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GENERALIZED QUOTIENTS OF HEMIRINGS [1]

by

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1.

This paper is concerned with generalizing some results of ring theory to semiring theory. Using the Iizuka [5] congruence relation, the existence of a quotient semiring for a commutative semisubtractive hemiring is proven. The bulk of section 3 is devoted to stating and proving analogues to several well-known theorems in the theory of generalized quotients of rings.

In ring theory Nagata [9], Zariski and Samuel [11], Dieudonné [3], Lambek [6], and others constructed mathematical concepts through the development and use of certain quotient structures. The primary purpose of this paper is to generalize these results to semirings and make available these new methods for application.

A *hemiring* as LaTorre [7] defined it is a semiring with commutative addition as well as a zero.

CONVENTION. In this paper all semirings considered will be assumed to be hemirings with commutative multiplication.

A *k-ideal* K is an ideal such that if x, $x+y \in K$ then $y \in K$. An *h-ideal* H of R is an ideal of R in which, if x, $z \in R$ and h, $k \in H$ with $x+h+z = k+z$, then $x \in H$. A *prime ideal* of R is a proper ideal A of R in which $x \in A$ or $y \in A$ whenever $xy \in A$. A *primary ideal* A of R is a proper ideal of R such that, if $xy \in A$ an $x \notin A$, then $y^n \in A$ for some positive integer n.

If A is an ideal of R, then the *radical* of A, denoted by $R(A)$, is the set of all $x \in R$ for which $x^n \in A$ for some positive integer n. This is an ideal of R, contains A, and if $1 \in R$ is the intersection of all the prime ideals of R that contain A (see Allen [1]).

An ideal Q of R is *primary to* P if Q is a primary ideal that is contained in a prime ideal of R and if $P = R(Q)$. It is to be noted that P is prime. Under these circumstances P is the *associated prime ideal of* Q.

[1] This paper is taken from the author's dissertation, written under the direction of Professor Ben T. Goldbeck at Texas Christian University.

If $1 \in R$, then R is *Noetherian* if any non-empty set of k-ideals of R has a maximal member with respect to set inclusion. This definition is equivalent to the ascending chain condition on k-ideals and as well to the condition that each k-ideal of R be finitely generated. If R is Noetherian, any homorphic image of R is Noetherian.

A semiring is *semisubtractive* if, for each x and y in the semiring, $x+h = y$ or $x = y+h$ for some h in the semiring.

The *zeroid* of R as introduced by Bourne and Zassenhaus [2] is the set Z of all $x \in R$ such that $x+z = z$ for some $z \in R$, and is an h-ideal of R.

PROPOSITION 1. If $Z = (0)$ and R is semisubtractive, then R satisfies the additive law of cancellation.

The proof is trivial.

2.

An ideal of R is *irreducible* if it is not a finite intersection of k-ideals of R that properly contain it. Clearly a prime ideal is irreducible.

LEMMA 2. *If R is Noetherian, then every k-ideal is a finite intersection of irreducible k-ideals.*

The proof is trivial.

If A is an ideal of R, let $[A]$ denote the intersection of all k-ideals of R that contain A. According to Henriksen [4], $[A]$ *is a k-ideal. If* $x-y = \{z \in R | z+y = x\}$, then $[A] = \cup \{x-y | x, y \in A\}$.

LEMMA 3. *If R is Noetherian and semisubtractive and if $Z = (0)$, then every irreducible k-ideal is primary.*

PROOF. Assume Q is a k-ideal that is not primary, implying there exist $b, c \in R$ with $bc \in Q$, $c \notin Q$, and $b^m \notin Q$ for all m. There exists a positive integer n such that $Q : (b^n) = Q : (b^{n+1})$. Clearly $Q \subset [Q+(b^n)] \cap [Q+(c)] = T$. If $x \in T$, then $q+sb^n+x = q'+s'b^n$ and $p+rc+x = p'+r'c$ for some $p, q, p', q' \in Q$ and $r, s, r', s' \in R$. In multiplying the second equation by b, it follows that $bx \in Q$. There exists $t \in R$ such that $s'+t = s$ or $s' = s+t$. It is sufficient to let $s'+t = s$. Hence $q+s'b^n+tb^n+x = q'+s'b^n$, and thus $q+tb^n+x = q'$. Multiplying by b, it is true that $tb^{n+1} \in Q$, so that $t \in Q : (b^n)$ and $tb^n \in Q$. Thus $x \in Q$ and $Q = T$. Since Q is properly contained in $[Q+(b^n)]$ and in $[Q+(c)]$, Q cannot be irreducible. This proves the lemma.

Let D be a set of primary ideals of R whose intersection is an ideal A. The set D is called a representation for A. Such a representation is *irredundant* if each $Q \in D$ is in a prime ideal of R, if the Q's have distinct associated prime ideals (called associated prime ideals of A), and if no Q

contains the intersection of the others. If D is finite, such a representation is a *finite irredundant primary representation.*

THEOREM 4. *If R is Noetherian and semisubtractive with $Z = (0)$, then every k-ideal of R has a finite irredundant primary representation in which each member of the representation is a k-ideal.*

The proof follows easily from the lemmas.

3.

If R has a multiplicatively cancellable element, then with R^* denoting the multiplicative subsemigroup of multiplicatively cancellable elements of R a *quotient semiring* F of R by R^* is a semiring with the following properties: (1) R is a subsemiring of F, (2) $1 \in F$, (3) each element of R^* has a multiplicative inverse in F, and (4) each element of F has the form $r\rho^{-1}$ with $r \in R$ and $\rho \in R^*$.

Weinert [10] and Murata [8] proved that if R has a multiplicatively cancellable element, then such a quotient structure exists for each R^*. Since R is a commutative hemiring, F is also. If R has additive cancellation, then F does also. The zeroid of R is (0) if and only if the zeroid of F is (0). Note that $r\rho^{-1} = s\sigma^{-1}$ if $r\sigma = \rho s$, $r\rho^{-1}+s\sigma^{-1} = (r\sigma+\rho s)(\rho\sigma)^{-1}$, and $(r\rho^{-1})(s\sigma^{-1}) = (rs)(\rho\sigma)^{-1}$.

Let M be a multiplicative subsemigroup of R such that $M \cap Z = \emptyset$. If $N = \{x \in R | mx \in Z \text{ for some } m \in M\}$, then from the fact that Z is an h-ideal it follows that N is an h-ideal. Define $x[\equiv]y$ (N) if there exist $n, m \in N$ and $r \in R$ such that $x+n+r = y+m+r$. Iizuka [5] introduced this congruence. If $R[/]N$ denotes the corresponding set of equivalence classes, then $R[/]N$ is a commutative hemiring with additive cancellation. The map $x \to C_x$, where C_x is the class that contains x, is the natural homomorphism of R onto $R[/]N$, which will be denoted by ϕ. According to LaTorre [7], N is the zero of $\phi(R)$. The following lemmas and proposition hold trivially.

LEMMA 8. *If R is semisubtractive and has additive cancellation, then each element that is not a zero divisor is multiplicatively cancellable.*

LEMMA 9. *If R is semisubtractive, then any homomorphic image of R is semisubtractive.*

PROPOSITION 10. If R is semisubtractive, then every element of $\phi(M)$ is multiplicatively cancellable.

Hence, if R is semisubtractive, then there exists a quotient semiring of $\phi(R)$ by $\phi(M)$ which will be denoted by R_M.

CONVENTION. It will be assumed that R is semisubtractive. It is to be observed that this implies that R_M is semisubtractive.

If A is an ideal of R, then A^e denotes the ideal it R_M generated by $\phi(A)$. If A^* is an ideal of R_M, then $(A^*)^c$ denotes the ideal $\phi^{-1}(A^* \cap \phi(R))$. Clearly $A^e = \{(a\phi)(m\phi)^{-1} | a \in A, m \in M\}$.

THEOREM 11. *If f is a homomorphism of R to an additively cancellative hemiring S in which mf has an inverse in S for each $m \in M$, then there is a homomorphism g of R_M to S with $f = \phi g$.*

PROOF. It is true that $N \subset \ker(f)$. The map g of R_M to S defined by $((r\phi)(m\phi)^{-1})g = (rf)(mf)^{-1}$ is a homomorphism such that $f = \phi g$.

THEOREM 12. *If A is an h-ideal of R such that $A \cap M = \emptyset$, and if ϕ_1 is the natural homorphism of R onto $R[/]A$, then $R_M[/]A^e$ is isomorphic to $\phi_1(R)_{\phi_1(M)}$.*

PROOF. Let ϕ_2 be the natural homomorphism of $\phi_1(R)$ onto $\phi_1(R)[/]N_1$ where $N_1 = \{x\phi_1 | (mx)\phi_1 = 0 \text{ for some } m \in M\}$. A quotient semiring $\phi_1(R)_{\phi_1(M)}$ of $\phi_2(\phi_1(R))$ by $\phi_2(\phi_1(M))$ exists. By Theorem 11, the map g defined by $((x\phi)(m\phi)^{-1})g = (x\phi_1\phi_2)(m\phi_1\phi_2)^{-1}$ is a homomorphism of R_M onto $\phi_1(R)_{\phi_1(M)}$ such that $\phi g = \phi_1\phi_2$. If ϕ_3 is the natural homomorphism of R_M onto $R_M[/]A^e$, then $A^e = \ker(\phi_3) = \ker(g)$. By LaTorre [7], Theorem 2.5, there is a semi-isomorphism h of $R_M[/]A^e$ onto $\phi_1(R)_{\phi_1(M)}$ with $g = \phi_3 h$. Since a semi-isomorphism of a semisubtractive semiring into a semiring whose zeroid is zero is an isomorphism, h is an isomorphism.

An ideal A of R is a *contracted* ideal if $A^{ec} = A$. An ideal A^* of R_M is an *extended* ideal if $(A^*)^{ce} = A^*$.

THEOREM 13. *Every ideal of R_M is an extended ideal.*

PROOF. Clearly $(A^*)^{ce} \subset A^*$. If $(x\phi)(m\phi)^{-1} \in A^*$ then $x\phi \in A^*$, so that $x \in (A^*)^c$. Hence $(x\phi)(m\phi)^{-1} \in (A^*)^{ce}$ and $A^* = (A^*)^{ce}$.

THEOREM 14. *If R is Noetherian, then R_M is Noetherian.*

The proof is trivial.

An element x in a hemiring is prime to an ideal A if $A : \{x\} = A$. A subset E is prime to A if each element of E is prime to A.

THEOREM 15. *If A is an ideal of R, and if $Z = (0)$, then* (1) $A^{ec} = \{x \in R | mx \in A$ *for some* $m \in M\}$ *and* (2) $A = A^{ec}$ *if and only if M is prime to A.*

PROOF. Let $D = \{x \in R | mx \in A \text{ for some } m \in M\}$. If $b \in A^{ec}$ then $b\phi \in A^e$, so that $b\phi = (a\phi)(m\phi)^{-1}$ with $a \in A$. Hence $bm\phi = a\phi$, and

$bm+n_1 = a+n_2$ for some $n_1, n_2 \in N$. Now $m_1n_1 = m_2n_2 = 0$ for some $m_1, m_2 \in M$. Hence $bmm_1m_2 = am_1m_2 \in A$, implying $b \in D$. Since $D \subset A^{ec}$ trivially, $D = A^{ec}$.

Statement (2) follows easily from (1).

THEOREM 16. *Let $Z = (0)$ and let P be an ideal of R. If P meets M, then $P^e = R_M$. If P is prime and disjoint from M, and if Q is primary to P, then* (1) $N \subset Q$, (2) $Q \cap M = \emptyset$, (3) $Q^{ec} = Q$ and $P^{ec} = P$, and (4) *Q^e is primary to P^e in R_M.*

PROOF. If $P \cap M \neq \emptyset$, then P^e contains a unit and is R_M. Statements (1) and (2) follow easily. If $m \in M$ then $x \in Q : \{m\}$ implies $xm \in Q$ and, since $m \notin P$, $x \in Q$. Thus $Q : \{m\} = Q$. By Theorem 15, $Q^{ec} = Q$. Similarly $P^{ec} = P$.

Let $((r\phi)(m\phi)^{-1})((s\phi)(n\phi)^{-1}) \in Q^e$ and $(r\phi)(m\phi)^{-1} \notin Q^e$, so that $r \notin Q$ and $(rs\phi)(mn\phi)^{-1} = (q\phi)(p\phi)^{-1}$ where $q \in Q$. Hence $rsp\phi = mnq\phi$, which implies $rsp+n_1 = mnq+n_2$ for some n_1, $n_2 \in N$. Also $m_1n_1 = m_2n_2 = 0$ for some m_1, $m_2 \in M$. Hence $mnqm_1m_2 = rspm_1m_2 \in Q$ and thus $rs \in Q$. For some positive integer t, $s^t \in Q$. Thus $((s\phi)(n\phi)^{-1})^t \in Q^e$. Clearly $Q^e \neq R_M$, so that Q^e is primary. Similarly P^e is prime. Further $R(Q^e) = P^e$.

THEOREM 17. *If $Z = (0)$ and if P is a prime ideal of R, then P^e is a maximal ideal of R_M if and only if P is maximal with respect to M.*

PROOF. Let P be maximal with respect to M. Since $P = P^{ec}$, $P^e \neq R_M$. There is a maximal ideal A^* of R_M with $P^e \subset A^*$. Hence $P \subset (A^*)^c$. Since $(A^*)^c$ does not meet M, $P = (A^*)^c$ and hence $P^e = A^*$.

Conversely P does not meet M. There is an ideal A of R which is maximal with respect to M and contains P. Now $A^{ec} = A$ and either $A^e = P^e$ or $A^e = R_M$. The latter implies $A = R$, a contradiction. Hence $A^e = P^e$ and thus $A = P$.

COROLLARY 18. If $Z = (0)$ and if A is an ideal of R, then $A^e = R_M$ if and only if A meets M.

Part of the proof is in Theorem 16 and the other is clear from Theorem 17.

THEOREM 19. *If $Z = (0)$, and if $Q_1, \cdots, Q_n$ are primary ideals of R such that, for some $r \in \{0, 1, \cdots, n\}$, $Q_i \cap M = \emptyset$ for $i \leqq r$ and $Q_i \cap M \neq \emptyset$ for $i > r$, then $(\bigcap_{i=1}^n Q_i)^e = \bigcap_{i=1}^n Q_i^e$. For $r \geqq 1$, $\bigcap_{i=1}^r Q_i^e = \bigcap_{i=1}^n Q_i^e$. Further, if $\bigcap_{i=2}^n Q_i \not\subset Q_1$ and $Q_1^e \neq R_M$, then $\bigcap_{i=2}^n Q_i^e \not\subset Q_i^e$.*

The proof follows analogously to the corresponding theorem in ring theory.

Corollary 20. If $Z = (0)$, if $Q_1, \cdots, Q_n$ are primary ideals of R such that, for some $r \in \{1, \cdots, n\}$, $Q_i \cap M = \emptyset$ for $i \leqq r$ and $Q_i \cap M \neq \emptyset$ for $i > r$, and if $\bigcap_{i=1}^{n} Q_i = (0)$, then $N = \bigcap_{i=1}^{r} Q_i$.
The proof is clear.

Corollary 21. If R is Noetherian and if $Z = (0)$, then the following ideals of R are equal: N, the intersection N' of all primary ideals of R which are disjoint from M, and the intersection N'' of all primary components of (0) that are disjoint from M.

Proof. By Theorem 4, N and (0) are finite intersections of primary k-ideals, say $N = \bigcap Q_i$ and $(0) = \bigcap Q'_j$. Some Q_i is disjoint from M and some Q'_j is also disjoint from M. By Theorem 16, $N \subset N'$. Clearly $N' \subset N''$. By Corollary 20, $N = N''$.

Theorem 22. *Suppose M_1 is a multiplicative subsemigroup of R_M with $0 \notin M_1$. If M_2 is the multiplicative subsemigroup of R generated by M and all $r \in R$ such that $(r\phi)(m\phi)^{-1} \in M_1$ for some $m \in M$, then R_{M_2} is semi-isomorphic to $(R_M)_{M_1}$.*

Proof. Let $N_1 = \{x \in R_M | xm = 0 \text{ for some } m \in M_1\}$, let $N_2 = \{x \in R | xm \in Z \text{ for some } m \in M_2\}$, let ϕ_1 be the natural homomorphism of R_M onto $R_M[/]N_1$, and let ϕ_2 be the natural homomorphism of R onto $R[/]N_2$. For each $m \in M_2$, $m\phi\phi_1$ has an inverse in $(R_M)_{M_1}$. Also $(R_M)_{M_1}$ is generated by $\phi_1(\phi(R))$ and the inverses of elements of $\phi_1(\phi(M_2))$. By Theorem 11, the map f of R_{M_2} to $(R_M)_{M_1}$ defined by $((x\phi_2)(m\phi_2)^{-1})f = (x\phi\phi_1)(m\phi\phi_1)^{-1}$ is a homomorphism with $\phi\phi_1 = \phi_2 f$. Clearly f is onto and $\ker(\phi\phi_1) \subset N_2 = \ker(\phi_2)$. Hence $\ker(f) = (0)$ and R_{M_2} is semi-isomorphic to $(R_M)_{M_1}$.

Theorem 23. *Let $Z = (0)$, let every element of M be multiplicatively cancellable in R, let R' be a semisubtractive commutative hemiring such that $R \subset R'$, $R' \subset R_M$, and its zeroid is (0). Then $R_M = R'_M$.*
The proof is trivial.

Theorem 24. *If $1 \in R$ and M' is a multiplicative subsemigroup of R such that $M \subset M'$, $0 \notin M'$, and every element of M' is the product of an element of M and a unit of R, then $R_M \cong R_{M'}$.*
The proof follows easily.

REFERENCES

P. Allen
[1] Ideal theory in semirings, Dissertation, Texas Christian University, 1967.

S. Bourne and H. Zassenhaus
[2] On the semiradical of a semiring, Proc. Nat. Acad. Sci. U.S.A., Vol. 44 (1958), 907–914.

J. Dieudonné
[3] Topics in Local Algebra, Notre Dame Mathematical Lectures Number 10, Notre Dame, Indiana: Notre Dame Press, 1967.

M. Henriksen
[4] Ideals in semirings with commutative addition, Notices A.M.S., Vol. 5 (1958) 321.

K. Iizuka
[5] On the Jacobson radical of a semiring, Tohoku Math. J., Vol. 2 (1959) 409–421.

F. Lambek
[6] Lectures on Rings and Modules, Toronto: Blaisdell Publishing Co., 1966.

D. LaTorre
[7] On *h*-ideals and *k*-ideals in hemirings, Publ. Math. Debrecen, Vol. 12 (1965) 219–226.

K. Murata
[8] On the quotient semigroup of a non-commutative semigroup, Osaka Math. J., Vol. 2 (1950) 1–5.

M. Nagata
[9] Local Rings, Interscience Tracts in Pure and Applied Mathematics Number 13, New York: Interscience Publishers, 1962.

H. Weinert
[10] Über Halbringe und Halbkörper II, Acta Math. Acad. Sci. Hungary, Vol. 14 (1963), 209–227.

O. Zariski and P. Samuel
[11] Commutative Algebra, Vol. 1, Princeton: D. Van Nostrand Co., Inc., 1959.

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