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*Compositio Mathematica*, tome 22, nº 3 (1970), p. 261-264
<http://www.numdam.org/item?id=CM_1970__22_3_261_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 3, 1970, pag. 261-264
Wolters-Noordhoff Publishing
Printed in the Netherlands

# HOMOTOPY TYPE OF MAPPING TRACKS

by

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## 1. Introduction

Let $(E, e_0)$ and $(B, b_0)$ be pointed spaces and $p : (E, e_0) \to (B, b_0)$ a continuous function. If $(E,p,B)$ has the weak covering homotopy property, it follows that the basic fiber $F = p^{-1}(b_0)$ and the mapping track $\Sigma p = \{(e, \alpha) \in E \times B^I : p(e) = \alpha(0)$ and $\alpha(1) = b_0\}$ have the same homotopy type, but necessary and sufficient conditions for the existence of such a homotopy equivalence are not known. For the case in which $(E, p, B)$ is a principal fiber structure, this paper gives necessary and sufficient conditions in terms of a lifting function that the fiber structures $(F, i, E)$ and $(\Sigma p, \pi, E)$ be $H$-isomorphic. Here $i : F \to E$ is the inclusion map and $\pi : \Sigma p \to E$ is the projection on the first component.

## 2. Preliminaries

DEFINITION. A pair $(A, q, C)$ and $(A', q', C)$ of fiber structures over the same base $C$ have the *same homotopy type* or are *homotopy equivalent* means that there is a homotopy equivalence $h : A \to A'$ such that $q'h \sim q$ (homotopic).

DEFINITION. A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of topological spaces with base points and continuous maps is *exact* means that

(1) The composition $gf$ is null-homotopic (i.e. homotopic to the constant map whose only value is the base point of $Z$); and

(2) for each space $W$ and continuous map $h$: $W \to Y$ such that $gh$ is null-homotopic, there is a continuous map $h' : W \to X$ such that $fh' \sim h$.

NOTE. The functions involved in this paper are not assumed to be base point preserving unless specifically stated. All function spaces are assigned the compact-open topology.

DEFINITION. The fiber structure $(E, p, B)$ is *principal* means that the basic fiber $F$ is an $H$-group which operates on $E$ in the following sense: There is a continuous map $\mu : E \times F \to E$ such that the restriction of $\mu$ to $F \times F$ is homotopic to the composition of the multiplication on $F$ and the inclusion of $F$ in $E$.

DEFINITION. A *quasi-lifting function* for $(E, p, B)$ is a continuous map $\lambda : \Sigma p \to E^I$ with the following properties:

(1) $\lambda(e, \alpha)(0) = e, \quad \lambda(e, \alpha)(1) \in F \quad (e, \alpha) \in \Sigma p,$

(2) the map $l$: $\Omega B \to \Omega(E, F)$ defined by

$$l(\beta) = \lambda(e_0, \beta) \quad \beta \in \Omega B$$

is a homotopy equivalence, and

(3) There is a continuous map $\theta : \Omega E \to \Omega E$ such that the diagram

$$\begin{array}{ccc} \Omega E & \xhookrightarrow{\ i\ } & \Omega(E, F) \\ {\scriptstyle \theta}\downarrow & & \uparrow{\scriptstyle l} \\ \Omega E & \xrightarrow[\Omega_p]{} & \Omega B \end{array}$$

commutes up to homotopy.

Here $\Omega B$ is the space of based loops in $B$, $\Omega(E, F)$ is the space of paths in $E$ beginning at $e_0$ and ending in $F$ and $\Omega_p$ is the natural map induced by $p$. If $\lambda : \Sigma p \to E^I$ is a quasi-lifting function, there is a continuous map $\lambda^* : \Sigma p \to F$ defined by

$$\lambda^*(e, \alpha) = \lambda(e, \alpha)(1) \quad (e, \alpha) \in \Sigma p.$$

THEOREM 1. *If $(E, p, B)$ has the weak covering homotopy property, then it has a quasi-lifting function.*

PROOF. Let

$$\rho : \Delta = \{(e, \alpha) \in E \times B^I : p(e) = \alpha(0)\} \to E^I$$

be a weak lifting function and $G : \Delta \times I \to E$ a homotopy such that

$$G(e, \alpha, 0) = e, \; G(e, \alpha, 1) = \rho(e, \alpha)(0),$$
$$pG(e, \alpha, t) = p(e) \qquad (e, \alpha) \in \Delta, \; t \in I.$$

Let $(PE, \pi_E, E)$ denote the usual path fibration ($PE$ is the space of paths in $E$ with initial point $e_0$ and $\pi_E$ is defined by evaluation at the terminal point). Then $(PE, p\pi_E, B)$ and $(PB, \pi_B, B)$ have the weak covering homotopy property and both total spaces are contractible.

Define $\mu : PB \to PE$ by

$$\mu(\beta) = G(e_0, \beta, \cdot) * \rho(e_0, \beta) \qquad \beta \in PB$$

where $*$ denotes the usual operation of juxtaposition of paths. Since $\mu$ is a fiber map, it is a fiber homotopy equivalence between $(PB, \pi_B, B)$ and $(PE, p\pi_E, B)$ [2, Theorem 6.1]. In particular, the restriction of $\mu$ to $\Omega B$ is a homotopy equivalence between $\Omega B$ and $\Omega(E, F)$.

A quasi-lifting function for $(E, p, B)$ is then defined by

$$\lambda(e, \alpha) = G(e, \alpha, \cdot) * \rho(e, \alpha) \qquad (e, \alpha) \in \Sigma p.$$

In this case we take $\theta = id_{\Omega E}$.

## 3. Homotopy type of $\Sigma p$

THEOREM 2. *If $(E, p, B)$ is a principal fiber structure such that $\Sigma p$ is an $H$-group, then $(F, i, E)$ and $(\Sigma p, \pi, E)$ are $H$-isomorphic if and only if there is a quasi-lifting function $\lambda : \Sigma p \to E^I$ such that $\lambda^*$ is an $H$-homomorphism.*

PROOF. (Sufficiency) The sequence

$$\Omega\Sigma p \xrightarrow{\Omega\pi} \Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{q} \Sigma p \xrightarrow{\pi} E \xrightarrow{p} B$$

is exact where $q(\beta) = (e_0, \beta)$ for each $\beta \in \Omega B$ [1, Theorem 3]. The existence of a quasi-lifting function implies the exactness of the sequence

$$\Omega F \xrightarrow{i} \Omega E \xrightarrow{i} \Omega(E, F) \xrightarrow{m} F \xrightarrow{i} E \xrightarrow{p} B$$

where $i$ denotes inclusion maps and $m : \Omega(E, F) \to F$ is the evaluation at the terminal point.

Let $t : \Omega(E, F) \to \Omega B$ be a homotopy inverse for $l$ and define $v : F \to \Sigma p$ by

$$v(x) = (x, c(b_0)), \; c(b_0)(I) = b_0.$$

Then $i\lambda^* v \sim i$ so that $i(\lambda^* v \cdot j)$ is null-homotopic where $\cdot$ denotes the $H$-group operation and $j$ denotes inversion. Hence there is a continuous map $s : F \to \Omega(E, F)$ such that

$$ms \cdot id_F \sim \lambda^* v.$$

Then

$$id_F \sim jms \cdot \lambda^* v \sim j\lambda^* qts \cdot \lambda^* v \sim \lambda^*(jqts \cdot v)$$

so the map $\rho = jqts \cdot v$ is a right homotopy inverse for $\lambda^*$.

Now consider $\rho\lambda^* \colon \Sigma p \to \Sigma p$ and observe that

$$0 \sim i\lambda^*(\rho\lambda^* \cdot j) \sim \pi(\rho\lambda^* \cdot j).$$

Then there is a continuous map $\sigma : \Sigma p \to \Omega B$ such that

$$q\sigma \sim \rho\lambda^* \cdot j.$$

Hence

$$ml\sigma \sim \lambda^* q\sigma \sim \lambda^*(\rho\lambda^* \cdot j) \sim 0$$

so there is a continuous map $\sigma' : \Sigma p \to \Omega E$ such that

$$i\sigma' \sim l\sigma.$$

Let $\theta : \Omega E \to \Omega E$ denote the map specified in the definition of quasi-lifting function. Since $l\Omega_p\theta \sim i$, then

$$qti \sim qtl\Omega_p\theta \sim q\Omega_p\theta \sim 0.$$

Hence

$$\rho\lambda^* \cdot j \sim q\sigma \sim qtl\sigma \sim qti\sigma' \sim 0$$

so that $\rho\lambda^*$ is homotopic to the identity on $\Sigma p$. Since $\lambda^*$ is an $H$-homomorphism and $i\lambda^* \sim \pi$, it follows that $(\Sigma p, \pi, E)$ and $(F, i, E)$ are $H$-isomorphic.

(Necessity) Suppose now that $h : F \to \Sigma p$ is an $H$-isomorphism such that $\pi h \sim i$. Since $(\Sigma p, \pi, E)$ has the weak covering homotopy property, there is a continuous map $r = (r_1, r_2) : F \to \Sigma p$ such that $r$ is homotopic to $h$ and $r_1 = i$. Let $\delta : \Sigma p \to F$ be a homotopy inverse for $r$ and $R = (R^1, R^2) : \Sigma p \times I \to \Sigma p$ a homotopy such that

$$R_0 = id_{\Sigma p}, \quad R_1 = r\delta.$$

Define $\lambda : \Sigma p \to E^I$ and $t : \Omega(E, F) \to \Omega B$ by

$$\lambda(e, \alpha)(s) = R^1(e, \alpha, s) \quad (e, \alpha) \in \Sigma p, \quad s \in I$$
$$t(\sigma) = p\sigma * r_2\sigma(1) \quad \sigma \in \Omega(E, F).$$

Then $t$ is a homotopy inverse for the induced map $l : \Omega B \to \Omega(E, F)$ and the composite $qti : \Omega E \to \Sigma p$ is null-homotopic. Hence there is a continuous map $\theta : \Omega E \to \Omega E$ such that $\Omega_p\theta \sim ti$. Then

$$l\Omega_p\theta \sim lti \sim i$$

so that $\lambda$ is a quasi-lifting function such that $\lambda^* = \delta$ is an $H$-homomorphism.

## Acknowledgment

The author is grateful to the referee for his helpful suggestions during the preparation of this paper.

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(Oblatum 8–VIII–69, 23–III–70)

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