite rank. G is nilpotent since $\mathscr{L}(G)$ is.

Conversely suppose G is nilpotent of finite rank. Let

$$1 = Z_0 \leqq Z_1 \leqq \cdots \leqq Z_s = G$$

be the upper central series of G. From lemma 2.4.3 corollary 2 each term Z_i is complete, so is isolated in G. Therefore Z_{i+1}/Z_i is complete, torsion-free, abelian, and of finite rank (since G is of finite rank). By standard abelian group theory, Z_{i+1}/Z_i is isomorphic to a finite direct sum of copies of $\mathbf{Q}$. Hence $\mathscr{L}(Z_{i+1}/Z_i) \in \mathfrak{F}$, so $\mathscr{L}(G) \in \mathfrak{F}$ as required.

This proves the lemma.

REMARK

Let $rr(G)$ denote the *rational rank* of G as defined in the Plotkin survey [10] p. 69. Then under the above circumstances we easily see that $\dim(\mathscr{L}(G)) = rr(G)$. According to [10] p. 72 Gluškov [3] has proved that for locally nilpotent torsion-free groups G the rank of $G = rr(G)$. Consequently $\dim(\mathscr{L}(G)) = \operatorname{rank}(G)$, a stronger result than lemma 4.3.1 (which, however, is sufficient for our purposes and easier to prove).

Applying the correspondence of chapter 2 and using the results of the present chapter, we clearly have

THEOREM 4.3.2

Let G be a complete locally nilpotent torsion-free group. Then the

following conditions are equivalent:

1) *G is nilpotent of finite rank.*

2) *G satisfies the minimal condition for complete subnormal subgroups.*

3) *G satisfies the minimal condition for complete subnormal subgroups of defect* $\leq$ 2.

4) *G satisfies the maximal condition for complete normal subgroups.*

On the other hand *G* may satisfy the minimal condition for complete normal subgroups without being either nilpotent or of finite rank.

(Some of these results have been obtained by Gluškov in [3]).

REFERENCES

R. BAER
[1] Nilgruppen, Math. Z. 62 (1955) 402–437.

CHAO CHONG-YUN
[2] Some characterisations of nilpotent Lie algebras, Math. Z. 103 (1968) 40–42.

V. M. GLUŠKOV
[3] On some questions in the theory of nilpotent and locally nilpotent torsion-free groups, Mat. Sbornik 30 (952) 79–104 (Russian).

P. HALL
[4] Some sufficient conditions for a group to be nilpotent, Illinois. J. Math. 2 (1958) 787–801.

B. HARTLEY
[5] Locally nilpotent ideals of a Lie algebra, Proc. Camb. Phil. Soc. 63 (1967) 257–272.

H. HEINEKEN AND I. J. MOHAMED
[6] A group with trivial centre satisfying the normalizer condition, J. Algebra 10 (1968) 368–376.

N. JACOBSON
[7] Lie algebras, Interscience, New York, 1962.

A. G. KUROŠ
[8] Theory of groups vol. II, Chelsea, New York, 1956. Translated by K. A. Hirsch.

R. E. PHILLIPS AND C. R. COMBRINK
[9] A note on subsolvable groups, Math. Z. 92 (1966) 349–352.

B. I. PLOTKIN
[10] Generalised soluble and generalised nilpotent groups, Amer. Math. Soc. Translations ser. 2 17 (1961) 29–116.

J. E. ROSEBLADE
[12] On groups in which every subgroup is subnormal, J. Algebra 2 (1965) 402–412.

E. SCHENKMAN
[13] A theory of subinvariant Lie algebras, Amer. J. Math. 73 (1951) 453–474.

A. G. R. STEWART
[14] On the class of certain nilpotent groups, Proc. Roy. Soc. Ser. (A) 292 (1966) 374–379.

I. N. STEWART
[15] The minimal condition for subideals of Lie algebras, Math. Z. 111 (1969) 301–310.