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CONVEXITY OF BALLS AND FIXED-POINT THEOREMS FOR MAPPINGS WITH NONEXPANSIVE SQUARE

by

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Introduction

Recently F. E. Browder, M. Edelstein, W. A. Kirk and others investigated the conditions under which the nonexpansive mapping of a closed, bounded and convex subset of a Banach space has a fixed point. The Kirk's result is fairly strong and states that it is sufficient to assume that the set is weakly compact and the space has the normal structure.

We shall try to extend this result to the case of mappings with the nonexpansive second iteration. To investigate these problems we shall use the method based on the property of the modulus of convexity and on some classification of Banach spaces with respect to convexity of balls.

Notations and definitions

Let B be an arbitrary Banach space and let $\|\ \|$, Θ be the norm and the zero element in B. The elements of B will be denoted $x, y, z, \cdots, d(X)$ will denote the diameter of a set $X \subset B$ and K_r will denote the ball with the radius r centered at Θ.

First of all we quote some classical definitions connected with the various classes of B spaces.

DEFINITION 1 [4]. The space B is called uniformly convex iff for every positive number ε, there exist a positive number δ such that for arbitrary $x, y \in K_1$, the inequality

$$\|x-y\| \geqq \varepsilon \tag{1}$$

implies

$$\left\|\frac{x+y}{2}\right\| \leqq 1-\delta. \tag{2}$$

DEFINITION 2 [2]. The space B has normal structure iff every bounded and convex subset X of B containing more than one point contains a nondiametral point, i.e. such point that $\sup\ [\|x-y\| : y \in X] < d(X)$.

DEFINITION 3 [5]. The space is called uniformly non-square iff there is a positive number δ such that there do not exist $x, y \in K_1$ for which

$$\left\|\frac{x-y}{2}\right\| > 1-\delta$$

and

$$\left\|\frac{x+y}{2}\right\| > 1-\delta.$$

DEFINITION 4 [4]. The space B is called strictly convex iff there are not segments laying on the boundary of K_1.

It is known that every uniformly convex space is strictly convex and has normal structure [7]. Moreover every uniformly nonsquare space is reflexive [5].

Modulus and characteristic of convexity

DEFINITION 5 [7]. The modulus of convexity of the space B is the function $\delta : [0, 2] \to [0, 1]$ defined by the following formula

$$\delta(\varepsilon) = \inf\left[1-\left\|\frac{x+y}{2}\right\| : x, y \in K_1 \|x-y\| \geqq \varepsilon\right].$$

LEMMA 1. *The function $\delta(\varepsilon)$ is nondecreasing and convex.*

PROOF. Monotonicity of $\delta(\varepsilon)$ is obvious. Let $u, v \in B$ be two arbitrary points such that $u \neq \Theta$, $v \neq \Theta$. Denote $N(u, v)$ the set of all pairs (x, y) such that $x \in K_1$, $y \in K_1$ and $x-y = au$, $x+y = bv$ for some real numbers a, b.

Consider the function:

$$\delta(u, v, \varepsilon) = \inf\left[1-\left\|\frac{x+y}{2}\right\| : x, y \in N(u, v), \|x-y\| \geqq \varepsilon\right].$$

Let $(x_1, y_1) \in N(u, v)$, $(x_2, y_2) \in N(u, v)$, $\|x_1-y_1\| \geqq \varepsilon_1$, $\|x_2-y_2\| \geqq \varepsilon_2$. Put

$$x_3 = \tfrac{1}{2}(x_1+x_2), \quad y_3 = \tfrac{1}{2}(y_1+y_2).$$

Obviously $(x_3, y_3) \in N(u, v)$ and hence

$$\|x_3-y_3\| = \tfrac{1}{2}\left(\|x_1-y_1\|+\|x_2-y_2\|\right) \geqq \tfrac{1}{2}(\varepsilon_1+\varepsilon_2)$$

and

$$(3) \quad 1-\left\|\frac{x_3+y_3}{2}\right\| = \tfrac{1}{2}\left[\left(1-\left\|\frac{x_1+y_1}{2}\right\|\right) + \left(1-\left\|\frac{x_2+y_2}{2}\right\|\right)\right].$$

Taking the infimum of right hand side of (3) and using the definition of $\delta(u, v, \varepsilon)$ we obtain

$$\delta(u, v, \tfrac{1}{2}(\varepsilon_1+\varepsilon_2)) \leqq \tfrac{1}{2}\,[\delta(u, v, \varepsilon_1)+\delta(u, v, \varepsilon_2)]$$

so $\delta(u, v, \varepsilon)$ is convex. Because each pair (x, y), $x \in K_1$, $y \in K_1$ belongs to some set $N(u, v)$ than

$$\delta(\varepsilon) = \inf\,[\delta(u, v, \varepsilon) : u, v \in B, u \neq \Theta, v \neq \Theta]$$

Obviously the infimum of arbitrary family of convex functions is convex.

Monotonicity and convexity of $\delta(\varepsilon)$ imply that $\delta(\varepsilon)$ is continuous except in at most one point $\varepsilon = 2$. Moreover for arbitrary $x, y \in K_r$ and for arbitrary number a such that $0 \leqq a \leqq 2r$ and $||x-y|| \geqq a$, the inequality

$$\left\|\frac{x+y}{2}\right\| \leqq \left(1-\delta\left(\frac{a}{r}\right)\right) r \tag{4}$$

holds.

Definition 6. The characteristic of convexity of the space B is the number $\varepsilon_0 = \sup\,[\varepsilon : \delta(\varepsilon) = 0]$.

Some of the mentioned above classes of B-spaces can be fully characterised by the number ε_0 and the modulus of convexity. The following lemmas can be easily proved:

Lemma 2. *B is uniformly convex iff* $\varepsilon_0 = 0$.

Lemma 3. *B is uniformly non-square iff* $\varepsilon_0 < 2$.

Lemma 4. *B is strictly convex iff* $\delta(2) = 1$.

An interesting subclass of the class of B-spaces with normal structure can also be described by the characteristic of convexity.

Lemma 5. *If* $\varepsilon_0 < 1$, *then B has normal structure.*

Proof. Suppose X is a convex subset of B, containing more than one point and such that all its points are diametral. Let x, y, z are any points of X satisfying the conditions

$$||x-y|| \geqq d-\alpha$$

$$\left\|z-\frac{x+y}{2}\right\| \geqq d-\alpha$$

where $d = d(X)$ and α is an arbitrary number from the interval $(0, d)$.

Put $u = z-x$, $v = z-y$. Because $||u|| \leqq d$, $||v|| \leqq d$, $||u-v|| = ||x-y|| \geqq d-\alpha$ so in view of (4) we have

$$d-\alpha \leqq \left\| z - \frac{x+y}{2} \right\| = \left\| \frac{u+v}{2} \right\| \leqq \left(1-\delta\left(\frac{d-\alpha}{d}\right)\right) d$$

what is the contradiction for sufficiently small α.

Fixed-points theorems

Let C be a closed and convex subset of B and let F be a continuous transformation of C into C. By F^n we denote the n-th iteration of F and by I the identity transformation of C.

THEOREM 1. *If $F^2 = I$ and if for arbitrary $x, y \in C$ we have*

$$\|Fx - Fy\| \leqq k\|x-y\| \tag{5}$$

where k is a constant such that

$$\frac{k}{2}\left(1-\delta\left(\frac{2}{k}\right)\right) < 1 \tag{6}$$

then F has at least one fixed point.

PROOF. Let x be an arbitrary point of C. Because of

$$\left\| F\left(\frac{x+Fx}{2}\right) - x \right\| = \left\| F\left(\frac{x+Fx}{2}\right) - F^2x \right\| \leqq k \left\| \frac{x+Fx}{2} - Fx \right\| = \frac{k}{2}\|x-Fx\|,$$

$$\left\| F\left(\frac{x+Fx}{2}\right) - Fx \right\| \leqq k \left\| \frac{x+Fx}{2} - x \right\| = \frac{k}{2}\|x-Fx\|$$

we obtain

$$\left\| \frac{x+Fx}{2} - F\left(\frac{x+Fx}{2}\right) \right\| \leqq \left(1-\delta\left(\frac{2}{k}\right)\right)\frac{k}{2}\|x-Fx\|.$$

Now if we put $G = I+F/2$ we see that

$$\|G^2x - Gx\| \leqq \left(1-\delta\left(\frac{2}{k}\right)\right)\frac{k}{2}\|x-Gx\|$$

hence the sequence $x_n = G^n x$ is convergent. Let $y = \lim x_n$. Obviously $y \in C$ and $y = Gy = Fy$.

Let us notice that the inequality (6) holds for $k < 2$ in arbitrary Banach space. The necessary and sufficient condition to satisfy (6) with some $k \geqq 2$ is $\varepsilon_0 < 1$.

For example in l^2 space as well as in arbitrary Hilbert space we have $\delta(\varepsilon) = 1-\sqrt{1-(\varepsilon/2)^2}$ so (6) is satisfied for $k < \sqrt{5}$.

It is not known whether the evaluation (6) is sharp for Theorem 1. Although if we assume that F is only continuous our theorem become false even in Hilbert space. To show it we use the known result [1] that every infinitely dimensional Hilbert space H is homeomorphic with $H \setminus \theta$. If h is such homeomorphism then $F = h^{-1}(-h)$ is the fixed-point-free involution of H.

THEOREM 2. *Suppose C is as in Theorem* 1 *but bounded and suppose B is such that $\varepsilon_0 < 1$ and $\delta(2) = 1$. If F satisfies the conditions* (5), (6) *and if*

$$\|F^2x - F^2y\| \leqq \|x-y\| \tag{7}$$

for $x, y \in C$, then there exist a fixed point of F.

PROOF. Since $\varepsilon_0 < 1$ B is uniformly non-square and in view of [5], it is reflexive. Moreover, in view of Lemma 5, B has normal structure. According to (7) and Kirk's fixed-point theorem [6] the set $C^* = [x : x = F^2x]$ is nonempty. The strict convexity of B implies that C^* is convex [7]. Obviously we have $F(C^*) = C^*$ and $F^2 = I$ on C^*. Hence using Theorem 1 we obtain our result.

Some unsolved problems

The following questions seem to be interesting:

1°. Does exist the B-space with $\varepsilon_0 = 1$ and without normal structure?

2°. Is the condition (6) exact? It means, does exist the space B, the convex set $C \subset B$ and the involution F of C satisfying the condition (6) with k such that

$$\frac{k}{2}\left(1-\delta\left(\frac{2}{k}\right)\right) = 1$$

and without fixed points?

3°. What are the sufficient condition for existence of the fixed points for the involutions of higher order (i.e. such mappings F that $F^n = I$ for some integer n)?

4°. It is not known whether Kirk's theorem is true in arbitrary reflexive B-space. Is it true in the spaces with $\varepsilon_0 = 1$? If yes, is it true in the space with $\varepsilon_0 < 2$? What is the greatest lower bound of such numbers ε that Kirk's theorem is true for all spaces with $\varepsilon_0 < \varepsilon$?

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